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Isometric composition operators on weighted Dirichlet-type spaces

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Abstract

In this article, we characterize the surjective isometric composition operator C_ϕ on the weighted Dirichlet-type spaces of the unit disk $\mathbb{D}$, where ϕ is an analytic self-map of $\mathbb{D}$, and show that C_ϕ is a surjective isometry if and only if ϕ is a rotation map.

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1. Introduction

The Lebesgue area measure on the unit disk $\mathbb{D}$ in the complex plane is defined by $dA(z) = rdrdt = dxdy$. Denote by $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the collection of all the analytic self mappings of $\mathbb{D}$.

Given real numbers $p > 0$ and $\alpha > -1$, the weighted Bergman space A^p_α is defined as the space of f in $H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha (\alpha+1)dA\right)^{1/p} < \infty.$$

The weighted Dirichlet-type space $\mathcal{D}^p_\alpha$ is the space of f in $H(\mathbb{D})$ such that $f' \in A^p_\alpha$, equipped with the "norm":

$$\|f\|_{\mathcal{D}^p_\alpha} = |f(0)| + \|f'\|_{A^p_\alpha}.$$

It is not a true norm for $0 < p < 1$, but it satisfies

$$\|f+g\|_{\mathcal{D}^p_\alpha} \le C_p(\|f\|_{\mathcal{D}^p_\alpha} + \|g\|_{\mathcal{D}^p_\alpha}),$$

where the constant C_p depends only on p. We write $\mathcal{D}^p = \mathcal{D}^p_0$. The space $\mathcal{D} = \mathcal{D}^2$ is the classical Dirichlet space of analytic functions whose image Riemann surface has finite area. Clearly, $\mathcal{D}^p \subset \mathcal{D}^q$ when $q < p < \infty$.

For $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the composition operator C_ϕ induced by ϕ is defined as $C_\phi f = f \circ \phi$ for $f \in H(\mathbb{D})$; the multiplication operator M_u induced by u is defined by $M_u f(z) = u(z)f(z)$; and the weighted composition operator $W_{u,\phi}$ induced by ϕ and u is defined by $(W_{u,\phi}f)(z) = u(z)f(\phi(z))$ for $z \in \Omega$ and $f \in H(\Omega)$. If we let $u \equiv 1$, then

$W_{u,\phi} = C_\phi$; if we let $\phi = id$, then $W_{u,\phi} = M_u$. So we can regard weighted composition operator as a generalization of a multiplication operator and a composition operator. These operators are linear. In [1], Hibschweiler studied the boundedness and compactness of composition operators on $\mathcal{D}^p$. In [2], Roan studied the boundness of composition operators on S^p, where $S^p = \{f \in H(\mathbb{D}) : f' \in H^p\}$.

An operator T on a normed space $\mathbf{X}$ is said to be an isometric operator if $||Tf||_\mathbf{X} = ||f||_\mathbf{X}$, for any $f \in \mathbf{X}$.

The isometric composition operator on analytic functions spaces has been studied by many authors. In [3], Martín and Vukotić studied the isometric composition operators on H^p, A^p_α (see also Kolaski [4,5]) for $1 \le p < \infty$ and the analytic Besov spaces B^p for $1 < p < \infty$. They obtained that C_ϕ is an isometry of H^p if and only if ϕ is inner and $\phi(0) = 0$, and C_ϕ is an isometry of A^p_α if and only if ϕ is a rotation. Also Carswell and Hammond [6] obtained that $C_\varphi : A^2_\alpha \to A^2_\alpha$ is an isometry if and only if ϕ is a rotation; this fact differs somewhat from the analogs results that are known for other Hilbert spaces.

The isometric composition operators on the Bloch spaces in the unit disk were discussed by Martín and Vukotić [7], Colonna [8], Allen and Colonna [9,10], Li and Zhou [11]. The same problems were studied on the Bloch spaces in the unit polydisk by Cohen and Colonna [12], in unit ball by Li [13], and Li and Ruan [14]. For the BMOA space, see [15]. In [16], Martín and Vukotić also studied the isometric composition operators on the classical Dirichlet spaces in the unit disk. They obtained that C_ϕ is an isometric operator if and only if ϕ is a univalent map of the unit disk such that $A[\mathbb{D} \backslash \varphi(\mathbb{D})] = 0$ and $\phi(0) = 0$. In [17], Novinger and Oberlin studied the isometric composition operators on S^p with different norms.

The present article continues this line of research and discusses the isometric composition operators on the weighted Dirichlet-type space in the unit disk.

2. Main results

Our proofs will depend upon a characterization of the linear isometries of weighted Bergman spaces due to Kolaski [4]. Although not stated by Kolaski explicitly, the following result is a direct consequence of Theorems 1 and 4 in [4], which are much more general results than we will need here.

Theorem 2.1. *Let $0 < p < \infty$, $p \neq 2$, and $\alpha > -1$. Then every linear isometry $T : A^p_\alpha \to A^p_\alpha$ takes the form $(Tf)(z) = g(z) \cdot f(\phi(z))$, for all $f \in A^p_\alpha$ and $z \in \mathbb{D}$, for some function ϕ which maps $\mathbb{D}$ conformally onto a dense subset of $\mathbb{D}$, and where $g = T1$. Moreover, if T is surjective, then ϕ is a disk automorphism and $g = \lambda(\phi')^{(2+\alpha)/p}$ for some $|\lambda| = 1$.*

Following the ideas of Theorem 4.5.1 in [18], we investigate the surjective isometric composition operators on the weighted Dirichlet-type space.

Theorem 2.2. *Let ϕ be a self-map of the unit disk. Then the induced composition operator C_ϕ is a surjective isometry on the weighted Dirichlet-type space $\mathcal{D}^p_\alpha$, $1 \le p < \infty$, $p \neq 2$, $-1 < \alpha < \infty$ if and only if ϕ is a rotation map.*

Proof. Since the composition operator induced by a rotation is clearly an isometry, it suffices to show that if C_ϕ is a surjective isometry, then ϕ is a rotation.

Suppose that C_ϕ is a surjective isometric composition operator. Let n be a positive integer and t be a real number. We define the function $p_n(z) = z^n$ for all $z \in \mathbb{D}$. The weighted Dirichlet-type space contains the polynomials and thus $1 + tp_n \in \mathcal{D}_\alpha^p$ for every real number t and positive integer n.

Using the definition of norm of $\mathcal{D}_\alpha^p$ and that C_ϕ is an isometry, we have

$$\left\|C_\varphi 1\right\|_{\mathcal{D}_\alpha^p} = \|1\|_{\mathcal{D}_\alpha^p}, \quad \left\|C_\varphi p_n\right\|_{\mathcal{D}_\alpha^p} = \left\|p_n\right\|_{\mathcal{D}_\alpha^p}$$

and

$$\left\|C_\varphi(1 + tp_n)\right\|_{\mathcal{D}_\alpha^p} = \left\|1 + tp_n\right\|_{\mathcal{D}_\alpha^0}.$$

It follows that

$$\begin{aligned}\left\|1 + tp_n\right\|_{\mathcal{D}_\alpha^p} &= 1 + \left(\int_{\mathbb{D}} \left|(tp_n)'\right|^p (1 - |z|^2)^\alpha (\alpha + 1) dA\right)^{\frac{1}{p}} \\ &= \|1\|_{\mathcal{D}_\alpha^p} + \left\|tp_n\right\|_{\mathcal{D}_\alpha^p} = \|1\|_{\mathcal{D}_\alpha^p} + |t| \left\|p_n\right\|_{\mathcal{D}_\alpha^p} \\ &= \left\|C_\varphi 1\right\|_{\mathcal{D}_\alpha^p} + |t| \left\|C_\varphi p_n\right\|_{\mathcal{D}_\alpha^p} \\ &= \left|C_\varphi 1(0)\right| + |t| \left|C_\varphi p_n(0)\right| + \left\|(C_\varphi 1)'\right\|_{A_\alpha^p} + |t| \left\|(C_\varphi p_n)'\right\|_{A_\alpha^p}\end{aligned}$$

and

$$\left\|C_\varphi(1 + tp_n)\right\|_{\mathcal{D}_\alpha^p} = \left|C_\varphi 1(0) + tC_\varphi p_n(0)\right| + \left\|(C_\varphi 1)' + t(C_\varphi p_n)'\right\|_{A_\alpha^p}.$$

So

$$\begin{aligned}&\left|C_\varphi 1(0) + tC_\varphi p_n(0)\right| + \left\|(C_\varphi 1)' + t(C_\varphi p_n)'\right\|_{A_\alpha^p} \\ &= \left|C_\varphi 1(0)\right| + |t| \left|C_\varphi p_n(0)\right| + \left\|(C_\varphi 1)'\right\|_{A_\alpha^p} + |t| \left\|(C_\varphi p_n)'\right\|_{A_\alpha^p}.\end{aligned}$$

A simple application of the triangle inequality for the norms shows that

$$\left|C_\varphi 1(0) + tC_\varphi p_n(0)\right| \le \left|C_\varphi 1(0)\right| + |t| \left|C_\varphi p_n(0)\right|$$

and

$$\left\|(C_\varphi 1)' + t(C_\varphi p_n)'\right\|_{A_\alpha^p} \le \left\|(C_\varphi 1)'\right\|_{A_\alpha^p} + |t| \left\|(C_\varphi p_n)'\right\|_{A_\alpha^p}.$$

Consequently,

$$\left|C_\varphi 1(0) + tC_\varphi p_n(0)\right| = \left|C_\varphi 1(0)\right| + |t| \left|C_\varphi p_n(0)\right|$$

and

$$\left\|(C_\varphi 1)' + t(C_\varphi p_n)'\right\|_{A_\alpha^p} = \left\|(C_\varphi 1)'\right\|_{A_\alpha^p} + |t| \left\|(C_\varphi p_n)'\right\|_{A_\alpha^p}.$$

In particular, for $n = 1$, we get $|1 + t\phi(0)| = 1 + |t||\phi(0)|$ which implies, since t is an arbitrary real number, that $\phi(0) = 0$.

Therefore $\left\|C_\varphi f\right\|_{\mathcal{D}_\alpha^p} = \left\|f\right\|_{\mathcal{D}_\alpha^p}$ is equivalent to $\left\|(C_\varphi f)'\right\|_{A_\alpha^p} = \left\|f'\right\|_{A_\alpha^p}$.

Let $\mathcal{D}_{\alpha,0}^p$ denotes the subspace of functions in $\mathcal{D}_\alpha^p$ that vanish at the origin. The differentiation operator D maps $\mathcal{D}_{\alpha,0}^p$ isometrically onto A_α^p and its inverse I is given by

$$Ig(z) = \int_0^z g(\xi) d\xi$$

and maps A_α^p isometrically onto $\mathcal{D}_{\alpha,0}^p$.

Since C_ϕ is a surjective isometry on $\mathcal{D}_\alpha^p$, for every $f \in \mathcal{D}_{\alpha,0}^p$, there exists $g \in \mathcal{D}_\alpha^p$ such that $C_\phi g = f$. Because

$$0 = f(0) = (C_\varphi g)(0) = g(\varphi(0)) = g(0),$$

then $g \in \mathcal{D}_\alpha^p$. Thus the isometric composition operator C_ϕ maps the subspace $\mathcal{D}_{\alpha,0}^p$ onto itself. So the composition $DC_\varphi I : A_\alpha^p \to A_\alpha^p$ is a surjective isometric operator. Set $T = DC_\phi I$. Then by Theorem 2.1, there exists an automorphism ϕ of $\mathbb{D}$ such that $Tf(z) = g(z)f(\phi(z))$, for all $f \in A_\alpha^p$, $z \in \mathbb{D}$, where $g = T1$. Then, noting that $I(f') = f - f(0)$, we obtain

$$g(z)f'(\phi(z)) = Tf'(z) = DC_\varphi If'(z) = DC_\varphi(f(z) - f(0)) = f'(\varphi(z))\varphi'(z).$$

Therefore,

$$f'(\varphi(z))\varphi'(z) = g(z)f'(\phi(z)) \tag{2.1}$$

for any $f \in \mathcal{D}_\alpha^p$.

Letting $f = id$ in (2.1), then $\phi'(z) = g(z)$ (Alternatively, one can see that $\phi' = g$ since $g(z) = (T1)(z) = DC_\phi I_1(z) = \phi'(z)$). Letting $f = p_2$, then $\phi(z)\phi'(z) = g(z)\phi(z)$, so from $g(z) \neq 0$, we have $\phi(z) = \phi(z)$. Since ϕ is an automorphism of $\mathbb{D}$ and $\phi(0) = 0$, it follows that $\phi(z) = \lambda z$, where $|\lambda| = 1$, that is ϕ is a rotation.

From the proof of the above theorem, it is easy to see that if C_ϕ is an isometry on $\mathcal{D}_\alpha^p$, $1 \leq p < \infty$ then $\phi(0) = 0$. We can get the following corollary.

Corollary 2.3. *Let* $\varphi \in \mathrm{Aut}(\mathbb{D})$. *Then the induced composition operator* C_ϕ *is an isometry on the weighted Dirichlet-type space* $\mathcal{D}_\alpha^p$, $1 \leq p < \infty$ *and* $-1 < \alpha < \infty$, *if and only if* ϕ *is a rotation map.*

Clearly, if both C_ϕ and M_u are isometries, then $W_{u,\phi}$ is an isometry, the following theorem will show that the converse is also true.

Theorem 2.4. *The weighted composition operator* $W_{u,\phi}$ *is an isometric operator on* $\mathcal{D}_\alpha^p$, $1 \leq p < \infty$ *and* $\alpha > -1$, *if and only if both* C_ϕ *and* M_u *are isometric operators.*

Proof. Suppose $W_{u,\phi}$ is an isometry. Replacing C_ϕ by $W_{u,\phi}$ and using the same methods and the same assumption in Theorem 2.2, we get

$$\left\|(W_{u,\varphi}1)' + t(W_{u,\varphi}p_n)'\right\|_{A_\alpha^p} = \left\|(W_{u,\varphi}1)'\right\|_{A_\alpha^p} + |t|\left\|(W_{u,\varphi}p_n)'\right\|_{A_\alpha^p} \tag{2.2}$$

It follows from (2.2) that the real valued function

$$p(t) = \left\|(W_{u,\varphi}1)' + t(W_{u,\varphi}p_n)'\right\|_{A_\alpha^p}$$

is not a differentiable function of t at $t = 0$. However, in the terminology of [19], the L^p norm is weakly differentiable at every point except the zero vector, that is $\|\cdot\|_{A_\alpha^p}$ is not weakly differentiable at $(W_{u,\phi}1)'$. Consequently

$$(W_{u,\varphi}1)'(z) = 0$$

for every $z \in \mathbb{D}$. So $(W_{u,\phi}1)' \equiv 0$, which implies that $u = W_{u,\phi}1$ is a constant.

Since $W_{u,\phi}$ is an isometry, $\|u\|_{\mathcal{D}_\alpha^p} = \|W_{u,\varphi}1\|_{\mathcal{D}_\alpha^p} = \|1\|_{\mathcal{D}_\alpha^p} = 1$, and consequently, $u \equiv \lambda$, for some $|\lambda| = 1$. Hence the multiplication operator M_u is an isometry. Now,

$$\|f\|_{\mathcal{D}_\alpha^p} = \|W_{u,\varphi}f\|_{\mathcal{D}_\alpha^p} = \|(uC_\varphi)(f)\|_{\mathcal{D}_\alpha^p} = \|\lambda C_\varphi(f)\|_{\mathcal{D}_\alpha^p} = \|C_\varphi(f)\|_{\mathcal{D}_\alpha^p}$$

for every $f \in \mathcal{D}_\alpha^p$. Hence the composition operator C_ϕ is an isometry.

Remark. From the above the theorem, we can get that the multiplier operator M_u is an isometric operator on $\mathcal{D}_\alpha^p$, $1 \le p < \infty$, $-1 < \alpha < \infty$ if and only if u is a constant of modulus one. In [20], Aleman et al. characterized the nontrivial isometric multipliers on the Dirichlet-type space $\mathcal{D}_w^p$, $1 \le p < \infty$ (here w is the weighted function). For the sake of completeness, we state their results as follows.

Theorem 2.5. *Let $1 \le p < \infty$. Suppose that the Dirichlet-type space $\mathcal{D}_w^p$ is complete and that point-evaluations are bounded. Then $\mathcal{D}_w^p$ has nonconstant isometric pointwise multipliers if and only if $p = 2$ and $w(z) = -2\log|z|$ a.e. in $\mathbb{D}$. In this case $\mathcal{D}_w^p = H^2$ and the isometric multipliers are precisely the inner functions.*

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Authors' contributions
All authors conceived and drafted the manuscript, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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