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Stability results in $\mathcal{L}$-fuzzy normed spaces for a cubic functional equation

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Abstract
We establish some stability results concerning the functional equation

$$nf(x+ny)+f(nx-y)=\frac{n(n^2+1)}{2}\big[f(x+y)+f(x-y)\big]+(n^4-1)f(y),$$

where $n \geq 2$ is a fixed integer, in the setting of $\mathcal{L}$-fuzzy normed spaces that in turn generalize a Hyers-Ulam stability result in the framework of classical normed spaces.
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1 Introduction and preliminaries

The theory of fuzzy sets was introduced by Zadeh [1] in 1965. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [2–9]. One of problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces. In 1984, Katsaras [10] defined a fuzzy norm on a linear space, and at the same year Wu and Fang [11] also introduced a fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. Some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [2, 3, 9, 12–14]. In 1994, Cheng and Mordeson introduced the definition of a fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [15]. In 2003, Bag and Samanta [16] modified the definition of Cheng and Mordeson [17] by removing a regular condition. In 2004, Park [18] introduced and studied the notion of intuitionistic fuzzy metric spaces. In 2006, Saadati and Park introduced and studied the notion of intuitionistic fuzzy normed spaces.

On the other hand, the study of stability problems for a functional equation is related to a question of Ulam [19], concerning the stability of group homomorphisms, affirmatively answered for Banach spaces by Hyers [20]. Subsequently, the result of Hyers was generalized by Aoki [21] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. We refer the interested readers for more information on such problems to the papers [19, 23–30].

Let X and Y be real linear spaces and let $f : X \to Y$ be a mapping. If $X = Y = \mathbb{R}$, the cubic function $f(x) = cx^3$, where c is a real constant, clearly satisfies the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x).$$

Hence, the above equation is called the *cubic functional equation*. Recently, Cho, Saadati and Wang [31] introduced the functional equation

$$3f(x+3y)+f(3x-y)=15\big[f(x+y)+f(x-y)\big]+80f(y),$$

which has $f(x) = cx^3$ ($x \in \mathbb{R}$) as a solution for $X = Y = \mathbb{R}$.

In this paper, we investigate the Hyers-Ulam stability of the functional equation as follows:

$$nf(x+ny)+f(nx-y)=\frac{n(n^2+1)}{2}\big[f(x+y)+f(x-y)\big]+\big(n^4-1\big)f(y), \tag{1.1}$$

where $n \geq 2$ is a fixed integer.

We recall some definitions and results for our main result in this paper.

A triangular norm (shorter t-norm) is a binary operation on the unit interval $[0,1]$, *i.e.*, a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following four axioms: for all $a, b, c \in [0,1]$,

(i) $T(a,b) = T(b,a)$ (commutativity);
(ii) $T(a, T(b,c)) = T(T(a,b),c)$ (associativity);
(iii) $T(a,1) = a$ (boundary condition);
(iv) $T(a,b) \leq T(a,c)$ whenever $b \leq c$ (monotonicity).

Basic examples are the Łukasiewicz t-norm T_L and the t-norms T_P, T_M and T_D, where $T_L(a,b) := \max\{a+b-1, 0\}$, $T_P(a,b) := ab$, $T_M(a,b) := \min\{a,b\}$ and

$$T_D(a,b) := \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\} = 1, \\ 0 & \text{otherwise} \end{cases}$$

for all $a, b \in [0,1]$.

For all $x \in [0,1]$ and all t-norms T, let $x_T^{(0)} := 1$. For all $x \in [0,1]$ and all t-norms T, define $x_T^{(n)}$ by the recursion equation $x_T^{(n)} = T(x_T^{(n-1)}, x)$ for all $n \in \mathbb{N}$. A t-norm T is said to be of *Hadžić type* (we denote it by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$ (see [12]).

Other important triangular norms are as follows (see [32]):

- The *Sugeno-Weber family* $\{T_\lambda^{\mathrm{SW}}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{\mathrm{SW}} := T_D$, $T_\infty^{\mathrm{SW}} := T_P$ and

  $$T_\lambda^{\mathrm{SW}}(x,y) := \max\left\{0, \frac{x+y-1+\lambda xy}{1+\lambda}\right\}$$

  if $\lambda \in (-1, \infty)$.
- The *Domby family* $\{T_\lambda^{\mathrm{D}}\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

  $$T_\lambda^{\mathrm{D}}(x,y) := \frac{1}{1+((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda)^{1/\lambda}}$$

  if $\lambda \in (0, \infty)$.

- The *Aczel-Alsina family* $\{T_\lambda^{\mathrm{AA}}\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

  $$T_\lambda^{\mathrm{AA}}(x,y) := e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}$$

  if $\lambda \in (0,\infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation by taking, for any $(x_1,\dots,x_n) \in [0,1]^n$, the value $T(x_1,\dots,x_n)$ defined by

$$T_{i=1}^0 x_i := 1, \qquad T_{i=1}^n x_i := T\big(T_{i=1}^{n-1} x_i, x_n\big) = T(x_1,\dots,x_n).$$

A t-norm T can also be extended to a countable operation by taking, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$, the value

$$T_{i=1}^\infty x_i := \lim_{n\to\infty} T_{i=1}^n x_i. \tag{1.2}$$

The limit on the right-hand side of (1.2) exists since the sequence $\{T_{i=1}^n x_i\}_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

Proposition 1.1 [32]

(1) *For $T \ge T_L$, the following equivalence holds*:

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \quad\Longleftrightarrow\quad \sum_{n=1}^\infty (1-x_n) < \infty.$$

(2) *If T is of Hadžić type, then*

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$$

for all sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$.

(3) *If $T \in \{T_\lambda^{\mathrm{AA}}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{\mathrm{D}}\}_{\lambda\in(0,\infty)}$, then*

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \quad\Longleftrightarrow\quad \sum_{n=1}^\infty (1-x_n)^\alpha < \infty.$$

(4) *If $T \in \{T_\lambda^{\mathrm{SW}}\}_{\lambda\in[-1,\infty)}$, then*

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \quad\Longleftrightarrow\quad \sum_{n=1}^\infty (1-x_n) < \infty.$$

We give some definitions and related lemmas for our main result.

Definition 1.2 [17] Let $\mathcal{L} = (L, \le_L)$ be a complete lattice and U a non-empty set called universe. An $\mathcal{L}$*-fuzzy set* $\mathcal{A}$ on U is defined by a mapping $\mathcal{A} : U \to L$. For any $u \in U$, $\mathcal{A}(u)$ represents the *degree* (in L) to which u satisfies $\mathcal{A}$.

Lemma 1.3 [16] *Consider the set L^* and the operation $\le_{L^*}$ defined by*

$$L^* = \big\{(x_1,x_2) : (x_1,x_2) \in [0,1]^2 \textit{ and } x_1 + x_2 \le 1\big\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, \qquad x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Definition 1.4 [33] An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ on a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ for all $u \in U$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, further, they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

We presented the classical definition of t-norms which can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} := \inf L$ and $1_{\mathcal{L}} := \sup L$.

Definition 1.5 A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)$ $(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (boundary condition);
(ii) $(\forall (x,y) \in L^2)$ $(\mathcal{T}(x,y) = \mathcal{T}(y,x))$ (commutativity);
(iii) $(\forall (x,y,z) \in L^3)$ $(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z))$ (associativity);
(iv) $(\forall (x,x',y,y') \in L^4)$ $(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x,y) \leq_L \mathcal{T}(x',y'))$ (monotonicity).

A t-norm can also be defined recursively an $(n+1)$-array operation for each $n \in \mathbb{N} \setminus \{0\}$ by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_{(1)}, \ldots, x_{(n+1)}) = \mathcal{T}\big(\mathcal{T}^{n-1}(x_{(1)}, \ldots, x_{(n)}), x_{(n+1)}\big)$$

for all $n \geq 2$ and $x_{(i)} \in L$.

The t-norm $\mathcal{M}$ defined by

$$\mathcal{M}(x,y) = \begin{cases} x, & \text{if } x \leq_L y, \\ y, & \text{if } y \leq_L x, \end{cases}$$

is a continuous t-norm.

Definition 1.6 A t-norm $\mathcal{T}$ on L^* is said to be *t-representable* if there exist a t-norm T and a t-conorm S on $[0,1]$ such that

$$\mathcal{T}(x,y) = \big(T(x_1,y_1), S(x_2,y_2)\big)$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$.

Definition 1.7

(1) A *negator* on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.
(2) If a negator $\mathcal{N}$ on $\mathcal{L}$ satisfies $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involution negator*.
(3) The negator $\mathcal{N}_s$ on $([0,1], \leq)$ defined as $\mathcal{N}_s(x) = 1 - x$ for all $x \in [0,1]$ is called the *standard negator* on $([0,1], \leq)$.

Definition 1.8 The 3-tuple $(V, \mathcal{P}, \mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy normed space* if V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, \infty)$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in (0, \infty)$,

(i) $0_{\mathcal{L}} <_L \mathcal{P}(x, t)$;
(ii) $\mathcal{P}(x, t) = 1_{\mathcal{L}} \Leftrightarrow x = 0$;
(iii) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(iv) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)$;
(v) $\mathcal{P}(x, \cdot) : (0, \infty) \to L$ is continuous;
(vi) $\lim_{t\to 0}\mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called an *$\mathcal{L}$-fuzzy norm*. If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set and the t-norm $\mathcal{T}$ is t-representable, then the 3-tuple $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space*.

Definition 1.9 (see [15]) Let $(V, \mathcal{P}, \mathcal{T})$ be an *$\mathcal{L}$-fuzzy normed space*.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(V, \mathcal{P}, \mathcal{T})$ is called a *Cauchy sequence* if, for any $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and for any $t > 0$, there exists a positive integer n_0 such that
$$\mathcal{N}(\varepsilon) <_L \mathcal{P}(x_{n+p} - x_n, t)$$
for all $n \geq n_0$ and $p > 0$, where $\mathcal{N}$ is a negator on $\mathcal{L}$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(V, \mathcal{P}, \mathcal{T})$ is said to be *convergent* to a point $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{P}} x$) if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$ wherever $n \to \infty$ for all $t > 0$.
(3) If every Cauchy sequence in $(V, \mathcal{P}, \mathcal{T})$ is convergent in V, then the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ is said to be *complete* and the $\mathcal{L}$-fuzzy normed space is called an *$\mathcal{L}$-fuzzy Banach space*.

Lemma 1.10 *Let $\mathcal{P}$ be an $\mathcal{L}$-fuzzy norm on V. Then we have the following.*

(1) $\mathcal{P}(x, t)$ *is non-decreasing with respect to* $t \in (0, \infty)$ *for all* x *in* V.
(2) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ *for all* x, y *in* V *and all* $t \in (0, \infty)$.

Definition 1.11 Let $(V, \mathcal{P}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space. For any $t \in (0, \infty)$, we define the *open ball* $B(x, r, t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as
$$B(x, r, t) = \{y \in V : \mathcal{N}(r) <_L \mathcal{P}(x - y, t)\}.$$

A subset $A \subseteq V$ is called *open* if, for all $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$.

Let $\tau_{\mathcal{P}}$ denote the family of all open subsets of V. Then $\tau_{\mathcal{P}}$ is called the *topology induced by the $\mathcal{L}$-fuzzy norm $\mathcal{P}$*.

2 Main results

In this section, we study the stability of the functional equation (1.1) in $\mathcal{L}$-fuzzy normed spaces.

Theorem 2.1 *Let X be a linear space and $(Y, \mathcal{P}, \mathcal{T})$ a complete $\mathcal{L}$-fuzzy normed space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and Q be an $\mathcal{L}$-fuzzy set on $X^2 \times (0, \infty)$ satisfying*

$$\mathcal{P}\left(nf(x+ny) + f(nx-y) - \frac{n(n^2+1)}{2}\big[f(x+y)+f(x-y)\big] - \big(n^4-1\big)f(y), t\right) \geq_L Q(x,y,t) \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. If

$$\mathcal{T}_{i=1}^{\infty}\big(Q\big(n^{r+i-1}x, n^{r+i-1}y, n^{3r+2i+1}t\big)\big) = 1_{\mathcal{L}}$$

and

$$\lim_{n\to\infty} Q\big(n^r x, n^r y, n^{3r}t\big) = 1_{\mathcal{L}}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}\big(f(x) - C(x), t\big) \geq_L \mathcal{T}_{i=1}^{\infty}\big(Q\big(n^{i-1}x, 0, n^{2i+1}t\big)\big) \tag{2.2}$$

for all $x \in X$ and $t > 0$.

Proof Putting $y = 0$ in (2.1), we have

$$\mathcal{P}\left(\frac{f(nx)}{n^3} - f(x), t\right) \geq_L Q\big(x, 0, n^3 t\big)$$

for all $x \in X$ and all $t > 0$. Therefore, it follows that

$$\mathcal{P}\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{t}{n^{3k}}\right) \geq_L Q\big(n^k x, 0, n^3 t\big),$$

which implies that

$$\mathcal{P}\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, t\right) \geq_L Q\big(n^k x, 0, n^{3(k+1)} t\big),$$

that is,

$$\mathcal{P}\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{t}{n^{k+1}}\right) \geq_L Q\big(n^k x, 0, n^{2(k+1)} t\big)$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. Since $n \geq 2$, we get $1 > \frac{1}{n} + \cdots + \frac{1}{n^r}$ for all $r \in \mathbb{N}$. By the triangle inequality, it follows that

$$\begin{aligned}
\mathcal{P}\left(\frac{f(n^r x)}{n^{3r}} - f(x), t\right) &\geq_L \mathcal{P}\left(\frac{f(n^r x)}{n^{3r}} - f(x), \sum_{k=0}^{r-1} \frac{t}{n^{k+1}}\right) \\
&\geq_L \mathcal{T}_{k=0}^{r-1}\left(\mathcal{P}\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{t}{n^{k+1}}\right)\right) \\
&\geq_L \mathcal{T}_{k=1}^{r}\big(Q\big(n^{k-1}x, 0, n^{2k}t\big)\big)
\end{aligned} \tag{2.3}$$

for all $x \in X$, $t > 0$ and all $r \in \mathbb{N}$.

In order to prove the convergence of the sequence $\{\frac{f(n^r x)}{n^{3r}}\}$, we replace x with $n^m x$ in (2.3) to find that for all $m, r > 0$,

$$\mathcal{P}\left(\frac{f(n^{r+m}x)}{n^{3(r+m)}} - \frac{f(n^m x)}{n^{3m}}, t\right) \geq_L \mathcal{T}_{k=1}^{r}\big(Q\big(n^{k+m-1}x, 0, n^{2k+3m}t\big)\big)$$

for all $x \in X$, $t > 0$ and all $r \in \mathbb{N}$. Since the right-hand side of the inequality tend to $1_{\mathcal{L}}$ as m tends to infinity, the sequence $\{\frac{f(n^r x)}{n^{3r}}\}$ is a Cauchy sequence. Thus, we may define

$$C(x) := \lim_{r\to\infty} \frac{f(n^r x)}{n^{3r}}$$

for all $x \in X$.

Now we show that C is a cubic mapping. Replacing x, y with $n^r x$ and $n^r y$, respectively, in (2.1), it follows that

$$\begin{aligned}
&\mathcal{P}\left(\frac{nf(n^r(x+ny))}{n^{3r}} + \frac{f(n^r(nx-y))}{n^{3r}} - \frac{n(n^2+1)f(n^r(x+y))}{2n^{3r}}\right.\\
&\quad\left. - \frac{n(n^2+1)f(n^r(x-y))}{2n^{3r}} - \frac{(n^4-1)f(n^r y)}{n^{3r}}, t\right) \geq_L Q\big(n^r x, n^r y, n^{3r}t\big)
\end{aligned}$$

for all $x, y \in X$, $t > 0$ and all $r \in \mathbb{N}$. Taking the limit as $r \to \infty$, we find that C satisfies (1.1) for all $x, y \in X$.

To prove (2.2), taking the limit as $r \to \infty$ in (2.3), we have (2.2).

Finally, to prove the uniqueness of the cubic mapping C subject to (2.2), let us assume that there exists another cubic mapping C' which satisfies (2.2). Obviously, we have

$$C\big(n^r x\big) = n^{3r}C(x), \qquad C'\big(n^r x\big) = n^{3r}C'(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Hence, it follows from (2.2) that

$$\begin{aligned}
\mathcal{P}\big(C(x) - C'(x), t\big) &= \mathcal{P}\big(C\big(n^r x\big) - C'\big(n^r x\big), n^{3r}t\big)\\
&\geq_L \mathcal{T}\big(\mathcal{P}\big(C\big(n^r x\big) - f\big(n^r x\big), n^{3r-1}t\big), \mathcal{P}\big(f\big(n^r x\big) - C'\big(n^r x\big), n^{3r-1}t\big)\big)\\
&\geq_L \mathcal{T}\big(\mathcal{T}_{i=1}^{\infty}\big(Q\big(n^{r+i-1}x, 0, n^{3r+2i}t\big)\big), \mathcal{T}_{i=1}^{\infty}\big(Q\big(n^{r+i-1}x, 0, n^{3r+2i}t\big)\big)\big)\\
&= \mathcal{T}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. This proves the uniqueness of C and completes the proof. □

Corollary 2.2 *Let $(X, \mathcal{P}', \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and $(Y, \mathcal{P}, \mathcal{T})$ be a complete $\mathcal{L}$-fuzzy normed space. If $f : X \to Y$ is a mapping such that*

$$\begin{aligned}
&\mathcal{P}\left(nf(x+ny) + f(nx-y) - \frac{n(n^2+1)}{2}\big[f(x+y) + f(x-y)\big] - \big(n^4-1\big)f(y), t\right)\\
&\quad \geq_L \mathcal{P}'(x+y, t)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. If

$$\mathcal{T}_{i=1}^{\infty}\big(\mathcal{P}'\big(x+y, n^{2r+i+2}t\big)\big) = 1_{\mathcal{L}}$$

and

$$\lim_{n\to\infty} \mathcal{P}'\big(x+y, n^{2r}t\big) = 1_{\mathcal{L}}$$

for all $x, y \in X$ *and all* $t > 0$, *then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$\mathcal{P}\big(f(x) - C(x), t\big) \geq_L \mathcal{T}_{i=1}^{\infty}\big(\mathcal{P}'\big(x, n^{i+2}t\big)\big)$$

for all $x \in X$ *and all* $t > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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