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An elementary operator and generalized Weyl's theorem

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Abstract

A Hilbert space operator T belongs to class A if $|T^2| - |T|^2 \geq 0$. Let d_{AB} denote either δ_{AB} or $\triangle_{AB}$, where δ_{AB} and $\triangle_{AB}$ denote the generalized derivation and the elementary operator on a Banach space $B(\mathcal{H})$ defined by $\delta_{AB}X = AX - XB$ and $\triangle_{AB}X = AXB - X$ respectively. If A and B^* are class A operators, we show that d_{AB} is polaroid and generalized Weyl's theorem holds for $f(d_{AB})$, generalized a-Weyl's theorem holds for $f((d_{AB})^*)$ for every $f \in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$, where $H(\sigma(d_{AB}))$ denotes the set of all analytic functions in a neighborhood of $\sigma(d_{AB})$.
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1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{C}$ be the set of complex numbers. Let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators and the ideal of compact operators acting on $\mathcal{H}$ respectively. For operators $A, B \in B(\mathcal{H})$, let $\delta_{AB} \in B(B(\mathcal{H}))$ denote the generalized derivation on a Banach space $B(\mathcal{H})$ defined by $\delta_{AB}X = AX - XB$; let $\triangle_{AB} \in B(B(\mathcal{H}))$ denote the elementary operator on a Banach space $B(\mathcal{H})$ defined by $\triangle_{AB} = AXB - X$. Let d_{AB} denote either δ_{AB} or $\triangle_{AB}$. d_{AB} has been studied by a number of authors [1–4]. Also let $\alpha(T) = \dim \ker T$, $\beta(T) = \dim \ker T^*$, and let $\sigma(T)$, $\sigma_a(T)$ denote the spectrum and approximate point spectrum of T. An operator $T \in B(\mathcal{H})$ is called upper (resp. lower) semi-Fredholm if $\operatorname{ran} T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). In the sequel, let $SF_+(\mathcal{H})$ denote the set of all upper semi-Fredholm operators. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator $T \in B(\mathcal{H})$ is called Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. Let $\sigma_e(T)$, $\sigma_w(T)$ and $\sigma_b(T)$ denote the essential spectrum, the Weyl spectrum and the Browder spectrum of $T \in B(\mathcal{H})$. Let $\operatorname{iso} \mathcal{K}$ denote the isolated points of $\mathcal{K} \subseteq \mathbb{C}$. We write $\pi_{00}(T) = \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$, $\pi_{00}^a(T) = \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}$, and $p_{00}(T) = \sigma(T) \setminus \sigma_b(T)$. It is evident that $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \bigcup \operatorname{acc} \sigma(T)$ and $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$, where $\operatorname{acc} \sigma(T) = \sigma(T) \setminus \operatorname{iso} \sigma(T)$.

We say that Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

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and that Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

By definition, $\sigma_{ea}(T) = \bigcap\{\sigma_a(T + K) : K \in K(\mathcal{H})\}$ is the essential approximate point spectrum of T, and $\sigma_{ab}(T) = \bigcap\{\sigma_a(T + K) : K \in K(\mathcal{H})$ and $KT = TK\}$ is the Browder approximate point spectrum of T.

We say that a-Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T),$$

and that a-Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

For a bounded linear operator T and a nonnegative integer n, define T_n to be the restriction of T to ran T^n viewed as a map from ran T^n into ran T^n (in particular $T_0 = T$). If for some integer n, the range space ran T^n is closed and T_n is a Fredholm operator, then T is called a B-Fredholm operator. If T is a B-Fredholm operator of index zero, then T is called a B-Weyl operator. The B-Fredholm spectrum $\sigma_{\mathrm{BF}}(T)$ and B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ of T are defined by $\sigma_{\mathrm{BF}}(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not a B-Fredholm operator$\}$ and $\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not a B-Weyl operator$\}$. An operator $T \in B(\mathcal{H})$ satisfies generalized Weyl's theorem [5, Definition 2.13] if

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus \pi_0(T),$$

where $\pi_0(T)$ is the set of all isolated eigenvalues of T, and $T \in B(\mathcal{H})$ satisfies generalized Browder's theorem [5, Definition 2.13] if

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus p_0(T),$$

where $p_0(T)$ is the set of all poles of the resolvent of T.

Let $\mathrm{SBF}_+(\mathcal{H})$ be the class of all the upper semi-B-Fredholm operators and $\mathrm{SBF}_+^-(\mathcal{H})$ be the class of all $T \in \mathrm{SBF}_+(\mathcal{H})$ such that $\mathrm{ind}(T) \leq 0$. Let

$$\sigma_{\mathrm{SBF}_+^-}(T) = \left\{\lambda \in \mathbb{C} : T - \lambda \notin \mathrm{SBF}_+^-(\mathcal{H})\right\}$$

be called the semi-B-essential approximate point spectrum of T. We say that $T \in B(\mathcal{H})$ satisfies generalized a-Weyl's theorem [5, Definition 2.13] if

$$\sigma_{\mathrm{SBF}_+^-}(T) = \sigma_a(T) \setminus \pi_0^a(T),$$

where $\pi_0^a(T)$ is the set of all eigenvalues of T which are isolated points of $\sigma_a(T)$.

The following implications are known to hold:

generalized a-Weyl's theorem

$\implies$ generalized Weyl's theorem $\implies$ Weyl's theorem

$\implies$ Browder's theorem;

generalized a-Weyl's theorem

$\implies$ a-Weyl's theorem $\implies$ a-Browder's theorem

$\implies$ Browder's theorem.

In this paper, we shall study the generalized Weyl's theorem for the elementary operator and the generalized derivation with class A operators as entries. Recall that $T \in B(\mathcal{H})$ is called p-hyponormal for $p > 0$ if $(T^*T)^p - (TT^*)^p \geq 0$ [6]; when $p = 1$, T is called hyponormal. And T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ [7, 8]. In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators (T is invertible and $\log T^*T \geq \log TT^*$), Furuta, Ito and Yamazaki [9] introduced a very interesting class of operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T, and they showed that class A is a subclass of paranormals and contains p-hyponormal and log-hyponormal operators.

Definition 1.1 An operator $T \in B(\mathcal{H})$ is said to have the single valued extension property (SVEP) at $\lambda \in \mathbb{C}$ if for every open neighborhood $\mathcal{G}$ of λ, the only function $f \in H(\mathcal{G})$ such that $(T - \mu)f(\mu) = 0$ on G is $0 \in H(\mathcal{G})$, where $H(\mathcal{G})$ means the space of all analytic functions on G. When T has SVEP at each $\lambda \in \mathbb{C}$, say that T has SVEP.

The single valued extension property dates back to the early days of local spectral theory; see the recent monograph of Laursen and Neumann [10] or Aiena [11]. In addition to the definition of SVEP, there are notions of property (β), property (δ) and condition (C). The interested reader is referred to [10] for more details.

2 The main results

For $T \in B(\mathcal{H})$, let L_T and R_T denote the operators of left and right multiplication by T respectively.

Chō and Yamazaki proved that class A operators have property β in [12] Theorem 3.1; unfortunately, there are some mistakes in the proof of this theorem; see details in [13]. So, Theorem 3.1 in [12] is still an open problem.

Lemma 2.1 *Let A and B* be class A operators satisfying property (β), then d_{AB} has SVEP.*

Proof By assumption and [10] Theorem 2.5.5, A satisfies property (β) and B satisfies property (δ). Hence, both L_A and R_B satisfy condition (C) by [10] Corollary 3.6.11. Clearly, L_A and R_B commute. By Theorem 3.6.3 and Note 3.6.19 on p.283 of [10], $L_A - R_B$ and $L_A R_B$ have SVEP, which implies that d_{AB} has SVEP. □

It is well known that the isolated points of the spectrum of a class A (indeed, paranormal) operator T are poles of the resolvent of the operator (hence, eigenvalues of the operator),

the restriction of T to an invariant subspace is again of class A (resp., paranormal), and that if T has countable spectrum then T is normal. (We shall use this information freely in the following without any further reference.)

Recall, [14], that $\sigma(\delta_{AB}) = \{\lambda : \lambda \in \sigma(A) - \sigma(B)\}$ and $\sigma(\triangle_{AB}) = \{\lambda : \lambda \in \sigma(A)\sigma(B) - 1\}$. If $\lambda \in \text{iso}\,\sigma(d_{AB})$, then we have one of the following two cases:

(1) $\lambda \neq -1$ if $d_{AB} = \triangle_{AB}$. Then there exist finite sequences $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$, where $\alpha_i \in \text{iso}\,\sigma(A)$ and $\beta_i \in \text{iso}\,\sigma(B)$ respectively, such that $\lambda = \alpha_i - \beta_i$ if $\lambda \in \text{iso}\,\sigma(\delta_{AB})$ and $\lambda = \alpha_i \beta_i - 1$ if $\lambda \in \text{iso}\,\sigma(\triangle_{AB})$, for all $1 \leq i \leq m$.

(2) $\lambda = -1$ and $d_{AB} = \triangle_{AB}$. Then either $0 \in \text{iso}\,\sigma(A)$ and $0 \in \text{iso}\,\sigma(B)$ or $0 \in \text{iso}\,\sigma(A)$ and $0 \notin \sigma(B)$ or $0 \in \text{iso}\,\sigma(B)$ and $0 \notin \sigma(A)$.

Theorem 2.2 *Let A and B^* be class A operators, then $H_0(d_{AB} - \lambda) = \ker(d_{AB} - \lambda)$ for all $\lambda \in \text{iso}\,\sigma(d_{AB})$.*

Proof We consider the case $d_{AB} = \delta_{AB}$ and $d_{AB} = \triangle_{AB}$ respectively.

(1) We consider the case $d_{AB} = \delta_{AB}$. The idea comes from [1]. If $\lambda \in \text{iso}\,\sigma(\delta_{AB})$, then there exist finite sets $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $\{\beta_1, \beta_2, \ldots, \beta_m\}$, where $\alpha_i \in \text{iso}\,\sigma(A)$ and $\beta_i \in \text{iso}\,\sigma(B)$ such that $\lambda = \alpha_i - \beta_i$ for all $1 \leq i \leq m$. Let

$$M_1 = \bigvee_{i=1}^m \ker(A - \alpha_i), \qquad M_2 = \mathcal{H} \ominus M_1$$

and

$$N_1 = \bigvee_{i=1}^m \ker\left(B^* - \overline{\beta_i}\right), \qquad N_2 = \mathcal{H} \ominus N_1.$$

Then A and B have representations $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ on $M_1 \oplus M_2$ and $B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$ on $N_1 \oplus N_2$ respectively, where A_{11} and B_{11} are normal, $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$, $\sigma(B) = \sigma(B_{11}) \cup \sigma(B_{22})$ and $\lambda \notin \sigma(\delta_{A_{ii}B_{jj}})$ for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Consider an $X \in H_0(\delta_{AB} - \lambda)$. Letting $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$, we have

$$(\delta_{AB} - \lambda)^n X = \begin{pmatrix} * & * \\ * & (\delta_{A_{22}B_{22}} - \lambda)^n X_{22} \end{pmatrix}$$

for some yet to be determined entries $*$.

Since

$$\lim_{n \longrightarrow \infty} \left\| (\delta_{AB} - \lambda)^n X \right\|^{\frac{1}{n}} = 0,$$

we have that

$$\lim_{n \longrightarrow \infty} \left\| (\delta_{A_{22}B_{22}} - \lambda)^n X_{22} \right\|^{\frac{1}{n}} = 0.$$

Since $\lambda \notin \sigma(\delta_{A_{22}B_{22}})$, we have that $\delta_{A_{22}B_{22}} - \lambda$ is invertible. Hence, $X_{22} = 0$. So, we have that

$$(\delta_{AB} - \lambda)^n X = \begin{pmatrix} * & (\delta_{A_{11}B_{22}} - \lambda)^n X_{12} \\ (\delta_{A_{22}B_{11}} - \lambda)^n X_{21} & 0 \end{pmatrix}.$$

Since

$$\lim_{n \longrightarrow \infty} \left\| (\delta_{A_{ii}B_{jj}} - \lambda)^n X_{ij} \right\|^{\frac{1}{n}} = 0$$

for all $1 \leq i, j \leq 2$ other than $i = j = 1$ and since $\lambda \notin \sigma(\delta_{A_{11}B_{22}})$ and $\lambda \notin \sigma(\delta_{A_{22}B_{11}})$, we have that $X_{12} = X_{21} = 0$. Hence,

$$(\delta_{AB} - \lambda)^n X = \begin{pmatrix} (\delta_{A_{11}B_{11}} - \lambda)^n X_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since A_{11} and B_{11} are normal,

$$\lim_{n \longrightarrow \infty} \left\| (\delta_{A_{11}B_{11}} - \lambda)^n X_{11} \right\|^{\frac{1}{n}} = 0$$

if and only if

$$(\delta_{A_{11}B_{11}} - \lambda) X_{11} = 0$$

by [15] Lemma 2. Hence, we have $H_0(\delta_{AB} - \lambda) \subseteq \ker(\delta_{AB} - \lambda)$. Since $\ker(\delta_{AB} - \lambda) \subseteq H_0(\delta_{AB} - \lambda)$ is always true, we have

$$H_0(\delta_{AB} - \lambda) = \ker(\delta_{AB} - \lambda).$$

(2) We consider the case $d_{AB} = \triangle_{AB}$. When $\lambda \neq -1$, the proof is similar to the proof of the first part. We omit the proof. When $\lambda = -1$, then either $0 \in \mathrm{iso}\,\sigma(A)$ and $0 \in \mathrm{iso}\,\sigma(B)$ or $0 \in \mathrm{iso}\,\sigma(A)$ and $0 \notin \sigma(B)$ or $0 \in \mathrm{iso}\,\sigma(B)$ and $0 \notin \sigma(A)$. If 0 is both in $\mathrm{iso}\,\sigma(A)$ and $\mathrm{iso}\,\sigma(B)$, then let $M_1 = \ker(A)$, $M_2 = \mathcal{H} \ominus M_1$ and $N_1 = \ker(B^*)$, $N_2 = \mathcal{H} \ominus N_1$.

Then we have $A = \begin{pmatrix} 0 & C_1 \\ 0 & A_2 \end{pmatrix}$ on $\mathcal{H} = M_1 \oplus M_2$ and $B = \begin{pmatrix} 0 & 0 \\ C_2 & B_2 \end{pmatrix}$ on $\mathcal{H} = N_1 \oplus N_2$ for some operators C_1, A_2 and C_2, B_2 respectively. Here both A_2 and B_2 are invertible. So, we have that $\triangle_{A_2B_2} - \lambda = L_{A_2}R_{B_2}$ is invertible. Let $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. If $X \in H_0(\triangle_{AB} - \lambda) = H_0(L_A R_B)$, it follows that $X_{22} = 0$ as in the proof of the first part. Hence, $L_A R_B X = 0$ for every $X \in H_0(\triangle_{AB} - \lambda) = H_0(L_A R_B)$. So, we have $H_0(\triangle_{AB} - \lambda) \subseteq \ker(\triangle_{AB} - \lambda)$. Since $\ker(\triangle_{AB} - \lambda) \subseteq H_0(\triangle_{AB} - \lambda)$ is always true, we have $H_0(\triangle_{AB} - \lambda) = \ker(\triangle_{AB} - \lambda)$. The proofs of the other remaining cases are similar, we consider $0 \in \mathrm{iso}\,\sigma(A)$ and $0 \notin \sigma(B)$. Here $\triangle_{AB} - \lambda = L_A R_B$. In the following, we shall prove that $H_0(L_A R_B) = H_0(L_A)$. If $X \in H_0(L_A R_B)$, then

$$\lim_{n \longrightarrow \infty} \left\| (L_A)^n X \right\|^{\frac{1}{n}} = \lim_{n \longrightarrow \infty} \left\| (L_A R_B)^n X B^{-n} \right\|^{\frac{1}{n}} \leq \left\| B^{-1} \right\| \lim_{n \longrightarrow \infty} \left\| (L_A R_B)^n X \right\|^{\frac{1}{n}} = 0.$$

On the other hand, if $X \in H_0(L_A)$, then

$$\lim_{n \longrightarrow \infty} \left\| (L_A R_B)^n X \right\|^{\frac{1}{n}} \leq \left\| B \right\| \lim_{n \longrightarrow \infty} \left\| (L_A)^n X \right\|^{\frac{1}{n}} = 0.$$

Hence, $H_0(L_A R_B) = H_0(L_A)$. Next, we shall prove that $H_0(L_A) = \ker(L_A)$. Let $M_1 = \ker A$, $M_2 = \mathcal{H} \ominus M_1$. We have $A = \begin{pmatrix} 0 & C_1 \\ 0 & A_2 \end{pmatrix}$ on $\mathcal{H} = M_1 \oplus M_2$, where A_2 is invertible. Let X have the following matrix representation: $X = [X_{ij}]_{i,j=1}^2$ on $\mathcal{H} = M_1 \oplus M_2$. If $X \in H_0(L_A)$, as in

the proof above, we have that $X_{21} = X_{22} = 0$. So, we have that $L_A X = 0$. Hence, $H_0(L_A) \subseteq \ker(L_A)$. Since $\ker(L_A) \subseteq H_0(L_A)$ is always true, we have that $H_0(L_A) = \ker(L_A)$. Since B is invertible, we have $\ker(L_A R_B) = \ker(L_A)$. Therefore, we have that

$$H_0(L_A R_B) = H_0(L_A) = \ker(L_A) = \ker(L_A R_B),$$

hence

$$H_0(L_A R_B) = \ker(L_A R_B).$$

That is,

$$H_0(\triangle_{AB} - \lambda) = \ker(\triangle_{AB} - \lambda).$$

This completes the proof. □

An operator $T \in B(\mathcal{H})$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid then it is isoloid.

Lemma 2.3 *Let A and B^* be class A operators, then d_{AB} and d_{AB}^* are polaroid. In particular, d_{AB} and d_{AB}^* are isoloid.*

Proof We only need to prove that d_{AB} is polaroid. Let $\mathcal{X} = B(\mathcal{H})$ and $\lambda \in \mathrm{iso}\,\sigma(d_{AB})$. Then we have that $H_0(d_{AB} - \lambda) = \ker(d_{AB} - \lambda)$ by Theorem 2.2. Hence,

$$\mathcal{X} = H_0(d_{AB} - \lambda) \oplus K(d_{AB} - \lambda) = \ker(d_{AB} - \lambda) \oplus K(d_{AB} - \lambda).$$

So, we have

$$(d_{AB} - \lambda)\mathcal{X} = 0 \oplus (d_{AB} - \lambda)\big(K(d_{AB} - \lambda)\big) = K(d_{AB} - \lambda).$$

Therefore,

$$\mathcal{X} = \ker(d_{AB} - \lambda) \oplus (d_{AB} - \lambda)\mathcal{X}.$$

Thus, isolated points of $\sigma(d_{AB})$ are simple poles of the resolvent of d_{AB}. Hence, d_{AB} is polaroid. So, we have that d_{AB} and d_{AB}^* are polaroid. Since polaroid operators are always isoloid, we have that d_{AB} and d_{AB}^* are isoloid. □

Theorem 2.4 *Let A and B^* be class A operators satisfying property (β). Then generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

Proof Suppose that A and B^* are class A operators. By Lemma 2.1 and Lemma 2.3, we have that d_{AB} has SVEP and d_{AB} is polaroid. So, we have that generalized Weyl's theorem

holds for d_{AB} by [16, Theorem 3.10(ii)]. Since d_{AB} has SVEP and d_{AB} is isoloid, we have that generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$ by [17, Theorem 2.2].

$\square$

Corollary 2.5 *Let A and B^* be class A operators satisfying property (β). Then Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$ and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

A bounded linear operator $T \in B(\mathcal{H})$ is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Note that every a-isoloid operator is isoloid and the converse is not true in general.

Lemma 2.6 *Let A and B^* be class A operators satisfying property (β), then d^*_{AB} is a-isoloid.*

Proof Let λ be an isolated point of $\sigma_a(d^*_{AB})$. Suppose that A and B^* are class A operators satisfying property (β). By Lemma 2.1 and Lemma 2.3, we have that d_{AB} has SVEP and d^*_{AB} is isoloid. Hence, $\sigma_a(d^*_{AB}) = \sigma(d^*_{AB})$ by [18, Corollary 7]. We have that λ is an isolated point of $\sigma(d^*_{AB})$. Since d^*_{AB} is isoloid, we have that λ is an eigenvalue of d^*_{AB}. Hence, d^*_{AB} is a-isoloid. $\square$

Theorem 2.7 *Let A and B^* be class A operators satisfying property (β). Then generalized a-Weyl's theorem holds for $f(d^*_{AB})$ for every $f \in H(\sigma(d_{AB}))$, and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

Proof Suppose that A and B^* are class A operators satisfying property (β). By Lemma 2.1 and Lemma 2.3, we have that d_{AB} has SVEP and d^*_{AB} is polaroid. By Corollary 2.5, Weyl's theorem holds for d_{AB}. Hence, d^*_{AB} satisfies Weyl's theorem by [19, Proposition 2.1]. Since d_{AB} has SVEP and d^*_{AB} is polaroid, generalized a-Weyl's theorem holds for d^*_{AB} by [16, Theorem 3.10]. T is a-isoloid by Lemma 2.6, hence generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$ by [17, Theorem 2.4]. $\square$

Corollary 2.8 *Let A and B^* be class A operators satisfying property (β). Then a-Weyl's theorem holds for $f(d^*_{AB})$ for every $f \in H(\sigma(d_{AB}))$, and f is not constant on each connected component of the open set U containing $\sigma(d_{AB})$.*

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