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Mean and uniform convergence of Lagrange interpolation with the Erdős-type weights

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Abstract

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^+ := [0, \infty)$ be an even function. We consider the exponential-type weights $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$. In this paper, we obtain a mean and uniform convergence theorem for the Lagrange interpolation polynomials $L_n(f)$ in L_p, $1 < p \leq \infty$ with the weight w.

MSC: 41A05

Keywords: exponential-type weight; Lagrange interpolation polynomial

1 Introduction and preliminaries

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^+ := [0, \infty)$ be an even function, and $w(x) = \exp(-Q(x))$ be the weight such that $\int_0^\infty x^n w^2(x)\,dx < \infty$ for all $n = 0, 1, 2, \ldots$. Then we can construct the orthonormal polynomials $p_n(x) = p_n(w^2; x)$ of degree n with respect to $w^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w^2(x)\,dx = \delta_{mn} \quad \text{(Kronecker's delta)}$$

and

$$p_n(x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0.$$

We denote the zeros of $p_n(x)$ by

$$-\infty < x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} < \infty.$$

We denote the Lagrange interpolation polynomial $L_n(f; x)$ based at the zeros $\{x_{k,n}\}_{k=1}^n$ as follows:

$$L_n(f; x) := \sum_{k=1}^n f(x_{k,n})l_{k,n}(x), \qquad l_{k,n}(x) := \frac{p_n(x)}{(x - x_{k,n})p_n'(x_{k,n})}.$$

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$ for $0 < x < y$.

We are interested in the following subclass of weights from [1].

Definition 1.1 Let $Q : \mathbb{R} \to \mathbb{R}^+$ be an even function satisfying the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R}\backslash\{0\}$.

(c) $\lim_{x\to\infty} Q(x) = \infty$.

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0,\infty)$ with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+\backslash\{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}\backslash\{0\}.$$

Then we write $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2)$. If there also exist a compact subinterval J $(\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}\backslash J,$$

then we write $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.

Example 1.2 (1) If $T(x)$ is bounded, then the weight $w = \exp(-Q)$ is called the Freud-type weight. The following example is the Freud-type weight:

$$Q(x) = |x|^\alpha, \quad \alpha > 1.$$

If $T(x)$ is unbounded, then the weight $w = \exp(-Q)$ is called the Erdős-type weight. The following examples give the Erdős-type weights $w = \exp(-Q)$.

(2) [2, Theorem 3.1] For $\alpha > 1$, $l = 1, 2, 3, \ldots$

$$Q(x) = Q_{l,\alpha}(x) = \exp_l\bigl(|x|^\alpha\bigr) - \exp_l(0),$$

where

$$\exp_l(x) = \exp\bigl(\exp(\exp \cdots \exp x) \cdots\bigr) \quad (l\text{-times}).$$

More generally, we define for $\alpha + u > 1$, $\alpha \geq 0$, $u \geq 0$ and $l \geq 1$,

$$Q_{l,\alpha,u}(x) := |x|^u \bigl(\exp_l\bigl(|x|^\alpha\bigr) - \alpha^* \exp_l(0)\bigr),$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$. (We note that $Q_{l,0,u}(x)$ gives a Freud-type weight.)

(3) We define $Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1$, $\alpha > 1$.

In this paper, we investigate the convergence of the Lagrange interpolation polynomials with respect to the weight $w \in \mathcal{F}(C^2+)$. When we consider the Erdős-type weights, the following definition follows from Damelin and Lubinsky [3].

Definition 1.3 Let $w(x) = \exp(-Q(x))$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and continuous. Q'' exists in $(0, \infty)$, $Q^{(j)} \geq 0$, in $(0, \infty)$, $j = 0, 1, 2$, and the function

$$T^*(x) := 1 + \frac{xQ''(x)}{Q'(x)}$$

is increasing in $(0, \infty)$ with

$$\lim_{x\to\infty} T^*(x) = \infty; \qquad T^*(0+) := \lim_{x\to 0+} T^*(x) > 1. \tag{1.1}$$

Moreover, we assume that for some constants $C_1, C_2, C_3 > 0$,

$$C_1 \leq T^*(x) \Big/ \left(\frac{xQ'(x)}{Q(x)}\right) \leq C_2, \quad x \geq C_3,$$

and for every $\varepsilon > 0$,

$$T^*(x) = O\big(Q(x)^\varepsilon\big), \quad x \to \infty. \tag{1.2}$$

Then we write $w \in \mathcal{E}$.

Damelin and Lubinsky [3] got the following results with the Erdős-type weights $w = \exp(-Q) \in \mathcal{E}$.

Theorem A ([3, Theorem 1.3]) *Let $w = \exp(-Q) \in \mathcal{E}$. Let $L_n(f, x)$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(w^2, x)$. Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\kappa > 0$. Then for*

$$\lim_{n\to\infty} \big\| \big(f - L_n(f)\big) w (1 + Q)^{-\Delta} \big\|_{L_p(\mathbb{R})} = 0$$

to hold for every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} \big| f(x) w(x) \big(\log |x|\big)^{1+\kappa} \big| = 0,$$

it is necessary and sufficient that

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}.$$

Our main purpose in this paper is to give mean and uniform convergence theorems with respect to $\{L_n(f)\}$, $n = 1, 2, \ldots$, in L_p-norm, $1 < p \leq \infty$. The proof for $1 < p < \infty$ will be shown by use of the method of Damelin and Lubinsky. In Section 2, we write the main theorems. In Section 3, we prepare some fundamental lemmas; and in Section 4, we will prove the theorem for $1 < p < \infty$. Finally, we will prove the theorem for the uniform convergence in Section 5.

For any nonzero real-valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exist constants $C_1, C_2 > 0$ independent of x such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all x. Similarly, for any two sequences of positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{=1}^{\infty}$, we define $c_n \sim d_n$. We denote the class of polynomials of degree at most n by $\mathcal{P}_n$.

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t, and polynomials of degree at most n. The same symbol does not necessarily denote the same constant in different occurrences.

2 Theorems

In the following, we introduce useful notations. Mhaskar-Rakhmanov-Saff numbers (MRS) a_x are defined as the positive roots of the following equations:

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{\frac{1}{2}}}\, du, \quad x > 0.$$

The function $\varphi_u(x)$ is defined as follows:

$$\varphi_u(x) = \begin{cases} \frac{a_u}{u} \frac{1-\frac{|x|}{a_{2u}}}{\sqrt{1-\frac{|x|}{a_u}+\delta_u}}, & |x| \le a_u, \\ \varphi_u(a_u), & a_u < |x|, \end{cases}$$

where

$$\delta_x = \left(xT(a_x)\right)^{-\frac{2}{3}}, \quad x > 0.$$

We define

$$\Phi(x) := \frac{1}{(1+Q(x))^{\frac{2}{3}} T(x)}$$

and

$$\Phi_n(x) := \max\left\{\delta_n, 1 - \frac{|x|}{a_n}\right\}.$$

Here we note that for $0 < d \le |x|$,

$$\Phi(x) \sim \frac{Q(x)^{\frac{1}{3}}}{xQ'(x)}$$

and we see

$$\Phi(x) \le C\Phi_n(x), \quad n \ge 1$$

(see Lemma 3.3 below). Moreover, we define

$$\Phi^{(\frac{1}{4}-\frac{1}{p})^+}(x) := \begin{cases} 1, & 0 < p < 4, \\ \Phi^{\frac{1}{4}-\frac{1}{p}}(x), & 4 \le p \le \infty. \end{cases}$$

Let $1 < p < \infty$. We give a convergence theorem as an analogy of Theorem A for $L_n(f)$ in L_p-norm. We need to prepare a lemma.

Lemma 2.1 ([4, Theorem 1.6]) *Let* $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

(a) *Let* $T(x)$ *be unbounded. Then for any* $\eta > 0$, *there exists a constant* $C(\eta) > 0$ *such that for* $t \geq 1$,

$$a_t \leq C(\eta)t^{\eta}.$$

(b) *Assume*

$$\frac{Q''(x)}{Q'(x)} \leq \lambda(b)\frac{Q'(x)}{Q(x)}, \quad |x| \geq b > 0, \tag{2.1}$$

where $b > 0$ *is large enough. Suppose that there exist constants* $\eta > 0$ *and* $C_1 > 0$ *such that* $a_t \leq C_1 t^{\eta}$. *If* $\lambda := \lambda(b) > 1$, *then there exists a constant* $C(\lambda, \eta)$ *such that for* $a_t \geq 1$,

$$T(a_t) \leq C(\lambda, \eta)t^{\frac{2(\eta+\lambda-1)}{\lambda+1}}. \tag{2.2}$$

If $0 < \lambda \leq 1$, *then for any* $\mu > 0$, *there exists* $C(\lambda, \mu)$ *such that*

$$T(a_t) \leq C(\lambda, \mu)t^{\mu}, \quad t \geq 1. \tag{2.3}$$

For a fixed constant $\beta > 0$, we define

$$\phi(x) := \left(1 + x^2\right)^{-\beta/2}. \tag{2.4}$$

Using this function, we have the following theorem. We suppose that the weight w is the Erdős-type weight.

Our theorem is as follows. Let $f \in C_0(\mathbb{R})$ mean that $f \in C(\mathbb{R})$ and $\lim_{|x|\to\infty} f(x) = 0$.

Theorem 2.2 *Let* $w = \exp(-Q) \in \mathcal{F}(C^2+)$, *and let* $T(x)$ *be unbounded. Let* $1 < p < \infty$ *and* $\beta > 0$, *and let us define* ϕ *as* (2.4), *and* $\lambda = \lambda(b) \geq 1$ *as* (2.1). *We suppose that for* $f \in C(\mathbb{R})$,

$$\phi^{-1}(x)w(x)f(x) \in C_0(\mathbb{R}),$$

and

$$\Delta > \frac{9}{4}\frac{\lambda - 1}{3\lambda - 1}. \tag{2.5}$$

Then we have

$$\lim_{n\to\infty} \left\| \left(f - L_n(f)\right)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} = 0.$$

We remark that if $w \in \mathcal{F}(C^2+)$ is the Erdős-type weight, then we have $\lambda = \lambda(b) \geq 1$ in (2.1). In fact, if $\lambda < 1$, then by Lemma 3.9 below, we see that for $x \geq b > 0$,

$$T(x) = \frac{xQ'(x)}{Q(x)} \leq \frac{x}{Q(x)}Q'(b)\left(\frac{Q(x)}{Q(b)}\right)^{\lambda} = \frac{Q'(b)}{Q(b)^{\lambda}}\frac{x}{Q(x)^{1-\lambda}} \to 0 \quad \text{as } x \to \infty.$$

This contradicts our assumption for $T(x)$. In Example 1.2, we consider the weight $w_{l,\alpha,m} = \exp(-Q_{l,\alpha,m})$. In (2.1), we set $Q := Q_{l,\alpha,m}$ and $\lambda := \lambda(b)$. If $w_{l,\alpha,m}$ is an Erdős-type weight, that is, $T(x) := T_{l,\alpha,m}(x)$ is unbounded, then it is easy to show

$$\lim_{b\to\infty} \lambda(b) = 1.$$

Therefore, when we give any $\Delta > 0$, there exists a constant b large enough such that

$$\Delta > \frac{9}{4}\frac{\lambda(b)-1}{3\lambda(b)-1}.$$

Hence, we have the following corollary.

Corollary 2.3 *Let* $1 < p < \infty$ *and* $\Delta > 0$. *Then for the weight* $w_{l,\alpha,m} = \exp(-Q_{l,\alpha,m})$ $(\alpha > 0)$, *we have*

$$\lim_{n\to\infty} \left\| \bigl(f - L_n(f)\bigr) w_{l,\alpha,m} \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} = 0.$$

We also consider the case of $p = \infty$.

Theorem 2.4 *Let* $w = \exp(-Q) \in \mathcal{F}(C^2+)$, *and let* $T(x)$ *be unbounded. For every* $f \in C_0(\mathbb{R})$ *and* $n \geq 1$, *we have*

$$\left\| \bigl(f - L_n(f)\bigr) w \Phi^{3/4} \right\|_{L_\infty(\mathbb{R})} \leq C E_{n-1}(w;f) \log n,$$

where

$$E_m(w;f) = \inf_{P_m\in\mathcal{P}_m} \max_{x\in\mathbb{R}} \bigl| \bigl(f(x) - P_m(x)\bigr) w(x) \bigr|, \quad m = 0,1,2,\ldots.$$

Moreover, if $f^{(r)}$, $r \geq 1$, *is an integer, then for* $n > r+1$ *we have*

$$\left\| \bigl(f - L_n(f)\bigr) w \Phi^{3/4} \right\|_{L_\infty(\mathbb{R})} \leq C \left(\frac{a_n}{n}\right)^r E_{n-r-1}\bigl(w; f^{(r)}\bigr) \log n.$$

3 Fundamental lemmas

To prove the theorems we need some lemmas.

Lemma 3.1 *Let* $w = \exp(-Q) \in \mathcal{F}(C^2+)$. *Then we have the following.*

(a) [1, Lemma 3.11(a), (b)] *Given fixed* $0 < \alpha$, $\alpha \neq 1$, *we have uniformly for* $t > 0$,

$$\left|1 - \frac{a_{\alpha t}}{a_t}\right| \sim \frac{1}{T(a_t)},$$

and we have for $t > 0$,

$$\left|1 - \frac{a_t}{a_s}\right| \sim \frac{1}{T(a_t)}\left|1 - \frac{t}{s}\right|, \quad \frac{1}{2} \leq \frac{t}{s} \leq 2.$$

(b) [1, Lemma 3.7 (3.38)] *For some* $0 < \varepsilon \leq 2$, *and for large enough* t,

$$T(a_t) \leq t^{2-\varepsilon}.$$

Lemma 3.2 *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Then we have the following.*

(a) [1, Lemma 3.5(a), (b)] *Let $L > 0$ be a fixed constant. Uniformly for $t > 0$,*

$$Q(a_{Lt}) \sim Q(a_t) \quad \textit{and} \quad Q'(a_{Lt}) \sim Q'(a_t).$$

Moreover,

$$a_{Lt} \sim a_t \quad \textit{and} \quad T(a_{Lt}) \sim T(a_t).$$

(b) [1, Lemma 3.4 (3.18), (3.17)] *Uniformly for $x > 0$ with $a_t := x$, $t > 0$, we have*

$$Q'(x) \sim \frac{t\sqrt{T(x)}}{a_t} \quad \textit{and} \quad Q(x) \sim \frac{t}{\sqrt{T(x)}}.$$

(c) [1, Lemma 3.8(a)] *For $x \in [0, a_t)$,*

$$Q'(x) \le C\frac{t}{a_t}\frac{1}{\sqrt{1-\frac{x}{a_t}}}.$$

Lemma 3.3 *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. For $x \in \mathbb{R}$, we have*

$$\Phi(x) \le C\Phi_n(x), \quad n \ge 1.$$

Proof Let $x = a_u$, $u \ge 1$. By Lemma 3.2(b), we have

$$u \sim Q(a_u)\sqrt{T(a_u)}.$$

So, we have

$$\delta_u^{-1} \sim Q^{\frac{2}{3}}(a_u)T(a_u) = \frac{a_u Q'(a_u)}{Q^{\frac{1}{3}}(a_u)} = \frac{xQ'(x)}{Q^{\frac{1}{3}}(x)}. \tag{3.1}$$

Now, if $u \le \frac{n}{2}$, then we have

$$\begin{aligned} 1-\frac{a_u}{a_n} &\ge 1-\frac{a_{n/2}}{a_n} \sim \frac{1}{T(a_n)} \quad \bigl(\text{by Lemma 3.1(a)}\bigr) \\ &\ge \frac{1}{(nT(a_n))^{\frac{2}{3}}} = \delta_n \quad \bigl(\text{by Lemma 3.1(b)}\bigr). \end{aligned}$$

So, we have

$$\begin{aligned} \Phi_n(x) &= 1-\frac{a_u}{a_n} \ge 1-\frac{a_u}{a_{2u}} \sim \frac{1}{T(a_u)} \quad \bigl(\text{by Lemma 3.1(a)}\bigr) \\ &\ge \frac{1}{(uT(a_u))^{\frac{2}{3}}} = \delta_u \sim \Phi(x) \quad \bigl(\text{by Lemma 3.2(b) and (3.1)}\bigr). \end{aligned}$$

Let $\frac{n}{2} < u < n$. Then we have

$$\Phi_n(x) \ge \delta_n \sim \delta_u \sim \Phi(x) \quad \bigl(\text{by Lemma 3.2(a), (b) and (3.1)}\bigr).$$

□

Lemma 3.4 *Let $w \in \mathcal{F}(C^2+)$. Then we have the following.*

(a) [1, Theorem 1.19(f)] *For the minimum positive zero $x_{[n/2],n}$,*

$$x_{[n/2],n} \sim \frac{a_n}{n},$$

and for the maximum zero $x_{1,n}$,

$$1 - \frac{x_{1,n}}{a_n} \sim \delta_n.$$

(b) [1, Theorem 1.19(e)] *For $n \geq 1$ and $1 \leq j \leq n-1$,*

$$x_{j,n} - x_{j+1,n} \sim \varphi_n(x_{j,n}).$$

(c) [1, p.329, (12.20)] *Uniformly for $n \geq 1$, $1 \leq k \leq n-1$,*

$$\varphi_n(x_{k,n}) \sim \varphi_n(x_{k+1,n}).$$

(d) *Let $\max\{|x_{k,n}|, |x_{k+1,n}|\} \leq a_{\alpha n}$, $0 < \alpha < 1$. Then we have*

$$w(x_{k,n}) \sim w(x_{k+1,n}) \sim w(x) \quad (x_{k+1,n} \leq x \leq x_{k,n}).$$

So, for given $C > 0$ and $|x| \leq a_{\beta n}$, $0 < \beta < \alpha$, if $|x - x_{k,n}| \leq C\varphi_n(x)$, then we have

$$w(x) \sim w(x_{k,n}).$$

Proof (d) Let $\max\{|x_{k,n}|, |x_{k+1,n}|\} = |x_{k,n}|$ (for the case of $\max\{|x_{k,n}|, |x_{k+1,n}|\} = |x_{k+1,n}|$, we also have the result similarly). By (b) there exists a constant $C > 0$ such that

$$|x_{k,n} - x_{k+1,n}| \leq C\varphi_n(x_{k,n}).$$

Then we see

$$\begin{aligned}
\varphi_n(x_{k,n}) &\sim \frac{a_n}{n}\frac{1 - \frac{|x_{k,n}|}{a_{2n}}}{\sqrt{1 - \frac{|x_{k,n}|}{a_n}}} = \frac{a_n}{n}\frac{1 - \frac{|x_{k,n}|}{a_n} + |x_{k,n}|\{\frac{1}{a_n} - \frac{1}{a_{2n}}\}}{\sqrt{1 - \frac{|x_{k,n}|}{a_n}}} \\
&= \frac{a_n}{n}\frac{1 - \frac{|x_{k,n}|}{a_n} + \frac{|x_{k,n}|}{a_n}(1 - \frac{a_n}{a_{2n}})}{\sqrt{1 - \frac{|x_{k,n}|}{a_n}}} \sim \frac{a_n}{n}\frac{1 - \frac{|x_{k,n}|}{a_n} + C\frac{|x_{k,n}|}{a_n}\frac{1}{T(a_n)}}{\sqrt{1 - \frac{|x_{k,n}|}{a_n}}} \\
&\sim \frac{a_n}{n}\sqrt{1 - \frac{|x_{k,n}|}{a_n}}.
\end{aligned} \tag{3.2}$$

Therefore, from (3.2) and Lemma 3.2(c), we have

$$\begin{aligned}
\bigl|Q(x_{k,n}) - Q(x_{k+1,n})\bigr| &= \bigl|Q'(\xi)(x_{k,n} - x_{k+1,n})\bigr| \leq C\bigl|Q'(\xi)\bigr|\varphi_n(x) \quad (x_{k+1,n} \leq \xi \leq x_{k,n}) \\
&\leq C\bigl|Q'(x_{k,n})\bigr|\frac{a_n}{n}\sqrt{1 - \frac{|x_{k,n}|}{a_n}} \leq C\frac{n}{a_n}\frac{1}{\sqrt{1 - \frac{|x_{k,n}|}{a_n}}}\frac{a_n}{n}\sqrt{1 - \frac{|x_{k,n}|}{a_n}} \leq C.
\end{aligned}$$

Consequently,

$$w(x_{k,n}) \sim w(x_{k+1,n}) \sim w(x) \quad (x_{k+1,n} \le x \le x_{k,n}).$$

Let $|x - x_{k,n}| \le C\varphi_n(x)$ and $|x| \le a_{\beta n}$. Then we see that there exists $n_0 > 0$ such that $|x_{k,n}| \le a_{\alpha n}$, $n \ge n_0$. In fact, we can show it as follows. We use Lemma 3.1(a) and (b). For $|x| \le a_{\beta n}$, we see

$$|x_{k,n}| \le |x| + C\varphi_n(x) \le |x| + C\frac{a_n}{n}\sqrt{1 - \frac{|x|}{a_n}},$$

and if we take n large enough, then we have

$$\begin{aligned}\frac{d}{dt}\left(t + C\frac{a_n}{n}\sqrt{1 - \frac{t}{a_n}}\right) &= 1 - C\frac{1}{n}\frac{1}{2\sqrt{1 - \frac{t}{a_n}}} \ge 1 - C\frac{1}{n}\frac{1}{2\sqrt{1 - \frac{a_{n/3}}{a_n}}} \\ &\ge 1 - C\frac{\sqrt{T(a_n)}}{2n} \ge 1 - C\frac{1}{2n^{\varepsilon/2}} > 0,\end{aligned}$$

that is, $g(t) = t + C\frac{a_n}{n}\sqrt{1 - \frac{t}{a_n}}$ is increasing. So, we see

$$|x_{k,n}| \le a_{\beta n} + C\frac{a_n}{n}\sqrt{1 - \frac{a_{\beta n}}{a_n}} \le a_{\beta n} + C\frac{a_n}{n}\frac{1}{\sqrt{T(a_n)}}.$$

Therefore, we have

$$\begin{aligned}a_{\alpha n} - \left(a_{\beta n} + C\frac{a_n}{n}\frac{1}{\sqrt{T(a_n)}}\right) &\sim \frac{a_n}{T(a_n)} - C\frac{a_n}{n}\frac{1}{\sqrt{T(a_n)}} \\ &= \frac{a_n}{T(a_n)}\left(1 - C\frac{\sqrt{T(a_n)}}{n}\right) \ge \frac{a_n}{T(a_n)}\left(1 - C\frac{1}{n^{\varepsilon/2}}\right) > 0.\end{aligned}$$

Now, we can show (d). Without loss of generality, we may assume $x \in [x_{j+1,n}, x_{j,n}] \subset \{x_{k,n} | |x - x_{k,n}| \le C\varphi_n(x)\}$. We define

$$x_{k_1,n} := \min\{x_{k,n} | |x - x_{k,n}| \le C\varphi_n(x)\}, \qquad x_{k_2,n} := \max\{x_{k,n} | |x - x_{k,n}| \le C\varphi_n(x)\}.$$

Here we note that k_1, k_2 are decided depending only on the constant C. Then by former result, we have

$$w(x_{k_1,n}) \sim w(x_{k_2,n}) \sim w(x) \quad (x_{k_1,n} \le x \le x_{k_2,n}). \qquad \square$$

Lemma 3.5 *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Then we have the following.*

(a) [1, Theorem 1.17] *Uniformly for $n \ge 1$,*

$$\sup_{x\in\mathbb{R}} \left|p_n(x)\right| w(x) \left|x^2 - a_n^2\right|^{\frac{1}{4}} \sim 1.$$

(b) [1, Theorem 1.19(a)] *Uniformly for $n \ge 1$ and $1 \le j \le n$,*

$$\left|\left(p_n' w\right)(x_{j,n})\right| \sim \varphi_n^{-1}(x_{j,n}) a_n^{-\frac{1}{2}} \left(1 - \frac{|x_{j,n}|}{a_n}\right)^{-\frac{1}{4}}.$$

(c) [1, Theorem 1.19(d)] *For* $x \in [x_{k+1,n}, x_{k,n}]$, *if* $k \le n-1$,

$$\left|p_n(x)w(x)\right| \sim \min\left\{|x-x_{k,n}|, |x-x_{k+1,n}|\right\} a_n^{1/2}\varphi_n(x)^{-1}\left(1-\frac{|x_{k,n}|}{a_n}\right)^{-1/4}.$$

Lemma 3.6 (*cf.* [5, Theorem 2.7]) *Let* $w \in \mathcal{F}(C^2+)$ *and* $0 < p \le \infty$. *Then uniformly* $n \ge 2$,

$$\left\| \Phi^{(\frac{1}{4}-\frac{1}{p})^+} p_n w \right\|_{L_p(\mathbb{R})} \le C a_n^{\frac{1}{p}-\frac{1}{2}} \begin{cases} 1, & 0 < p < 4 \text{ or } p = \infty; \\ \log(1+n), & 4 \le p, \end{cases}$$

where $x^+ = 0$ *if* $x \le 0$, $x^+ = x$ *if* $x > 0$.

Proof From Lemma 3.3, we know $\Phi(x) \le \Phi_n(x)$, then in [5, Theorem 2.7] we only exchange Φ_n with Φ. □

Let $f \in L_{p,w}(\mathbb{R})$. The Fourier-type series of f is defined by

$$\tilde{f}(x) := \sum_{k=0}^{\infty} a_k\left(w^2, f\right) p_k\left(w^2, x\right), \quad a_k\left(w^2, f\right) := \int_{-\infty}^{\infty} f(t) p_k\left(w^2, t\right) w^2(t)\, dt.$$

We denote the partial sum of $\tilde{f}(x)$ by

$$s_n(f, x) := s_n\left(w^2, f, x\right) := \sum_{k=0}^{n-1} a_k\left(w^2, f\right) p_k\left(w^2, x\right).$$

The partial sum $s_n(f)$ admits the representation

$$s_n(f, x) = \sum_{j=0}^{n-1} a_j p_j(x) = \int_{-\infty}^{\infty} f(t) K_n(x, t) w^2(t)\, dt,$$

where

$$K_n(x, t) := \sum_{j=0}^{n-1} p_j(x) p_j(t).$$

The Christoffel-Darboux formula

$$K_n(x, t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t} \tag{3.3}$$

is well known (see [6, Theorem 1.1.4]).

Lemma 3.7 ([6, Lemma 9.2.6]) *Let* $1 < p < \infty$ *and* $g \in L_p(\mathbb{R})$. *Then for the Hilbert transform*

$$H(g, x) := \lim_{\varepsilon \to 0+} \int_{|x-t| \ge \varepsilon} \frac{g(t)}{x-t}\, dt, \quad x \in \mathbb{R}, \tag{3.4}$$

we have

$$\|H(g)\|_{L_p(\mathbb{R})} \le C\|g\|_{L_p(\mathbb{R})},$$

where $C > 0$ *is a constant depending upon* p *only.*

Lemma 3.8 (see [7, Theorem 1.4, Theorem 1.6]) *Let* $w = \exp(-Q) \in \mathcal{F}(C^2)$, $1 \le p \le \infty$ *and* $\gamma \ge 0$. *Then for any* $\varepsilon > 0$, *there exists a polynomial* P *such that*

$$\left\|\left(f(x) - P(x)\right)\left(1 + x^2\right)^{\gamma} w(x)\right\|_{L_p(\mathbb{R})} < \varepsilon.$$

Lemma 3.9 *Let* $w \in \mathcal{F}(C^2+)$ *be an Erdős-type weight, that is,* $T(x)$ *is unbounded. Then for any* $M > 1$, *there exist* $x_M > 0$ *and* $C_M > 0$ *such that*

$$Q(x) \ge C_M x^M, \quad x \ge x_M.$$

Proof For every $M > 1$, there exists $x_M > 0$ such that $T(x) \ge M$ for $x \ge x_M$, so that $Q'(x)/Q(x) = T(x)/x \ge M/x$ for $x \ge x_M$. Hence, we see

$$\log \frac{Q(x)}{Q(x_M)} \ge \log\left(\frac{x}{x_M}\right)^M, \quad x \ge x_M,$$

that is,

$$Q(x) \ge \frac{Q(x_M)}{(x_M)^M} x^M, \quad x \ge x_M.$$

Let us put $C_M := Q(x_M)/(x_M)^M$. □

4 Proof of Theorem 2.2 by Damelin and Lubinsky methods

In this section, we assume $w \in \mathcal{F}(C^2+)$. To prove the theorem we need some lemmas, and we will use the Damelin and Lubinsky methods of [3].

Lemma 4.1 (*cf.* [3, Lemma 3.1]) *Let* $w \in \mathcal{F}(C^2+)$. *Let* $0 < \alpha < \frac{1}{4}$ *and*

$$\sum_n(x) := \sum_{|x_{k,n}| \ge a_{\alpha n}} \left|l_{k,n}(x)\right| w^{-1}(x_{k,n}).$$

Then we have for $|x| \le a_{\alpha n/2}$ *and* $|x| \ge a_{2n}$,

$$\sum_n(x) w(x) \le C.$$

Moreover, for $a_{\alpha n/2} \le |x| \le a_{2n}$,

$$\sum_n(x) w(x) \le C\left(\log n + a_n^{\frac{1}{2}}\left|p_n(x)w(x)\right|T^{-\frac{1}{4}}(a_n)\right).$$

Proof The proof of [3, Lemma 3.1] holds without the condition (1.2) and the second condition in (1.1) and under the assumption of the quasi-increasingness of $T(x)$. The conditions in Definition 1.1 contain all the conditions in Definition 1.3 except for (1.2) and the second condition in (1.1). We see that in [3, Lemma 3.1] we can replace $T^*(x)$ with $T(x)$. □

Lemma 4.2 ([3, Lemma 3.2]) *Let* $0 < \eta < 1$. *Let* $\psi : \mathbb{R} \mapsto (0, \infty)$ *be a continuous function with the following property*: *For* $n \geq 1$, *there exist polynomials* R_n *of degree* $\leq n$ *such that*

$$C_1 \leq \frac{\psi(t)}{R_n(t)} \leq C_2, \quad |t| \leq a_{4n}.$$

Then for $n \geq n_0$ *and* $P \in \mathcal{P}_n$,

$$\sum_{|x_{k,n}| \leq a_{\eta n}} \lambda_{k,n} \left|P(x_{k,n})\right| w^{-1}(x_{k,n}) \psi(x_{k,n}) \leq C \int_{-a_{4n}}^{a_{4n}} \left|P(t)w(t)\right| \psi(t)\, dt.$$

Remark 4.3 To prove Lemma 4.7 below, we apply this lemma with $\psi(t) = \phi(t) = (1 + t^2)^{-\beta/2}$, $\beta > 0$. In fact, when $\phi^*(x) = \phi(t)$, $t = a_{4n}x$, we can approximate ϕ^* by polynomials $R_n^* \in \mathcal{P}_n$ on $[-1,1]$, that is, for any $\varepsilon > 0$ there exists $R_n^* \in \mathcal{P}_n$ such that

$$\left|\phi^*(x) - R_n^*(x)\right| < \varepsilon, \quad x \in [-1,1].$$

Therefore,

$$\left|\frac{R_n^*(x)}{\phi^*(x)} - 1\right| < \frac{\varepsilon}{\phi^*(x)}, \quad x \in [-1,1],$$

and so there exist $C_1, C_2 > 0$ such that

$$C_1 \leq 1 - \frac{\varepsilon}{\phi^*(x)} \leq \left|\frac{R_n^*(x)}{\phi^*(x)}\right| < 1 + \frac{\varepsilon}{\phi^*(x)} \leq C_2, \quad x \in [-1,1].$$

Now, if we set $R_n(t) = R_n^*(x)$, then we have the result.

Lemma 4.4 (*cf.* [3, Lemma 4.1]) *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of measurable functions from* $\mathbb{R} \mapsto \mathbb{R}$ *such that for* $n \geq 1$,

$$f_n(x) = 0, \quad |x| < a_{\frac{n}{9}}; \qquad \left|f_n(x)\right| w(x) \leq \phi(x), \quad x \in \mathbb{R}.$$

Then for $1 \leq p \leq \infty$ *and* $\Delta > 0$, *we have*

$$\lim_{n\to\infty} \left\|L_n(f_n) w \Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+}\right\|_{L_p(\mathbb{R})} = 0. \tag{4.1}$$

Proof Let $|x| \leq a_{\frac{n}{18}}$ or $|x| \geq a_{2n}$. We use the first inequality of Lemma 4.1 with $\alpha = \frac{1}{9}$, then from the assumption with respect to f_n, we see that

$$\left|L_n(f_n;x)w(x)\right| \leq \phi(a_{\frac{n}{9}}) \sum_{|x_{k,n}| \geq a_{\frac{n}{9}}} \left|l_{k,n}(x)\right| w^{-1}(x_{k,n}) w(x) \leq C_1 \phi(a_{\frac{n}{9}}).$$

So,

$$\begin{aligned}\bigl\|L_n(f_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(|x|\le a_{\frac{n}{18}}\text{ or }|x|\ge a_{2n})} &\le \phi(a_{\frac{n}{9}})\bigl\|\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(\mathbb{R})}\\ &\le C_2\phi(a_{\frac{n}{9}}) = o(1)\end{aligned}\tag{4.2}$$

by Lemma 3.9 (note the definition of $\Phi(x)$) and the definition of ϕ in (2.4). Next, we let $a_{\frac{n}{18}} \le |x| \le a_{2n}$. From the second inequality in Lemma 4.1, we see that

$$\bigl|L_n(f_n;x)w(x)\bigr| \le \phi(a_{\frac{n}{9}})\bigl(\log n + a_n^{\frac{1}{2}}\bigl|p_n(x)\bigr|w(x)T^{-\frac{1}{4}}(a_n)\bigr).$$

Also, for this range of x, we see that

$$\Phi(x) = \frac{1}{(1+Q(x))^{\frac{2}{3}}T(x)} \sim \frac{1}{(1+Q(a_n))^{\frac{2}{3}}T(a_n)} \sim \frac{T^{\frac{1}{3}}(a_n)}{n^{\frac{2}{3}}T(a_n)} = \delta_n$$

by Lemma 3.2(b). So, for n large enough,

$$\begin{aligned}&\bigl\|L_n(f_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(a_{\frac{n}{18}}\le|x|\le a_{2n})}\\ &\quad\le \phi(a_{\frac{n}{9}})\log n\bigl\|\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(a_{\frac{n}{18}}\le|x|\le a_{2n})}\\ &\qquad+\phi(a_{\frac{n}{9}})a_n^{\frac{1}{2}}T^{-\frac{1}{4}}(a_n)\bigl\|p_n(x)w(x)\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(a_{\frac{n}{18}}\le|x|\le a_{2n})}.\end{aligned}$$

Then since $\Delta > 0$, using Lemma 3.1(a), Lemma 2.1(a), and Lemma 3.6, we have

$$\log n\bigl\|\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(a_{\frac{n}{18}}\le|x|\le a_{2n})} \le C\delta_n^{\Delta}(a_{2n}-a_{\frac{n}{18}})^{\frac{1}{p}}\log n \le C$$

and

$$\begin{aligned}&a_n^{\frac{1}{2}}T^{-\frac{1}{4}}(a_n)\bigl\|p_nw\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(a_{\frac{n}{18}}\le|x|\le a_{2n})}\\ &\quad\le T^{-\frac{1}{4}}(a_n)\delta_n^{\Delta}a_n^{\frac{1}{p}}\begin{cases}1, & 1\le p<4 \text{ or } p=\infty;\\ \log(1+n), & 4\le p,\end{cases} \le C.\end{aligned}$$

Therefore, we have by (2.4)

$$\bigl\|L_n(f_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\bigr\|_{L_p(a_{\frac{n}{18}}\le|x|\le a_{2n})} \le C_4\phi(a_{\frac{n}{9}}) = o(1).$$

Consequently, with (4.2) we have (4.1). □

Lemma 4.5 (*cf.* [3, Lemma 4.2]) *Let* $1 \le p \le \infty$. *Let* $\{g_n\}_{n=1}^{\infty}$ *be a sequence of measurable functions from* $\mathbb{R} \mapsto \mathbb{R}$ *such that for* $n \ge 1$,

$$g_n(x) = 0, \quad |x| \ge a_{\frac{n}{9}}; \qquad \bigl|g_n(x)\bigr|w(x) \le \phi(x), \quad x \in \mathbb{R}.\tag{4.3}$$

Let us suppose

$$\Delta > \frac{9}{4}\frac{\lambda - 1}{3\lambda - 1}, \tag{4.4}$$

where $\lambda \geq 1$ *is defined in Lemma* 2.1. *Then for* $1 \leq p \leq \infty$, *we have*

$$\lim_{n\to\infty} \left\| L_n(g_n) w \Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+} \right\|_{L_p(|x| \geq a_{\frac{n}{8}})} = 0. \tag{4.5}$$

Proof Using Lemma 3.5(b) and Lemma 3.4(b), we have for $x \geq a_{\frac{n}{8}}$,

$$\begin{aligned} \left| L_n(g_n; x) \right| &\leq \sum_{|x_{k,n}| \leq a_{\frac{n}{9}}} \left| l_{k,n}(x) \right| w^{-1}(x_{k,n}) \phi(x_{k,n}) \\ &\leq C_1 a_n^{\frac{1}{2}} \left| p_n(x) \right| \sum_{|x_{k,n}| \leq a_{\frac{n}{9}}} (x_{k,n} - x_{k+1,n}) \frac{(1 - \frac{|x_{k,n}|}{a_n} + \delta_n)^{\frac{1}{4}}}{|x - x_{k,n}|} \phi(x_{k,n}) \\ &\leq C_2 a_n^{\frac{1}{2}} \left| p_n(x) \right| \int_{-a_{\frac{n}{9}}}^{a_{\frac{n}{9}}} \frac{(1 - \frac{|t|}{a_n} + \delta_n)^{\frac{1}{4}}}{|x - t|} \phi(t)\, dt. \end{aligned} \tag{4.6}$$

Equation (4.6) is shown as follows: First, we see

$$|x - t| \sim |x - x_{k,n}|, \quad t \in [x_{k+1,n}, x_{k,n}]. \tag{4.7}$$

Let $|x| \geq a_{\frac{n}{8}}$ and $t \in [x_{k+1,n}, x_{k,n}]$. Then

$$\left| \frac{x - t}{x - x_{k,n}} - 1 \right| = \left| \frac{t - x_{k,n}}{x - x_{k,n}} \right| \leq \frac{x_{k,n} - x_{k+1,n}}{|x_{k\pm 2,n} - x_{k,n}|} \leq c < 1.$$

Now, we use the fact that $x + C\varphi(x)$, $x > 0$ is increasing for $0 < x \leq a_{n/2}$, and then

$$x_{k,n} + C\varphi_n(x_{k,n}) \leq a_{\frac{n}{9}} + C\varphi_n(a_{\frac{n}{9}}) \leq a_{\frac{n}{8}} \leq x.$$

Here, the second inequality follows from the definition of $\varphi_n(x)$ and Lemma 3.1(a), (b). Hence, we have (4.7). Now, we use the monotonicity of $(1 - \frac{|x|}{a_n} + \delta_n)^{\frac{1}{4}} \phi(x)$. From (4.7) there exists $C > 0$ such that for $t \in [x_{k+1,n}, x_{k,n}]$,

$$\begin{aligned} (x_{k,n} - x_{k+1,n}) \frac{(1 - \frac{|x_{k,n}|}{a_n} + \delta_n)^{\frac{1}{4}}}{|x - x_{k,n}|} \phi(x_{k,n}) &\leq \int_{x_{k+1,n}}^{x_{k,n}} \frac{(1 - \frac{|t|}{a_n} + \delta_n)^{\frac{1}{4}}}{|x - x_{k,n}|} \phi(t)\, dt \\ &\leq \frac{1}{C} \int_{x_{k+1,n}}^{x_{k,n}} \frac{(1 - \frac{|t|}{a_n} + \delta_n)^{\frac{1}{4}}}{|x - t|} \phi(t)\, dt. \end{aligned}$$

Hence, (4.6) holds. Next, for $t \in [0, a_{\frac{n}{9}}]$ and $x \geq a_{\frac{n}{8}}$, we know by Lemma 3.1(a),

$$1 \leq \frac{a_n - t}{x - t} \leq 1 + \frac{a_n - a_{\frac{n}{8}}}{a_{\frac{n}{8}} - t} \leq 1 + \frac{a_n - a_{\frac{n}{8}}}{a_{\frac{n}{8}} - a_{\frac{n}{9}}} \leq 1 + C \frac{a_{\frac{n}{8}}}{a_{\frac{n}{9}}} \frac{T(a_{\frac{n}{9}})}{T(a_{\frac{n}{8}})} \leq C_3$$

and

$$1-\frac{|t|}{a_n}\geq C_4\frac{1}{T(a_n)}\geq \delta_n.$$

So, we have

$$\left|L_n(g_n;x)\right|\leq C_6 a_n^{\frac{1}{4}}\left|p_n(x)\right|\int_0^{a_{\frac{n}{9}}}(x-t)^{-\frac{3}{4}}\phi(t)\,dt.$$

Let $t=a_s$, $\frac{n}{9}\geq s\geq 1$. Then, since we know for $x\geq a_{\frac{n}{8}}$,

$$x-t=x\left(1-\frac{t}{x}\right)\geq a_{\frac{n}{8}}\left(1-\frac{a_s}{a_{\frac{9}{8}s}}\right)\geq C_7\frac{a_n}{T(a_s)},$$

we obtain

$$\left|L_n(g_n;x)\right|\leq C_8 a_n^{-\frac{1}{2}}\left|p_n(x)\right|\int_0^{a_{\frac{n}{9}}}T^{\frac{3}{4}}(t)\phi(t)\,dt\leq C_8 a_n^{\frac{1}{2}}T^{\frac{3}{4}}(a_n)\left|p_n(x)\right|.$$

Hence, if $1\leq\lambda$, then using Lemma 3.6, (3.1) and (2.2), we have

$$\begin{aligned}
&\left\|L_n(g_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|_{L_p(|x|\geq a_{\frac{n}{8}})}\\
&\quad\leq C_9 a_n^{\frac{1}{2}}T^{\frac{3}{4}}(a_n)\Phi^{\Delta}(a_{\frac{n}{8}})\left\|\Phi^{(\frac{1}{4}-\frac{1}{p})^+}wp_n\right\|_{L_p(\mathbb{R})}\\
&\quad\leq C_{10}a_n^{\frac{1}{p}}T^{\frac{3}{4}}(a_n)\left(\frac{1}{nT(a_n)}\right)^{\frac{2}{3}\Delta}\begin{cases}1, & 0<p<4 \text{ or } p=\infty;\\ \log(1+n), & 4\leq p\end{cases}\\
&\quad\leq C_{11}C(\lambda,\eta)a_n^{\frac{1}{p}}\left(\frac{1}{n}\right)^{\frac{2}{3}\frac{3\lambda+2\eta-1}{\lambda+1}(\Delta-\frac{9}{4}\frac{\lambda+\eta-1}{3\lambda+2\eta-1})}\begin{cases}1, & 1\leq p<4 \text{ or } p=\infty;\\ \log(1+n), & 4\leq p.\end{cases}
\end{aligned}$$

Here, we may consider that above estimations hold under the condition (4.4), because that $\eta>0$ can be taken small enough. Then we have (4.5), that is, for $\Delta>\frac{9}{4}\frac{\lambda-1}{3\lambda-1}$,

$$\lim_{n\to\infty}\left\|L_n(g_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|_{L_p(|x|\geq a_{\frac{n}{8}})}=0. \qquad\square$$

Lemma 4.6 (*cf.* [3, Lemma 4.3]) *Let $1<p<\infty$. Let $\sigma:\mathbb{R}\mapsto\mathbb{R}$ be a bounded measurable function. Let $\lambda=\lambda(b)\geq 1$ be defined in Lemma 2.1, and then we suppose*

$$\Delta>\begin{cases}0, & 1<p\leq 2;\\ \frac{3}{2}\frac{(\lambda-1)}{3\lambda-1}\frac{p-2}{p}, & 2<p\leq 4;\\ \max\{\frac{\lambda-1}{3\lambda-1}\frac{p-1}{p}-\frac{1}{4}\frac{\lambda+1}{3\lambda-1}\frac{p-4}{p},0\}, & 4<p.\end{cases}\tag{4.8}$$

Then for $1<p<\infty$ and the partial sum s_n of the Fourier series, we have

$$\left\|s_n\left[\sigma\phi w^{-1}\right]w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|_{L_p(|x|\leq a_{\frac{n}{8}})}\leq C\|\sigma\|_{L_\infty(\mathbb{R})}\tag{4.9}$$

for $n\geq 1$. Here C is independent of σ and n.

Proof We may suppose that $\|\sigma\|_{L_\infty(\mathbb{R})} = 1$. By (3.3), (3.4) and Lemma 3.5(a),

$$\left|s_n\left[\sigma\phi w^{-1}\right](x)\right|w(x) \le a_n^{\frac{1}{2}}\left(1-\frac{|x|}{a_n}\right)^{-\frac{1}{4}}\sum_{j=n-1}^{n}\left|H[\sigma\phi p_j w](x)\right|. \tag{4.10}$$

Let us choose $l := l(n)$ such that $2^l \le \frac{n}{8} \le 2^{l+1}$. Then we know

$$2^{l+3} \le n \le 2^{l+4}. \tag{4.11}$$

Define

$$\mathcal{I}_k = [a_{2^k}, a_{2^{k+1}}], \quad 1 \le k \le l+2.$$

For $j = n-1, n$ and $x \in \mathcal{I}_k$, we split

$$\begin{aligned} H[\sigma\phi p_j w](x)w(x) &= \left(\int_{-\infty}^{0} + \int_{0}^{a_{2^{k-1}}} + P.V.\int_{a_{2^{k-1}}}^{a_{2^{k+2}}} + \int_{a_{2^{k+2}}}^{\infty}\right)\frac{(\sigma\phi p_j w)(t)}{x-t}\,dt \\ &:= I_1(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned} \tag{4.12}$$

Here $P.V.$ stands for the principal value. First, we give the estimations of I_1 and I_2 for $x \in \mathcal{I}_k$. Let $x \in \mathcal{I}_k$. Then we have by Lemma 3.5(a) and Lemma 3.6 with $p = 1$,

$$\begin{aligned} \left|I_1(x)\right| &\le \int_0^\infty \frac{|(p_j w\phi)(-t)|}{t+x} \le C_1 a_n^{-\frac{1}{2}}\int_0^{\frac{a_n}{2}}\frac{\phi(t)}{t+a_2}\,dt + C_2 a_n^{-1}\int_{\frac{a_n}{2}}^{\infty}\left|p_j(t)\right|w(t)\,dt \\ &\le C_2\left(a_n^{-\frac{1}{2}} + a_n^{-1}a_n^{1-\frac{1}{2}}\right) \le C_3 a_n^{-\frac{1}{2}}. \end{aligned} \tag{4.13}$$

Here we have used

$$\int_0^\infty \frac{\phi(t)}{1+t}\,dt < \infty. \tag{4.14}$$

By Lemma 3.5(a), and noting $1 - x/a_n \le 1 - t/a_n$ for $x \in \mathcal{I}_k$,

$$\begin{aligned} \left|I_2(x)\right| &\le \int_0^{a_{2^{k-1}}}\frac{|(p_j w\phi)(t)|}{x-t}\,dt \le C_4 a_n^{-\frac{1}{2}}\int_0^{a_{2^{k-1}}}\frac{(1-\frac{t}{a_n})^{-\frac{1}{4}}}{x-t}\,dt \\ &\le C_4 a_n^{-\frac{1}{2}}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\int_0^{a_{2^{k-1}}}\frac{dt}{x-t} \\ &= C_4 a_n^{-\frac{1}{2}}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\log\left(1-\frac{a_{2^{k-1}}}{x}\right)^{-1}. \end{aligned}$$

Using

$$1-\frac{a_{2^{k-1}}}{x} \ge 1-\frac{a_{2^{k-1}}}{a_{2^k}} \ge C\frac{1}{T(a_{2^k})} \ge C\frac{1}{T(x)},$$

we can see

$$\left|I_2(x)\right| \le C_6 a_n^{-\frac{1}{2}}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}}\log\left(\frac{T(x)}{C}\right). \tag{4.15}$$

Next, we give an estimation of I_4 for $x \in \mathcal{I}_k$. Let $x \in \mathcal{I}_k$. From Lemma 3.5(a) again,

$$\begin{aligned}
\bigl|I_4(x)\bigr| &\le \int_{a_{2^{k+2}}}^{2a_{2^{k+2}}} \frac{|(p_j w\phi)(t)|}{t-x}\,dt + C\int_{2a_{2^{k+2}}}^{\infty} \frac{|(p_j w\phi)(t)|}{t}\,dt \quad \bigl(\text{by } t \le 2(t-x)\bigr) \\
&\le C_7\biggl(a_n^{-\frac{1}{2}}\int_{a_{2^{k+2}}}^{2a_{2^{k+2}}} \left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}} \frac{dt}{t-x} \\
&\quad + a_n^{-\frac{1}{2}}\int_{2a_{2^{k+2}}}^{\max\{2a_{2^{k+2}},\frac{1}{2}a_n\}} \frac{\phi(t)}{t}\,dt + \int_{\frac{1}{2}a_n}^{\infty} \frac{|(p_j w)(t)|}{t}\,dt\biggr) \\
&\le C_7\biggl(a_n^{-\frac{1}{2}}\int_{a_{2^{k+2}}}^{2a_{2^{k+2}}} \left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}} \frac{dt}{t-x} + Ca_n^{-\frac{1}{2}} + a_n^{-1}a_n^{1-\frac{1}{2}}\biggr) \\
&\quad \bigl(\text{by (4.14) and Lemma 3.6 with } p=1\bigr) \\
&\le C_8 a_n^{-\frac{1}{2}}[J+1], \qquad (4.16)
\end{aligned}$$

where

$$J := \int_{a_{2^{k+2}}}^{2a_{2^{k+2}}} \left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}} \frac{dt}{t-x}.$$

Since, if

$$\left|1-\frac{t}{a_n}\right| \le \frac{1}{2}\left(1-\frac{x}{a_n}\right),$$

then we see

$$|t-x| = a_n\left|\left(1-\frac{x}{a_n}\right)-\left(1-\frac{t}{a_n}\right)\right| \ge \frac{a_n}{2}\left(1-\frac{x}{a_n}\right).$$

Now, we have

$$\begin{aligned}
J &\le C_9\Biggl(\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \int_{\substack{|1-\frac{t}{a_n}|\ge\frac{1}{2}(1-\frac{x}{a_n}),\\ t\in[a_{2^{k+2}},2a_{2^{k+2}}]}} \frac{1}{t-x}\,dt \\
&\quad + a_n^{-1}\left(1-\frac{x}{a_n}\right)^{-1} \int_{\substack{|1-\frac{t}{a_n}|\le\frac{1}{2}(1-\frac{x}{a_n}),\\ t\in[a_{2^{k+2}},2a_{2^{k+2}}]}} \left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}} dt\Biggr) \\
&\le C_{10}\Biggl(\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\left(1+\frac{a_{2^{k+2}}}{a_{2^{k+2}}-a_{2^{k+1}}}\right) \\
&\quad + \left(1-\frac{x}{a_n}\right)^{-1} \int_{|1-s|\le\frac{1}{2}(1-\frac{x}{a_n})} |1-s|^{-\frac{1}{4}}\,ds\Biggr) \\
&\le C_{10}\Biggl(\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\bigl(1+CT(a_{2^{k+2}})\bigr) + \frac{4}{3}\left(\frac{1}{2}\left(1-\frac{x}{a_n}\right)\right)^{\frac{3}{4}}\left(1-\frac{x}{a_n}\right)^{-1}\Biggr) \\
&\le C_{11}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\bigl(CT(x)\bigr).
\end{aligned}$$

So, from (4.16) we have

$$\left|I_4(x)\right| \le C_{12} a_n^{-\frac{1}{2}} \left(1 - \frac{x}{a_n}\right)^{-\frac{1}{4}} \log\bigl(CT(x)\bigr). \tag{4.17}$$

Therefore, from (4.13), (4.15) and (4.17), we have

$$|I_1 + I_2 + I_4| \le C_{13} a_n^{-\frac{1}{2}} \left(1 - \frac{x}{a_n}\right)^{-\frac{1}{4}} \log\bigl(CT(x)\bigr).$$

Hence, with (4.10), (4.12) we have

$$\begin{aligned}
&\left\| s_n\bigl[\sigma\phi w^{-1}\bigr] w \Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+} \right\|_{L_p(\mathcal{I}_k)} \\
&\quad\le C_{14} \Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+}(a_{2^k}) \left(\left(1 - \frac{a_{2^{k+1}}}{a_n}\right)^{-\frac{1}{2}} \log\bigl(CT(a_{2^{k+1}})\bigr)(a_{2^{k+1}} - a_{2^k})^{\frac{1}{p}} \right. \\
&\qquad \left. + a_n^{\frac{1}{2}} \left(1 - \frac{a_{2^{k+1}}}{a_n}\right)^{-\frac{1}{4}} \sum_{j=n-1}^{n} \left\| P.V. \int_{a_{2^{k-1}}}^{a_{2^{k+2}}} \frac{(\sigma\phi p_j w)(t)}{x - t}\, dt \right\|_{L_p(\mathcal{I}_k)} \right).
\end{aligned} \tag{4.18}$$

We must estimate the L_p-norm with respect to I_3, that is, $\|P.V. \int_{a_{2^{k-1}}}^{a_{2^{k+2}}} \frac{(\sigma\phi p_j w)(t)}{x-t}\, dt\|_{L_p(\mathcal{I}_k)}$. We use M. Riesz's theorem on the boundedness of the Hilbert transform from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ (Lemma 3.7) to deduce that by Lemma 3.5(a) and the boundedness of $|\sigma\phi|$,

$$\begin{aligned}
\left\| P.V. \int_{a_{2^{k-1}}}^{a_{2^{k+2}}} \frac{(\sigma\phi p_j w)(t)}{x - t}\, dt \right\|_{L_p(\mathcal{I}_k)} &\le C_{15} \left(\int_{a_{2^{k-1}}}^{a_{2^{k+2}}} \left|(\sigma\phi p_j w)(t)\right|^p dt \right)^{\frac{1}{p}} \\
&\le C_{16} a_n^{-\frac{1}{2}} \left(1 - \frac{a_{2^{k+2}}}{a_n}\right)^{-\frac{1}{4}} (a_{2^{k+2}} - a_{2^{k-1}})^{\frac{1}{p}}.
\end{aligned} \tag{4.19}$$

So, by (4.18) and (4.19) we conclude

$$\begin{aligned}
&\left\| s_n\bigl[\sigma\phi w^{-1}\bigr] w \Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+} \right\|_{L_p(\mathcal{I}_k)} \\
&\quad\le C_{18} \Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+}(a_{2^k}) \left(1 - \frac{a_{2^{k+1}}}{a_n}\right)^{-\frac{1}{2}} \log\bigl(CT(a_{2^{k+1}})\bigr)(a_{2^{k+1}} - a_{2^k})^{\frac{1}{p}}.
\end{aligned} \tag{4.20}$$

Noting (4.11), we see $n \ge 2^{l+3}$ for $k \le l$, so

$$1 - \frac{a_{2^{k+1}}}{a_n} \ge 1 - \frac{a_{2^{k+1}}}{a_{2^{k+3}}} \ge C_{19} \frac{1}{T(a_{2^k})} \quad \text{and} \quad a_{2^{k+1}} - a_{2^k} \le C_{20} \frac{a_{2^k}}{T(a_{2^k})}.$$

On the other hand, using Lemma 3.2(b), we see $\Phi(a_t) \sim \delta_t$. Hence, we have

$$\begin{aligned}
\Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+}(a_{2^k}) \sim \delta_{2^k}^{\Delta + (\frac{1}{4} - \frac{1}{p})^+} &= \left(\frac{1}{2^k T(a_{2^k})} \right)^{\frac{2}{3}(\Delta + (\frac{1}{4} - \frac{1}{p})^+)} \\
&= \begin{cases} (\frac{1}{2^k T(a_{2^k})})^{\frac{2}{3}\Delta}, & 0 < p \le 4; \\ (\frac{1}{2^k T(a_{2^k})})^{\frac{2}{3}(\Delta + (\frac{1}{4} - \frac{1}{p}))}, & 4 < p. \end{cases}
\end{aligned}$$

Hence, from (4.20) we have

$$\begin{aligned}
&\left\| s_n\left[\sigma\phi w^{-1}\right] w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|_{L_p(\mathcal{I}_k)} \\
&\quad\le C_{19}\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}(a_{2^k})T^{\frac{1}{2}}(a_{2^k})\log\bigl(CT(a_{2^{k+1}})\bigr)\left(\frac{a_{2^k}}{T(a_{2^k})}\right)^{\frac{1}{p}} \\
&\quad\le C_{19}\log\bigl(CT(a_{2^{k+1}})\bigr)a_{2^k}^{\frac{1}{p}}\begin{cases} (\frac{1}{2^k})^{\frac{2}{3}\Delta}T^{-\frac{2}{3}\Delta+\frac{1}{2}-\frac{1}{p}}(a_{2^k}), & 1<p\le 4; \\ (\frac{1}{2^k})^{\frac{2}{3}(\Delta+(\frac{1}{4}-\frac{1}{p}))}T^{-\frac{2}{3}\Delta+\frac{1}{3}(1-\frac{1}{p})}(a_{2^k}), & 4<p. \end{cases}
\end{aligned}$$

From Lemma 2.1 (2.2), we know

$$T^{-\frac{2}{3}\Delta+\frac{1}{2}-\frac{1}{p}}(a_{2^k})\le C_1C(\lambda,\eta)\bigl(2^k\bigr)^{\frac{2(\eta+\lambda-1)}{\lambda+1}\max\{-\frac{2}{3}\Delta+\frac{1}{2}-\frac{1}{p},0\}},$$

and

$$T^{-\frac{2}{3}\Delta+\frac{1}{3}(1-\frac{1}{p})}(a_{2^k})\le C_2C(\lambda,\eta)\bigl(2^k\bigr)^{\frac{2(\eta+\lambda-1)}{\lambda+1}\max\{-\frac{2}{3}\Delta+\frac{1}{3}(1-\frac{1}{p}),0\}}.$$

Therefore, we continue with Lemma 2.1(a) as

$$\begin{aligned}
&\le C_{20}C(\lambda,\eta)\log\bigl(CT(a_{2^{k+1}})\bigr) \\
&\quad\times\begin{cases} (\frac{1}{2^k})^{\frac{2}{3}\Delta-\frac{\eta}{p}-\frac{2(\eta+\lambda-1)}{\lambda+1}\max\{-\frac{2}{3}\Delta+\frac{1}{2}-\frac{1}{p},0\}}, & 1<p\le 4; \\ (\frac{1}{2^k})^{\frac{2}{3}(\Delta+(\frac{1}{4}-\frac{1}{p}))-\frac{\eta}{p}-\frac{2(\eta+\lambda-1)}{\lambda+1}\max\{-\frac{2}{3}\Delta+\frac{1}{3}(1-\frac{1}{p}),0\}}, & 4<p. \end{cases}
\end{aligned}\tag{4.21}$$

First, let $1<p\le 4$. Then (4.8), that is,

$$\Delta>\begin{cases} 0, & 1<p\le 2; \\ \frac{3}{2}\frac{\lambda-1}{3\lambda-1}\frac{p-2}{p}, & 2<p\le 4 \end{cases}$$

implies

$$\Delta>\frac{3}{2}\frac{\lambda-1}{3\lambda-1}\frac{p-2}{p}\quad\text{and}\quad\Delta>0$$

iff

$$\frac{2}{3}\Delta-\frac{2(\lambda-1)}{\lambda+1}\left(-\frac{2}{3}\Delta+\frac{1}{2}-\frac{1}{p}\right)>0\quad\text{and}\quad\Delta>0$$

iff

$$\frac{2}{3}\Delta-\frac{2(\lambda-1)}{\lambda+1}\max\left\{-\frac{2}{3}\Delta+\frac{1}{2}-\frac{1}{p},0\right\}>0.$$

This means that there exists a positive constant $\eta_1>0$ small enough such that

$$A(\eta_1):=\frac{2}{3}\Delta-\frac{\eta_1}{p}-\frac{2(\eta_1+\lambda-1)}{\lambda+1}\max\left\{-\frac{2}{3}\Delta+\frac{1}{2}-\frac{1}{p},0\right\}>0.$$

Now, let $p > 4$. Then (4.8), that is,

$$\Delta > \frac{\lambda - 1}{3\lambda - 1}\frac{p-1}{p} - \frac{1}{4}\frac{\lambda + 1}{3\lambda - 1}\frac{p-4}{p}$$

implies

$$\Delta > \frac{\lambda - 1}{3\lambda - 1}\left(1 - \frac{1}{p}\right) - \frac{\lambda + 1}{3\lambda - 1}\left(\frac{1}{4} - \frac{1}{p}\right) \quad \text{and} \quad \Delta + \frac{1}{4} - \frac{1}{p} > 0$$

iff

$$\frac{2}{3}\left(\Delta + \left(\frac{1}{4} - \frac{1}{p}\right)\right) - \frac{2(\lambda - 1)}{\lambda + 1}\left(-\frac{2}{3}\Delta + \frac{1}{3}\left(1 - \frac{1}{p}\right)\right) > 0$$

and

$$\frac{2}{3}\left(\Delta + \left(\frac{1}{4} - \frac{1}{p}\right)\right) > 0$$

iff

$$\frac{2}{3}\left(\Delta + \left(\frac{1}{4} - \frac{1}{p}\right)\right) - \frac{2(\lambda - 1)}{\lambda + 1}\max\left\{-\frac{2}{3}\Delta + \frac{1}{3}\left(1 - \frac{1}{p}\right), 0\right\} > 0.$$

Similarly to the previous case, this means that there exists a positive constant $\eta_2 > 0$ small enough such that

$$B(\eta_2) := \frac{2}{3}\left(\Delta + \left(\frac{1}{4} - \frac{1}{p}\right)\right) - \frac{\eta_2}{p} - \frac{2(\eta_2 + \lambda - 1)}{\lambda + 1}\max\left\{-\frac{2}{3}\Delta + \frac{1}{3}\left(1 - \frac{1}{p}\right), 0\right\} > 0.$$

Now, we estimate $I_{p,k}$. From (4.21), we have

$$\begin{aligned}
&\left\| s_n\left[\sigma\phi w^{-1}\right] w\Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+}\right\|_{L_p(\mathcal{I}_k)} \\
&\quad \leq C_{20} C(\lambda, \eta)\log\left(CT(a_{2^{k+1}})\right)\begin{cases} (\frac{1}{2^k})^{A(\eta)}, & 1 < p \leq 4; \\ (\frac{1}{2^k})^{B(\eta)}, & 4 < p. \end{cases}
\end{aligned}$$

For $\eta > 0$ small enough, we can see $A(\eta) > A(\eta_1) > 0$ and $B(\eta) > B(\eta_2) > 0$. Let $\tau := \min\{A(\eta_1), B(\eta_2)\}/2$. Then for small enough $\eta > 0$, we have

$$\begin{aligned}
\left\| s_n\left[\sigma\phi w^{-1}\right] w\Phi^{\Delta + (\frac{1}{4} - \frac{1}{p})^+}\right\|_{L_p(\mathcal{I}_k)} &\leq C_{20} C(\lambda, \eta)\log\left(CT(a_{2^{k+1}})\right)\left(\frac{1}{2^k}\right)^{2\tau} \\
&\leq C_{21} C(\lambda, \eta)\left(\frac{1}{2^k}\right)^{\tau},
\end{aligned}$$

because we see that for all $k > 0$,

$$\log\left(CT(a_{2^{k+1}})\right)\left(\frac{1}{2^k}\right)^{\tau} < C_{22}.$$

Therefore, under the conditions (4.8) we have

$$\begin{aligned}\left\|s_n\left[\sigma\phi w^{-1}\right]w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|^p_{L_p(a_2\le|x|\le a_{\frac{n}{8}})} &\le \sum_{k=1}^{l}\left\|s_n\left[\sigma\phi w^{-1}\right]w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|^p_{L_p(\mathcal{I}_k)}\\ &\le C_{21}C(\lambda,\eta)\sum_{k=1}^{l}\left(\frac{1}{2^k}\right)^{\tau}\le C_{23}C(\lambda,\eta). \qquad (4.22)\end{aligned}$$

The estimation of

$$\left\|s_n\left[\sigma\phi w^{-1}\right]w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|^p_{L_p(|x|\le a_2)}$$

is similar. In fact, for $x\in[-a_2,a_2]$, we split

$$H[\sigma\phi p_j w](x)=\left(\int_{-\infty}^{-2a_2}+P.V.\int_{-2a_2}^{2a_2}+\int_{2a_2}^{\infty}\right)\frac{(\sigma\phi p_j w)(t)}{x-t}\,dt.$$

Here we see that

$$\begin{aligned}\left|\int_{-\infty}^{-2a_2}\frac{(\sigma\phi p_j w)(t)}{x-t}\,dt\right| &= \left|\int_{2a_2}^{\infty}\frac{(\sigma\phi p_j w)(-t)}{x+t}\,dt\right|\le\left|\int_{2a_2}^{\infty}\frac{(\sigma\phi p_j w)(-t)}{t-a_2}\,dt\right|\\ &= \left|\int_{0}^{\infty}\frac{(\sigma\phi p_j w)(-s-2a_2)}{s+a_2}\,dt\right|\end{aligned}$$

and

$$\begin{aligned}\left|\int_{2a_2}^{\infty}\frac{(\sigma\phi p_j w)(t)}{x-t}\,dt\right| &= \left|\int_{2a_2}^{\infty}\frac{(\sigma\phi p_j w)(t)}{t-x}\,dt\right|\le\left|\int_{2a_2}^{\infty}\frac{(\sigma\phi p_j w)(t)}{t-a_2}\,dt\right|\\ &= \left|\int_{0}^{\infty}\frac{(\sigma\phi p_j w)(s+2a_2)}{s+a_2}\,ds\right|.\end{aligned}$$

So, we can estimate $\int_{-\infty}^{-2a_2}$ and $\int_{2a_2}^{\infty}$ as we did I_1 before (see (4.12)). We can estimate the second integral as follows: By M. Riesz's theorem,

$$\left\|P.V.\int_{-2a_2}^{2a_2}\frac{(\sigma\phi p_j w)(t)}{x-t}\,dt\right\|^p_{L_p(|t|\le 2a_2)}\le C\int_{-2a_2}^{2a_2}\left|(\sigma\phi p_j w)(t)\right|^p dt\le Ca_n^{-\frac{p}{2}}\le C.$$

Now, under the assumption (4.8), we can select $\eta_0>0$ small enough such that

$$\Delta>\begin{cases}0, & 1<p\le 2;\\ \frac{3}{2}\frac{\lambda+\eta_0-1}{3\lambda+2\eta_0-1}\frac{p-2}{p}, & 2<p\le 4;\\ \max\{\frac{\lambda+\eta_0-1}{3\lambda+2\eta_0-1}\frac{p-1}{p}-\frac{1}{4}\frac{\lambda+1}{3\lambda+2\eta_0-1}\frac{p-4}{p},0\}, & 4<p.\end{cases}$$

Consequently, from (4.22) with η_0 we have the result (4.9). □

Let $0<\alpha<1$, then for g_n in Lemma 4.5 we estimate $L_n(g_n)$ over $[-a_{\alpha n},a_{\alpha n}]$.

Lemma 4.7 (*cf.* [3, Lemma 4.4]) *Let* $1 < p < \infty$ *and* $0 < \varepsilon < 1$. *Let* $\{g_n\}$ *be as in Lemma* 4.4, *but we exchange* (4.3) *with*

$$\left|g_n(x)w(x)\right| \le \varepsilon\phi(x), \quad x \in \mathbb{R}, n \ge 1.$$

Then for $1 < p < \infty$,

$$\limsup_{n\to\infty} \left\|L_n(g_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|_{L_p(|x|\le a_{\frac{n}{8}})} \le C\varepsilon.$$

Proof Let

$$\chi_n := \chi_{[-a_{\frac{n}{8}}, a_{\frac{n}{8}}]}; \qquad h_n := \operatorname{sign}\bigl(L_n(g_n)\bigr)\left|L_n(g_n)\right|^{p-1}\chi_n w^{p-2}\Phi^{(\Delta+(\frac{1}{4}-\frac{1}{p})^+)p}$$

and

$$\sigma_n := \operatorname{sign} s_n[h_n].$$

We shall show that

$$\left\|L_n(g_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|_{L_p(|x|\le a_{\frac{n}{8}})} \le \varepsilon\left\|s_n\left[\sigma_n\phi w^{-1}\right]w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|_{L_p(|x|\le a_{\frac{n}{8}})}. \tag{4.23}$$

Then from Lemma 4.5 we will conclude (4.22). Using orthogonality of $f - s_n[f]$ to $\mathcal{P}_{n-1}$, and the Gauss quadrature formula, we see that

$$\begin{aligned}
&\left\|L_n(g_n)w\Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right\|^p_{L_p(|x|\le a_{\frac{n}{8}})} \\
&\quad = \int_{\mathbb{R}} L_n(g_n)(x)h_n(x)w^2(x)\,dx \\
&\quad = \int_{\mathbb{R}} L_n(g_n)(x)s_n[h_n](x)w^2(x)\,dx = \sum_{j=1}^{n} \lambda_{j,n}g_n(x_{j,n})s_n[h_n](x_{j,n}) \\
&\quad = \sum_{|x_{j,n}|\le a_{\frac{n}{8}}} \lambda_{j,n}g_n(x_{j,n})s_n[h_n](x_{j,n}) \quad \bigl(\text{see (4.4), that is, the definition of } g_n\bigr) \\
&\quad \le \varepsilon \sum_{|x_{j,n}|\le a_{\frac{n}{8}}} \lambda_{j,n}w^{-1}(x_{j,n})\phi(x_{j,n})\left|s_n[h_n](x_{j,n})\right|.
\end{aligned}$$

Here, if we use Lemma 4.2 with $\psi = \phi$, we continue as

$$\begin{aligned}
&\le C\varepsilon \int_{\mathbb{R}} \left|s_n[h_n](x)\right|\phi(x)w(x)\,dx \\
&= C\varepsilon \int_{\mathbb{R}} s_n[h_n](x)\sigma_n\phi(x)w^{-1}(x)w^2(x)\,dx = C\varepsilon \int_{\mathbb{R}} h_n(x)s_n\left[\sigma_n\phi w^{-1}\right](x)w^2(x)\,dx \\
&= C\varepsilon \int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} h_n(x)s_n\left[\sigma_n\phi w^{-1}\right](x)w^2(x)\,dx.
\end{aligned}$$

Using Hölder's inequality with $q = p/(p-1)$, we continue this as

$$\begin{aligned}
&\leq C\varepsilon \left(\int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} \left| h_n(x) w(x) \Phi^{-(\Delta+(\frac{1}{4}-\frac{1}{p})^+)}(x) \right|^q dx \right)^{1/q} \left(\int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} \left| s_n\left[\sigma_n \phi w^{-1}\right] w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right|^p dx \right)^{\frac{1}{p}} \\
&= C\varepsilon \left\| L_n(g_n) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(|x|\leq a_{\frac{n}{8}})}^{p-1} \left\| s_n\left[\sigma_n \phi w^{-1} w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\right] \right\|_{L_p(|x|\leq a_{\frac{n}{8}})}.
\end{aligned}$$

Cancellation of $\|L_n(g_n) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+}\|_{L_p(|x|\leq a_{\frac{n}{8}})}^{p-1}$ gives (4.23). □

Proof of Theorem 2.2 In proving the theorem, we split our functions into pieces that vanish inside or outside $[-a_{\frac{n}{9}}, a_{\frac{n}{9}}]$. Throughout, we let χ_S denote the characteristic function of a set S. Also, we set for some fixed $\beta > 0$,

$$\phi(x) = \left(1 + x^2\right)^{-\beta/2},$$

and suppose (2.5). We note that (2.5) means (4.8). Let $0 < \varepsilon < 1$. We can choose a polynomial P such that

$$\left\| (f-P) w \phi^{-1} \right\|_{L_\infty(\mathbb{R})} \leq \varepsilon$$

(see Lemma 3.8). Then we have

$$\begin{aligned}
&\left\| \left(f - L_n(f)\right) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} \\
&\quad \leq \left\| (f-P) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} + \left\| L_n(P-f) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} \\
&\quad \leq \varepsilon \left\| \phi \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} + \left\| L_n(P-f) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} \\
&\quad \leq C\varepsilon + \left\| L_n(P-f) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})}. \qquad (4.24)
\end{aligned}$$

Here we used that

$$\left\| \phi \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(\mathbb{R})} < \infty,$$

because $\Delta > 0$ and Φ^{-1} grows faster than any power of x (see Lemma 3.9). Next, let

$$\chi_n := \chi[-a_{\frac{n}{9}}, a_{\frac{n}{9}}],$$

and write

$$P - f = (P-f)\chi_n + (P-f)(1-\chi_n) =: g_n + f_n.$$

By Lemma 4.4 we have

$$\lim_{n\to\infty} \left\| L_n(f_n) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{4})^+} \right\|_{L_p(\mathbb{R})} = 0.$$

By Lemma 4.5 we have

$$\lim_{n\to\infty} \left\| L_n(g_n) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{4})^+} \right\|_{L_p(|x|\geq a_{\frac{n}{8}})} = 0,$$

and by Lemma 4.7,

$$\limsup_{n\to\infty} \left\| L_n(g_n) w \Phi^{\Delta+(\frac{1}{4}-\frac{1}{p})^+} \right\|_{L_p(|x|\le a_{\frac{n}{8}})} \le C\varepsilon.$$

Here we take $\varepsilon > 0$ as $\varepsilon \to 0$, then with (4.24) we have the result. □

5 Proof of Theorem 2.4

Lemma 5.1 (*cf.* [3, Lemma 3.1]) *Let $w \in \mathcal{F}(C^2+)$. Let $0 < \alpha < \frac{1}{4}$ and*

$$\sum_n(x) := \sum_{|x_{k,n}|\ge a_{\alpha n}} \left|l_{k,n}(x)\right| w^{-1}(x_{k,n}).$$

Then we have for $x \in \mathbb{R}$,

$$\sum_n(x) w(x) \Phi^{1/4}(x) \le C \log n.$$

Proof From Lemma 4.1 and Lemma 3.6 with $p = \infty$, we have the result easily. □

Lemma 5.2 *Let $w \in \mathcal{F}(C^2+)$. Let $0 < \alpha < \frac{1}{4}$ and*

$$\sum_n{}'(x) := \sum_{|x_{k,n}|\le a_{\alpha n}} \left|l_{k,n}(x)\right| w^{-1}(x_{k,n}).$$

Then we have

$$\sum_n{}'(x) w(x) \Phi(x)^{3/4} \le C \log n.$$

Proof By Lemma 3.5(c), Lemma 3.4(d) and Lemma 3.5(b),

$$\begin{aligned}
\sum_n{}'(x) &= \sum_{|x_{k,n}|\le a_{\alpha n}} \left|l_{k,n}(x)\right| w^{-1}(x_{k,n}) \\
&= \frac{|p_n(x)|}{|x - x_{j_x,n}||P_n'(x_{j_x,n})|w(x_{j_x,n})} + \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \frac{|p_n(x)|}{|x - x_{k,n}||P_n'(x_{k,n})|w(x_{k,n})} \\
&\le C w(x)^{-1} + a_n^{1/2}\left|p_n(x)\right| \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \frac{\varphi_n(x_{k,n})(1 - \frac{|x_{k,n}|}{a_n})}{|x - x_{k,n}|} \\
&\sim C w(x)^{-1} + \frac{a_n^{3/2}}{n}\left|p_n(x)\right| \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \frac{1 - \frac{|x_{k,n}|}{a_{2n}}}{\sqrt{1 - \frac{|x_{k,n}|}{a_n}}} \left(1 - \frac{|x_{k,n}|}{a_n}\right)^{1/4} \frac{1}{|x - x_{k,n}|} \\
&\sim C w(x)^{-1} + \frac{a_n^{3/2}}{n}\left|p_n(x)\right| \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \left(1 - \frac{|x_{k,n}|}{a_n}\right)^{3/4} \frac{1}{|x - x_{k,n}|},
\end{aligned}$$

where we used the fact

$$1-\frac{|x_{k,n}|}{a_{2n}} \sim 1-\frac{|x_{k,n}|}{a_n}, \quad |x_{k,n}| \le a_{\alpha n}.$$

So,

$$\begin{aligned}
\sum_n{}'(x) &\le Cw(x)^{-1} + \frac{a_n^{3/2}}{n}\left|p_n(x)\right| \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \left(1-\frac{|x_{k,n}|}{a_n}\right)^{3/4} \frac{1}{|x_{j_x,n}-x_{k,n}|} \\
&\le Cw(x)^{-1} + \frac{a_n^{3/2}}{n}\left|p_n(x)\right| \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \left(1-\frac{|x_{k,n}|}{a_n}\right)^{3/4} \frac{1}{\sum_{j_x\leqq i\leqq k}\varphi_n(x_{i,n})} \\
&\le Cw(x)^{-1} + a_n^{1/2}\left|p_n(x)\right| \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \left(1-\frac{|x_{k,n}|}{a_n}\right)^{3/4} \frac{1}{\sum_{j_x\leqq i\leqq k}\sqrt{1-|x_{i,n}|/a_n}}.
\end{aligned}$$

Therefore we have by Lemma 3.6 with $p=\infty$,

$$\begin{aligned}
\sum_n{}'(x)w(x)\Phi(x)^{3/4} &\le C + Ca_n^{1/2}\left|p_n(x)\right|w(x)\Phi(x)^{1/4} \\
&\quad \times \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \left(1-\frac{|x_{k,n}|}{a_n}\right)^{3/4}\left(1-\frac{|x_{j_x,n}|}{a_n}\right)^{1/2} \frac{1}{\sum_{j_x\leqq i\leqq k}\sqrt{1-|x_{i,n}|/a_n}} \\
&\le C \sum_{\substack{|x_{k,n}|\le a_{\alpha n},\\ k\ne j_x}} \frac{1}{|j_x-k|} \sim \log n.
\end{aligned}$$

□

Lemma 5.3 ([8, Theorem 1]) *Let $w \in \mathcal{F}(C^2+)$. Then there exists a constant $C_0 > 0$ such that for every absolutely continuous function f with $wf' \in C_0(\mathbb{R})$ (this means $w(x)f'(x) \to 0$ as $|x| \to \infty$) and every $n \in \mathbb{N}$, we have*

$$E_n(w;f) \le C\frac{a_n}{n}E_{n-1}\left(w;f'\right).$$

Proof of Theorem 2.4 There exists $P_{n-1} \in \mathcal{P}_n$ such that

$$\left|\left(f(x)-P_{n-1}(x)\right)w(x)\right| \le 2E_{n-1}(w;f).$$

Therefore, by Lemma 5.1 and Lemma 5.2,

$$\begin{aligned}
&\left|\left(f(x)-L_n(f)(x)\right)w(x)\Phi^{3/4}(x)\right| \\
&\quad \le \left|\left(f(x)-P_{n-1}(x)\right)w(x)\Phi^{1/4}(x)\right| + \left|L_n(f-P_{n-1})(x)w(x)\Phi^{3/4}(x)\right| \\
&\quad = \left|\left(f(x)-P_{n-1}(x)\right)w(x)\Phi^{3/4}(x)\right| \\
&\qquad + \left|w(x)\Phi^{3/4}(x)\sum_{k=1}^{n}\left(f(x_{k,n})-P_{n-1}(x_{k,n})\right)w(x_{k,n})l_{k,n}(x)w^{-1}(x_{k,n})\right|
\end{aligned}$$

$$\leq 2E_{n-1}(w;f)\left\{1 + w(x)\Phi^{3/4}(x)\left|\sum_{k=1}^{n} l_{k,n}(x)w^{-1}(x_{k,n})\right|\right\}$$
$$\leq CE_{n-1}(w;f)\log n.$$

Let $wf^{(r)} \in C_0(\mathbb{R})$. If we repeatedly use Lemma 5.3, then we have

$$\left|\left(f(x) - L_n(f)(x)\right)w(x)\Phi^{3/4}(x)\right| \leq C_r\left(\frac{a_n}{n}\right)^r E_{n-r-1}\left(w;f^{(r)}\right)\log n. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript and participated in the sequence alignment. All authors read and approved the final manuscript.

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