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# Some properties concerning close-to-convexity of certain analytic functions

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**Abstract**

Let $f(z)$ be an analytic function in the open unit disk $\mathbb{D}$ normalized with $f(0) = 0$ and $f'(0) = 1$. With the help of subordinations, for convex functions $f(z)$ in $\mathbb{D}$, the order of close-to-convexity for $f(z)$ is discussed with some example.

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## 1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be convex of order $\alpha$ if it satisfies

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \alpha \quad \text{in } \mathbb{D}$$

for some real $\alpha$ ($0 \leqq \alpha < 1$). This family of functions was introduced by Robertson [1] and we denote it by $\mathcal{K}(\alpha)$.

A function $f(z) \in \mathcal{A}$ is called starlike of order $\alpha$ in $\mathbb{D}$ if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad \text{in } \mathbb{D}$$

for some real $\alpha$ ($0 \leqq \alpha < 1$).

This class was also introduced by Robertson [1] and we denote it by $\mathcal{S}^*(\alpha)$. By the definitions for the classes $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$, we know that $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$.

Marx [2] and Strohhäcker [3] showed that $f(z) \in \mathcal{K}(0)$ implies $f(z) \in \mathcal{S}^*(\frac{1}{2})$.

This estimate is sharp for an extremal function

$$f(z) = \frac{z}{1 - z}.$$

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Jack [4] posed a more general problem: What is the largest number $\beta = \beta(\alpha)$ so that

$$\mathcal{K}(\alpha) \subset \mathcal{S}^*\big(\beta(\alpha)\big).$$

MacGregor [5] determined the exact value of $\beta(\alpha)$ for each $\alpha$ ($0 \leqq \alpha < 1$) as the infimum over the disc $\mathbb{D}$ of the real part of a specific analytic function. It has been conjectured that this infimum is attained on the boundary of $\mathbb{D}$ at $z = -1$.

Wilken and Feng [6] asserted MacGregor's conjecture: If $0 \leqq \alpha < 1$ and $f(z) \in \mathcal{K}(\alpha)$, then $f(z) \in \mathcal{S}^*(\beta(\alpha))$, where

$$\beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2(1-\alpha)}-2} & \text{if } \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & \text{if } \alpha = \frac{1}{2}. \end{cases} \tag{1}$$

Ozaki [7] and Kaplan [8] investigated the following functions: If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > 0 \quad \text{in } \mathbb{D}$$

for some convex function $g(z)$, then $f(z)$ is univalent in $\mathbb{D}$. In view of Kaplan [8], we say that $f(z)$ satisfying the above inequality is close-to-convex in $\mathbb{D}$.

It is well known that the above definition concerning close-to-convex functions is equivalent to the following condition:

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } \mathbb{D}$$

for some starlike function $g(z) \in \mathcal{A}$.

Let us define a function $f(z) \in \mathcal{A}$ which satisfies

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \alpha \quad \text{in } \mathbb{D}$$

for some real $\alpha$ ($0 \leqq \alpha < 1$) and for some starlike function $g(z)$ in $\mathbb{D}$.

Then we call $f(z)$ close-to-convex of order $\alpha$ in $\mathbb{D}$ with respect to $g(z)$.

It is the purpose of the present paper to investigate the order of close-to-convexity of the functions which satisfy $f(z) \in \mathcal{K}(\alpha)$ and $0 \leqq \alpha < 1$.

## 2 Preliminary

To discuss our problems, we have to give here the following lemmas.

**Lemma 1** *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ and suppose that*

$$p(z) \prec \frac{1-\alpha z}{1+\beta z} \quad \text{in } \mathbb{D},$$

*where $\prec$ means the subordination, $0 < \alpha < 1$ and $0 < \beta < 1$.*

*Then we have*

$$\frac{1-\alpha}{1+\beta} < \operatorname{Re} p(z) < \frac{1+\alpha}{1-\beta}.$$

*This shows that*

$$\operatorname{Re} p(z) > 0 \quad in\ \mathbb{D}.$$

A proof is very easily obtained.

**Lemma 2** *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$, and suppose that there exists a point $z_0 \in \mathbb{D}$ such that*

$$\operatorname{Re} p(z) > c \quad for\ |z| < |z_0|$$

*and*

$$\operatorname{Re} p(z_0) = c, \qquad p(z_0) \neq c$$

*for some real $c$ $(0 < c < 1)$. Then we have*

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} \leqq \begin{cases} -\frac{1-c}{2c} & when\ \frac{1}{2} \leqq c < 1, \\ -\frac{c}{2(1-c)} & when\ 0 < c < \frac{1}{2}. \end{cases}$$

*Proof* Let us put

$$q(z) = \frac{p(z) - c}{1 - c}, \qquad q(0) = 1.$$

Then $q(z)$ is analytic in $\mathbb{D}$ and

$$\operatorname{Re} q(z) > 0 \quad for\ |z| < |z_0|$$

and

$$\operatorname{Re} q(z_0) = 0, \qquad q(z_0) \neq 0.$$

Then, from [9, Theorem 1], we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i\ell,$$

where

$$\ell \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \geqq 1 \quad when\ \arg q(z_0) = \frac{\pi}{2}$$

and

$$\ell \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leqq -1 \quad when\ \arg q(z_0) = -\frac{\pi}{2},$$

where $q(z_0) = \pm ia$ and $a > 0$.

For the case $\arg q(z_0) = \frac{\pi}{2}$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{z_0 p'(z_0)}{p(z_0) - c} = i\ell$$

and so

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{p(z_0) - c}{p(z_0)} i\ell,$$

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} = \operatorname{Re}\left(\frac{i(1-c)a}{c + i(1-c)a} i\ell\right) = \operatorname{Re}\left\{\left(\frac{c - i(1-c)a}{c^2 + (1-c)^2 a^2}\right)\left(-a(1-c)\ell\right)\right\}$$

$$= \frac{-(1-c)ca\ell}{c^2 + (1-c)^2 a^2} \leqq -\frac{c(1-c)}{2}\left(\frac{1 + a^2}{c^2 + (1-c)^2 a^2}\right).$$

If we put

$$h(x) = \frac{1 + x^2}{c^2 + (1-c)^2 x^2} \quad (x > 0),$$

then it is easy to see that

$$\frac{1}{c^2} < h(x) < \frac{1}{(1-c)^2} \quad \text{when } \frac{1}{2} \leqq c < 1$$

and

$$\frac{1}{(1-c)^2} < h(x) < \frac{1}{c^2} \quad \text{when } 0 < c < \frac{1}{2}.$$

This shows that

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} \leqq \begin{cases} -\frac{1-c}{2c} & \text{when } \frac{1}{2} \leqq c < 1, \\ -\frac{c}{2(1-c)} & \text{when } 0 < c < \frac{1}{2}. \end{cases}$$

For the case $\arg q(z_0) = -\frac{\pi}{2}$, applying the same method as above, we have the same conclusion

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} \leqq \begin{cases} -\frac{1-c}{2c} & \text{when } \frac{1}{2} \leqq c < 1, \\ -\frac{c}{2(1-c)} & \text{when } 0 < c < \frac{1}{2}. \end{cases}$$

This completes the proof of the lemma. □

Our next lemma is

**Lemma 3** *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ and suppose that there exists a point $z_0 \in \mathbb{D}$ such that*

$$\operatorname{Re} p(z) > c \quad \text{for } |z| < |z_0|$$

*and*

$$\operatorname{Re} p(z_0) = c, \qquad p(z_0) \neq c$$

*for some real $c$ ($c < 0$).*

*Then we have*

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} > -\frac{c}{2(1-c)} > 0. \tag{2}$$

*Proof* Let us put

$$q(z) = \frac{p(z) - c}{1 - c}, \qquad q(0) = 1.$$

Then $q(z)$ is analytic in $\mathbb{D}$. If $p(z)$ satisfies the hypothesis of the lemma, then there exists a point $z_0 \in \mathbb{D}$ such that

$$\operatorname{Re} q(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} q(z_0) = 0 \quad \text{and} \quad q(z_0) \neq 0,$$

then $p(z)$ satisfies the conditions of the lemma.

For the case $\arg q(z_0) = \frac{\pi}{2}$, applying the same method as in the proof of Lemma 2, we have

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} = -\frac{(1-c)ca\ell}{c^2 + (1-c)^2 a^2} \geqq -\frac{c(1-c)}{2} \left( \frac{1 + a^2}{c^2 + (1-c)^2 a^2} \right).$$

Putting

$$h(x) = \frac{1 + x^2}{c^2 + (1-c)^2 x^2} \quad (x > 0),$$

it follows that

$$h'(x) = \frac{(2c - 1)x}{(c^2 + (1-c)^2 x^2)^2} < 0 \quad (x > 0). \tag{3}$$

Therefore, from (3) we obtain (2).

For the case $\arg q(z_0) = -\frac{\pi}{2}$, applying the same method as above, we have the same conclusion as in the case $\arg q(z_0) = \frac{\pi}{2}$. □

## 3 The order of close-to-convexity

Now, we discuss the close-to-convexity of $f(z)$ with the help of lemmas.

**Theorem 1** *Let $f(z) \in \mathcal{A}$, and suppose that there exists a starlike function $g(z)$ such that*

(i) *for the case* $\frac{1}{2} \leqq c < 1$,

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \operatorname{Re} \frac{zg'(z)}{g(z)} - \frac{1-c}{2c} \quad in\ \mathbb{D},$$

$$\frac{zf'(z)}{g(z)} \neq c \quad in\ \mathbb{D}$$

*and*

(ii) *for the case* $0 < c < \frac{1}{2}$,

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \operatorname{Re} \frac{zg'(z)}{g(z)} - \frac{c}{2(1-c)} \quad in\ \mathbb{D},$$

$$\frac{zf'(z)}{g(z)} \neq c \quad in\ \mathbb{D}.$$

*Then we have*

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > c \quad in\ \mathbb{D}.$$

*This means that $f(z)$ is close-to-convex of order $c$ in $\mathbb{D}$.*

*Proof* Let us put

$$p(z) = \frac{zf'(z)}{g(z)}, \qquad p(0) = 1.$$

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}.$$

(i) For the case $\frac{1}{2} \leqq c < 1$, if there exists a point $z_0 \in \mathbb{D}$ such that

$$\operatorname{Re} p(z) > c \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = c,$$

then, applying Lemma 2 and the hypothesis of Theorem 1, we have

$$p(z_0) \neq c$$

and

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} \leqq -\frac{1-c}{2c}.$$

Thus, it follows that

$$1 + \mathrm{Re}\,\frac{z_0 f''(z_0)}{f'(z_0)} = \mathrm{Re}\,\frac{z_0 g'(z_0)}{g(z_0)} + \mathrm{Re}\,\frac{z_0 p'(z_0)}{p(z_0)}$$

$$\leqq \mathrm{Re}\,\frac{z_0 g'(z_0)}{g(z_0)} - \frac{1-c}{2c},$$

which contradicts the hypothesis of Theorem 1. (ii) For the case $0 < c < \frac{1}{2}$, applying the same method as above, we also have that

$$\mathrm{Re}\,\frac{zf'(z)}{g(z)} > c \quad \text{in } \mathbb{D}.$$

This completes the proof of the theorem. □

Applying Theorem 1, we have the following corollary.

**Corollary 1** *Let $f(z) \in \mathcal{A}$ be convex of order $\alpha$ $(0 < \alpha < 1)$, and suppose that there exists a starlike function $g(z)$ such that*
  (i) *for the case $\frac{1}{2} \leqq \alpha < c$,*

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} > \mathrm{Re}\,\frac{zg'(z)}{g(z)} - \frac{1-\beta(c)}{2\beta(c)} \quad \text{in } \mathbb{D},$$

$$\frac{zf'(z)}{g(z)} \neq \beta(c) \quad \text{in } \mathbb{D}$$

*and*
  (ii) *for the case $0 < \alpha < c \leqq \frac{1}{2}$,*

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} > \mathrm{Re}\,\frac{zg'(z)}{g(z)} - \frac{\beta(c)}{2(1-\beta(c))} \quad \text{in } \mathbb{D},$$

$$\frac{zf'(z)}{g(z)} \neq \beta(c) \quad \text{in } \mathbb{D}.$$

*Then we have*

$$\mathrm{Re}\,\frac{zf'(z)}{g(z)} > \beta(c) > \beta(\alpha) > \alpha \quad \text{in } \mathbb{D}.$$

**Remark 1** For the case $0 < \alpha < c < 1$, it is trivial that

$$\alpha < \beta(\alpha) < \beta(c) < 1.$$

**Example 1** Let $f(z) \in \mathcal{A}$ satisfy

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} > \mathrm{Re}\,\frac{1-Az}{1+Az} - \frac{1-\beta(\frac{1}{2})}{2\beta(\frac{1}{2})} > -\frac{1}{10} \quad \text{in } \mathbb{D}, \tag{4}$$

where

$$A = \frac{32\beta(\frac{1}{2}) - 10}{8\beta(\frac{1}{2}) + 10} \doteqdot 0.29605$$

and $\beta(\frac{1}{2}) = \frac{1}{2\log 2}$. If we consider the starlike function $g(z)$ given by

$$g(z) = \frac{z}{(1 + Az)^2},$$

then we have

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \beta\left(\frac{1}{2}\right) \fallingdotseq 0.7213,$$

which means that $f(z)$ is close-to-convex of order $\beta(\frac{1}{2})$ in $\mathbb{D}$.

Next we show

**Theorem 2** *Let $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$ be given by*

$$g(z) = \begin{cases} \dfrac{z}{(1 + \beta z)^{\frac{\alpha + \beta}{\beta}}} & (\beta \neq 0), \\ ze^{-\alpha z} & (\beta = 0) \end{cases}$$

*for some $\alpha$ $(0 \leqq \alpha < 1)$ and some $\beta$ $(0 \leqq \beta < 1)$. Further suppose that for arbitrary $r$ $(0 < r < 1)$,*

$$\min_{|z|=r}\left( \operatorname{Re} \frac{zf'(z)}{g(z)} \right) = \left( \operatorname{Re} \frac{z_0 f'(z_0)}{g(z_0)} \right)_{|z_0|=r} \neq \frac{z_0 f'(z_0)}{g(z_0)}$$

*and*

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} \leqq -\frac{c}{2(1 - c)} + \frac{1 - \alpha}{1 + \beta}$$

*for $c < 0$. Then we have*

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > c \quad \textit{in } \mathbb{D}.$$

*Proof* Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{g(z)}, \qquad p(0) = 1$$

for $c < 0$. If there exists a point $z_0 \in \mathbb{D}$ such that

$$\operatorname{Re} p(z) > c \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = c$$

for $c < 0$, then from the hypothesis of Theorem 2, we have

$$\operatorname{Re} p(z_0) \neq p(z_0).$$

Therefore, applying Lemma 1 and Lemma 3, we have

$$1 + \operatorname{Re} \frac{z_0 f''(z_0)}{f'(z_0)} = \operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} + \operatorname{Re} \frac{z_0 g'(z_0)}{g(z_0)}$$
$$> -\frac{c}{2(1-c)} + \frac{1-\alpha}{1+\beta}.$$

This is a contradiction, and therefore we have

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > c \quad \text{in } \mathbb{D}. \qquad \square$$

**Remark 2** In view of the definition for close-to-convex functions, if $f(z)$ satisfies

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } \mathbb{D},$$

then we can say that $f(z)$ is close-to-convex in $\mathbb{D}$. But $c$ should be a negative real number in Theorem 2. Therefore, we cannot say that $f(z)$ is close-to-convex in $\mathbb{D}$ in Theorem 2.

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