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Some new generalized retarded nonlinear integral inequalities with iterated integrals and their applications

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Abstract

Some new generalized retarded nonlinear integral inequalities with iterated integrals are discussed and upper bound estimations of unknown functions are given by analysis technique. These estimations can be used as tools in the study of differential-integral equations with the initial conditions.

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1 Introduction

Gronwall-Bellman inequalities [1, 2] can be used as important tools in the study of existence, uniqueness, boundedness, stability and other qualitative properties of solutions of differential equations and integral equations. There can be found a lot of generalizations of Gronwall-Bellman inequalities in various cases from literature (*e.g.*, [3–15]).

Agarwal *et al.* [5] studied the inequality

$$u(t) \le a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t,s) w_i\big(u(s)\big)\,ds, \quad t_0 \le t < t_1.$$

Agarwal *et al.* [6] obtained the explicit bound to the unknown function of the following retarded integral inequality:

$$\varphi\big(u(t)\big) \le c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)\big[f_i(s)\varphi_1\big(u(s)\big) + g_i(s)\varphi_2\big(\log\big(u(s)\big)\big)\big]\,ds.$$

In 2011, Abdeldaim and Yakout [4] studied the following integral inequalities:

$$u(t) \le u_0 + \int_0^t \big(f(s)u(s) + q(s)\big)\,ds + \int_0^t f(s)u(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\,d\lambda\right] ds,$$

$$u(t) \le u_0 + \int_0^t f(s)u(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\,d\lambda\right]^p ds.$$

However, the bound given on such an inequality in [4] is not directly applicable in the study of certain retarded nonlinear differential and integral equations. It is desirable to establish new inequalities of the above type, which can be used more effectively in the study of certain classes of retarded nonlinear differential and integral equations.

In this paper, we discuss some new retarded nonlinear integral inequalities with iterated integrals

$$u(t) \leq u_0 + \int_0^{\alpha(t)} \big(f(s)u(s) + q(s)\big)\, ds + \int_0^{\alpha(t)} f(s)u(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\, d\lambda\right] ds, \tag{1.1}$$

$$\begin{aligned} u(t) \leq{}& u_0 + \int_0^{\alpha(t)} \big(f(s)\varphi\big(u(s)\big) + q(s)\big)\, ds \\ &+ \int_0^{\alpha(t)} f(s)\varphi\big(u(s)\big)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\, d\lambda\right] ds, \end{aligned} \tag{1.2}$$

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)u(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\, d\lambda\right]^p ds, \tag{1.3}$$

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\phi_1\big(u(s)\big)\left[u(s) + \int_0^s g(\lambda)\phi_2\big(u(\lambda)\big)\, d\lambda\right]^p ds, \tag{1.4}$$

where u_0 is a positive constant, and give upper bound estimation of the unknown function by integral inequality technique. Furthermore, we apply our result to differential-integral equations for estimation.

2 Main result

In this section, we discuss some retarded integral inequalities with iterated integrals. Throughout this paper, let $I = [0,\infty)$.

Lemma 1 (Abdeldaim and Yakout [4]) *We assume that $u(t), f(t)$ and $g(t)$ are nonnegative real-valued continuous functions defined on $I = [0,\infty)$ and satisfy the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\, d\lambda\right]^p ds$$

for all $t \in I$, where u_0 and p are positive constants. Then

$$u(t) \leq u_0 \exp\left(\int_0^t f(s)B_1(s)\right) ds, \quad \forall t \in I,$$

where

$$B_1(t) = \frac{u_0^p \exp(p\int_0^t g(s)\, ds)}{1 - pu_0^p \int_0^t f(s)\exp(p\int_0^s g(\tau)\, d\tau)\, ds},$$

such that $pu_0^p \int_0^t f(s)\exp(p\int_0^s g(\tau)\, d\tau)\, ds \leq 1$ for all $t \in I$.

Lemma 2 (Abdeldaim and Yakout [4]) *We assume that $u(t), f(t)$ and $g(t)$ are nonnegative real-valued continuous functions defined on $I = [0,\infty)$ and satisfy the inequality*

$$u(t) \leq u_0 + \int_0^t \big(f(s)u(s) + q(s)\big)\, ds + \int_0^t f(s)u(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\, d\lambda\right] ds, \tag{2.1}$$

for all $t \in I$, where u_0 is a positive constant. Then

$$u(t) \leq \left(u_0 + \int_0^t q(s) \exp\bigl(-A(s)\bigr)\,ds \right) \exp\bigl(A(t)\bigr), \quad \forall t \in I, \tag{2.2}$$

where $A(t) = \int_0^t (f(s) + f(s)Q_1(s))\,ds$, and

$$Q_1(t) = L(t) - \frac{(L(0) - u_0)\exp(A_1(t))}{1 + (L(0) - u_0)\int_0^t f(s)\exp(A_1(s))\,ds},$$

where $A_1(t) = \int_0^t 2(f(s)L(s) + f(s) + g(s))\,ds$, and $L(t)$ is the maximal solution of the differential equation

$$\frac{dL(t)}{dt} = q(t) + f(t)L^2(t) + \bigl(f(t) + g(t)\bigr)L(t), \quad \forall t \in I,$$

such that $L(0) > u_0$.

Lemma 3 *Suppose that $\varphi(t)$ is a positive and increasing function on I with $\varphi(0) = 0$, $f_i \in C(I, I)$, $i = 1, 2$; $u(t)$ is a nonnegative real-valued continuous function defined on I with $u(0) = u_0 > 0$ and satisfies the inequality*

$$\frac{du(t)}{dt} \leq f_1(t) + f_2(t)\varphi\bigl(u(t)\bigr), \quad \forall t \in I, \tag{2.3}$$

then $u(t)$ has the following estimation:

$$u(t) \leq \Phi^{-1}\left(\Phi\left(u_0 + \int_0^t f_1(s)\,ds \right) + \int_0^t f_2(s)\,ds \right), \quad \forall t \in [0, T_1], \tag{2.4}$$

where

$$\Phi(r) := \int_1^r \frac{ds}{\varphi(s)}, \quad r > 0, \tag{2.5}$$

and T_1 is the largest number such that

$$\Phi\left(u_0 + \int_0^{T_1} f_1(s)\,ds \right) + \int_0^{T_1} f_2(s)\,ds \leq \int_1^{\infty} \frac{ds}{\varphi(s)}. \tag{2.6}$$

Remark 1 $\Phi(t) = \ln(t)$ when $\varphi(t) = t$.

Proof Integrating both sides of (2.3) from 0 to t,

$$\begin{aligned} u(t) &\leq u_0 + \int_0^t f_1(s)\,ds + \int_0^t f_2(s)\varphi\bigl(u(s)\bigr)\,ds \\ &\leq u_0 + \int_0^T f_1(s)\,ds + \int_0^t f_2(s)\varphi\bigl(u(s)\bigr)\,ds, \quad \forall t \in [0, T], \end{aligned} \tag{2.7}$$

where $T \in [0, T_1]$ is a positive constant chosen arbitrarily, T_1 is defined by (2.6). Let

$$R_1(t) = u_0 + \int_0^T f_1(s)\,ds + \int_0^t f_2(s)\varphi\bigl(u(s)\bigr)\,ds, \quad \forall t \in [0, T], \tag{2.8}$$

then $R_1(t)$ is a nonnegative and nondecreasing function on I with $R_1(0) = u_0 + \int_0^T f_1(s)\,ds$. Then (2.7) is equivalent to

$$u(t) \le R_1(t), \quad \forall t \in [0,T]. \tag{2.9}$$

Differentiating $R_1(t)$ with respect to t, from (2.8) and (2.9), we have

$$\frac{dR_1(t)}{dt} = f_2(t)\varphi\bigl(u(t)\bigr) \le f_2(t)\varphi\bigl(R_1(t)\bigr), \quad \forall t \in [0,T]. \tag{2.10}$$

Since $R_1(t) > 0$, from (2.10) we have

$$\frac{dR_1(t)}{\varphi(R_1(t))\,dt} \le f_2(t), \quad \forall t \in [0,T]. \tag{2.11}$$

By taking $t = s$ in (2.11) and integrating it from 0 to t, we get

$$R_1(t) \le \Phi^{-1}\biggl(\Phi\bigl(R_1(0)\bigr) + \int_0^t f_2(s)\,ds\biggr), \quad \forall t \in [0,T], \tag{2.12}$$

where Φ is defined by (2.5). From (2.9) we have

$$u(t) \le \Phi^{-1}\biggl(\Phi\biggl(u_0 + \int_0^T f_1(s)\,ds\biggr) + \int_0^t f_2(s)\,ds\biggr), \quad \forall t \in [0,T]. \tag{2.13}$$

Letting $t = T$, from (2.13) we get

$$u(T) \le \Phi^{-1}\biggl(\Phi\biggl(u_0 + \int_0^T f_1(s)\,ds\biggr) + \int_0^T f_2(s)\,ds\biggr).$$

Because $T \in [0,T_1]$ is chosen arbitrarily, this proves (2.4). □

Lemma 4 *Let $f_i \in C(I,I)$, $i = 1,2,3$; we assume that $u(t)$ is a nonnegative real-valued continuous function defined on I with $u(0) = u_0 > 0$ and satisfies the inequality*

$$\frac{du(t)}{dt} \le f_1(t) + f_2(t)u(t) + f_3(t)u^2(t), \quad \forall t \in I, \tag{2.14}$$

then $u(t)$ has the following estimation:

$$\begin{aligned} u(t) \le{}& \exp\biggl(\int_0^t f_2(s)\,ds\biggr)\biggl(\biggl(u_0 + \int_0^t f_1(s)\,ds\biggr)^{-1} \\ &- \int_0^t f_3(s)\exp\biggl(\int_0^s f_2(\tau)\,d\tau\biggr)\,ds\biggr)^{-1}, \end{aligned} \tag{2.15}$$

for all $t \in [0,T_2]$, where T_2 is the largest number such that

$$\biggl(u_0 + \int_0^t f_1(s)\,ds\biggr)^{-1} - \int_0^t f_3(s)\exp\biggl(\int_0^s f_2(\tau)\,d\tau\biggr)\,ds > 0, \quad \forall t \in [0,T_2]. \tag{2.16}$$

Proof Integrating both sides of (2.14) from 0 to t, we get

$$\begin{aligned} u(t) &\le u_0 + \int_0^t f_1(s)\,ds + \int_0^t f_2(s)u(s)\,ds + \int_0^t f_3(s)u^2(s)\,ds \\ &\le u_0 + \int_0^T f_1(s)\,ds + \int_0^t f_2(s)u(s)\,ds \\ &\quad + \int_0^t f_3(s)u^2(s)\,ds, \quad \forall t \in [0,T], \end{aligned} \tag{2.17}$$

where $T \in [0, T_2]$ is a positive constant chosen arbitrarily, T_2 is defined by (2.16). Let

$$\begin{aligned} R_2(t) &= u_0 + \int_0^T f_1(s)\,ds + \int_0^t f_2(s)u(s)\,ds \\ &\quad + \int_0^t f_3(s)u^2(s)\,ds, \quad \forall t \in [0,T], \end{aligned} \tag{2.18}$$

then $R_2(t)$ is a nonnegative and nondecreasing function on I with $R_2(0) = u_0 + \int_0^T f_1(s)\,ds$. Then (2.17) is equivalent to

$$u(t) \le R_2(t), \quad \forall t \in [0,T]. \tag{2.19}$$

Differentiating $R_2(t)$ with respect to t, from (2.18) and (2.19), we have

$$\frac{dR_2(t)}{dt} = f_2(t)u(t) + f_3(t)u^2(t) \le f_2(t)R_2(t) + f_3(t)R_2^2(t), \quad \forall t \in [0,T]. \tag{2.20}$$

Since $R_2(t) > 0$, from (2.20) we have

$$R_2^{-2}(t)\frac{dR_2(t)}{dt} \le f_2(t)R_2^{-1}(t) + f_3(t), \quad \forall t \in [0,T]. \tag{2.21}$$

Let $S_1(t) = R_2^{-1}(t)$, then $S_1(0) = (u_0 + \int_0^T f_1(s)\,ds)^{-1}$, from (2.21) we obtain

$$\frac{dS_1(t)}{dt} + f_2(t)S_1(t) \ge -f_3(t), \quad \forall t \in [0,T]. \tag{2.22}$$

Consider the ordinary differential equation

$$\begin{cases} \frac{dS_2(t)}{dt} + f_2(t)S_2(t) = -f_3(t), \quad \forall t \in [0,T], \\ S_2(0) = (u_0 + \int_0^T f_1(s)\,ds)^{-1}. \end{cases} \tag{2.23}$$

The solution of equation (2.23) is

$$\begin{aligned} S_2(t) &= \exp\left(-\int_0^t f_2(s)\,ds\right)\left(\left(u_0 + \int_0^T f_1(s)\,ds\right)^{-1} \right. \\ &\quad \left. - \int_0^t f_3(s)\exp\left(\int_0^s f_2(\tau)\,d\tau\right)ds\right) \end{aligned} \tag{2.24}$$

for all $t \in [0, T]$. Letting $t = T$ in (2.24), from (2.19), (2.22), (2.23) and (2.24), we obtain

$$\begin{aligned} u(T) &\leq R_2(T) = \frac{1}{S_1(T)} \leq \frac{1}{S_2(T)} \\ &\leq \exp\left(\int_0^T f_2(s)\,ds\right)\left(\left(u_0 + \int_0^T f_1(s)\,ds\right)^{-1} \right. \\ &\quad \left. - \int_0^T f_3(s)\exp\left(\int_0^s f_2(\tau)\,d\tau\right)ds\right)^{-1}. \end{aligned} \tag{2.25}$$

Because $T \in [0, T_2]$ is chosen arbitrarily, this proves (2.15). □

Lemma 5 *Suppose that $\varphi(t)$, $\varphi(t)/t$ are positive and increasing functions on I, $f_i \in C(I, I)$, $i = 1, 2, 3, 4$; $u(t)$ is a nonnegative real-valued continuous function defined on I with $u(0) = u_0 > 0$ and satisfies the inequality*

$$\begin{aligned} &\frac{du(t)}{dt} \leq f_1(t) + f_2(t)u(t) + f_3(t)u(t)\varphi\big(u(t)\big) + f_4(t)\varphi\big(u(t)\big), \\ &u(0) = u_0, \quad \forall t \in I, \end{aligned} \tag{2.26}$$

then $u(t)$ has the following estimation:

$$\begin{aligned} u(t) &\leq \exp\left(\Phi_1^{-1}\left(\Phi_1\left(\ln\left(u_0 + \int_0^t f_1(s)\,ds\right) + \int_0^t f_2(s)\,ds\right) + \int_0^t f_3(s)\,ds \right.\right. \\ &\quad \left.\left. + \int_0^t \frac{f_4(s)\,ds}{\exp(\ln(u_0 + \int_0^t f_1(\tau)\,d\tau) + \int_0^t f_2(\tau)\,d\tau)}\right)\right), \end{aligned} \tag{2.27}$$

for all $t \in [0, T_3]$, where

$$\Phi_1(r) := \int_1^r \frac{ds}{\varphi(\exp(s))}, \quad r > 0, \tag{2.28}$$

and T_3 is the largest number such that

$$\begin{aligned} &\Phi_1\left(\ln\left(u_0 + \int_0^T f_1(s)\,ds\right) + \int_0^T f_2(s)\,ds\right) + \int_0^T f_3(s)\,ds \\ &\quad + \int_0^T \frac{f_4(s)\,ds}{\exp(\ln(u_0 + \int_0^T f_1(\tau)\,d\tau) + \int_0^T f_2(\tau)\,d\tau)} \leq \int_1^\infty \frac{ds}{\varphi(\exp(s))}. \end{aligned} \tag{2.29}$$

Proof Integrating both sides of (2.26) from 0 to t, we get

$$\begin{aligned} u(t) &\leq u_0 + \int_0^t f_1(s)\,ds + \int_0^t f_2(s)u(s)\,ds + \int_0^t f_3(s)u(s)\varphi\big(u(s)\big)\,ds + \int_0^t f_4(s)\varphi\big(u(s)\big)\,ds \\ &\leq u_0 + \int_0^T f_1(s)\,ds + \int_0^t f_2(s)u(s)\,ds \\ &\quad + \int_0^t f_3(s)u(s)\varphi\big(u(s)\big)\,ds + \int_0^t f_4(s)\varphi\big(u(s)\big)\,ds, \end{aligned} \tag{2.30}$$

for all $t \in [0, T]$, where $T \in [0, T_3]$ is a positive constant chosen arbitrarily, T_3 is defined by (2.29). Let $R_3(t)$ denote the function on the right-hand side of (2.30), which is a positive and nondecreasing function on $[0, T]$ with

$$R_3(0) = u_0 + \int_0^T f_1(s)\,ds. \tag{2.31}$$

Then (2.26) is equivalent to

$$u(t) \leq R_3(t), \quad \forall t \in [0, T]. \tag{2.32}$$

Differentiating $R_3(t)$ with respect to t and using (2.32), we have

$$\frac{dR_3(t)}{dt} \leq f_2(t)R_3(t) + f_3(t)R_3(t)\varphi\bigl(R_3(t)\bigr) + f_4(t)\varphi\bigl(R_3(t)\bigr), \quad \forall t \in [0, T]. \tag{2.33}$$

From (2.33) we get

$$R_3^{-1}(t)\frac{dR_3(t)}{dt} \leq f_2(t) + f_3(t)\varphi\bigl(R_3(t)\bigr) + f_4(t)\varphi\bigl(R_3(t)\bigr)R_3^{-1}(t), \quad \forall t \in [0, T]. \tag{2.34}$$

Integrating both sides of (2.34) from 0 to t and using (2.31), we get

$$\begin{aligned} R_3(t) &\leq \exp\left(\ln R_3(0) + \int_0^t f_2(s)\,ds + \int_0^t f_3(s)\varphi\bigl(R_3(s)\bigr)\,ds + \int_0^t f_4(s)\varphi\bigl(R_3(s)\bigr)R_3^{-1}(s)\,ds\right) \\ &\leq \exp\left(\ln\left(u_0 + \int_0^T f_1(s)\,ds\right) + \int_0^T f_2(s)\,ds + \int_0^t f_3(s)\varphi\bigl(R_3(s)\bigr)\,ds \right. \\ &\quad \left. + \int_0^t f_4(s)\varphi\bigl(R_3(s)\bigr)R_3^{-1}(s)\,ds\right), \quad \forall t \in [0, T], \end{aligned} \tag{2.35}$$

here we use the monotonicity of $\varphi(t)$ and $\varphi(t)/t$. Let

$$\begin{aligned} R_4(t) &= \ln\left(u_0 + \int_0^T f_1(s)\,ds\right) + \int_0^T f_2(s)\,ds + \int_0^t f_3(s)\varphi\bigl(R_3(s)\bigr)\,ds \\ &\quad + \int_0^t f_4(s)\varphi\bigl(R_3(s)\bigr)R_3^{-1}(s)\,ds \end{aligned} \tag{2.36}$$

for all $t \in [0, T]$, then $R_4(t)$ is a positive and nondecreasing function on $[0, T]$ with

$$R_4(0) = \ln\left(u_0 + \int_0^T f_1(s)\,ds\right) + \int_0^T f_2(s)\,ds. \tag{2.37}$$

(2.36) is equivalent to

$$R_3(t) \leq \exp\bigl(R_4(t)\bigr), \quad \forall t \in [0, T]. \tag{2.38}$$

Differentiating $R_4(t)$ with respect to t and using (2.38), we have

$$\begin{aligned} \frac{dR_4(t)}{dt} &= f_3(t)\varphi\bigl(R_3(t)\bigr) + f_4(t)\varphi\bigl(R_3(t)\bigr)R_3^{-1}(t) \\ &\leq f_3(t)\varphi\bigl(\exp\bigl(R_4(t)\bigr)\bigr) + f_4(t)\varphi\bigl(\exp\bigl(R_4(t)\bigr)\bigr)\bigl(\exp\bigl(R_4(t)\bigr)\bigr)^{-1}, \quad \forall t \in [0, T]. \end{aligned} \tag{2.39}$$

From (2.39) we get

$$\varphi^{-1}\bigl(\exp\bigl(R_4(t)\bigr)\bigr)\frac{dR_4(t)}{dt} \le f_3(t) + f_4(t)\bigl(\exp\bigl(R_4(t)\bigr)\bigr)^{-1}, \quad \forall t \in [0,T]. \tag{2.40}$$

Integrating both sides of (2.40) from 0 to t,

$$\begin{aligned}\Phi_1\bigl(R_4(t)\bigr) &\le \Phi_1\bigl(R_4(0)\bigr) + \int_0^t f_3(s)\,ds + \int_0^t f_4(s)\bigl(\exp\bigl(R_4(0)\bigr)\bigr)^{-1}\,ds \\ &\le \Phi_1\left(\ln\left(u_0 + \int_0^T f_1(s)\,ds\right) + \int_0^T f_2(s)\,ds\right) + \int_0^t f_3(s)\,ds \\ &\quad + \int_0^t \frac{f_4(s)\,ds}{\exp(\ln(u_0 + \int_0^T f_1(\tau)\,d\tau) + \int_0^T f_2(\tau)\,d\tau)}\end{aligned} \tag{2.41}$$

for all $t \in [0,T]$, where Φ_1 is defined by (2.28). From (2.32), (2.38) and (2.41), we have

$$\begin{aligned}u(t) &\le R_3(t) \le \exp\bigl(R_4(t)\bigr) \\ &\le \exp\Biggl(\Phi_1^{-1}\Biggl(\Phi_1\left(\ln\left(u_0 + \int_0^T f_1(s)\,ds\right) + \int_0^T f_2(s)\,ds\right) + \int_0^t f_3(s)\,ds \\ &\quad + \int_0^t \frac{f_4(s)\,ds}{\exp(\ln(u_0 + \int_0^T f_1(\tau)\,d\tau) + \int_0^T f_2(\tau)\,d\tau)}\Biggr)\Biggr),\end{aligned} \tag{2.42}$$

for all $t \in [0,T]$. Letting $t = T$, from (2.42) we get

$$\begin{aligned}u(T) &\le \exp\Biggl(\Phi_1^{-1}\Biggl(\Phi_1\left(\ln\left(u_0 + \int_0^T f_1(s)\,ds\right) + \int_0^T f_2(s)\,ds\right) + \int_0^T f_3(s)\,ds \\ &\quad + \int_0^T \frac{f_4(s)\,ds}{\exp(\ln(u_0 + \int_0^T f_1(\tau)\,d\tau) + \int_0^T f_2(\tau)\,d\tau)}\Biggr)\Biggr).\end{aligned}$$

Because $T \in [0,T_3]$ is chosen arbitrarily, this gives the estimation (2.27) of the unknown function in the inequality (2.26). □

Theorem 1 *Suppose that* $\alpha \in C^1(I,I)$ *is an increasing function with* $\alpha(t) \le t$, $\alpha(0) = 0$, $\forall t \in I$; $u(t), f(t), q(t)$ *and* $g(t)$ *are nonnegative real-valued continuous functions defined on* $I = [0,\infty)$ *and satisfy the inequality* (1.1). *Then*

$$u(t) \le \exp\left(\ln\left(u_0 + \int_0^{\alpha(t)} q(s)\,ds\right) + \int_0^{\alpha(t)} \bigl(f(s) + f(s)L_1\bigl(\alpha^{-1}(s)\bigr)\bigr)\,ds\right) \tag{2.43}$$

for all $t \in [0,T_4]$, *where*

$$\begin{aligned}L_1(t) = \exp&\left(\int_0^{\alpha(t)} \bigl(f(s) + g(s)\bigr)\,ds\right)\Biggl(\left(u_0 + \int_0^{\alpha(t)} q(s)\,ds\right)^{-1} \\ &- \int_0^{\alpha(t)} f(s)\exp\left(\int_0^s \bigl(f(\tau) + g(\tau)\bigr)\,d\tau\right)ds\Biggr)^{-1},\end{aligned} \tag{2.44}$$

T_4 is the largest number such that

$$\left(u_0 + \int_0^{\alpha(t)} q(s)\,ds\right)^{-1} - \int_0^{\alpha(t)} f(s)\exp\left(\int_0^s \bigl(f(\tau) + g(\tau)\bigr)\,d\tau\right)ds > 0.$$

Remark 2 Theorem 1 gives the explicit estimation (2.43) for the inequality (1.1) which is just the inequality (2.1) when $\alpha(t) = t$. Lemma 1 gives the implicit estimation (2.2) for the inequality (2.1).

Proof Let $z_1(t)$ denote the function on the right-hand side of (1.1), which is a positive and nondecreasing function on I with $z_1(0) = u_0$. Then (1.1) is equivalent to

$$u(t) \le z_1(t), \qquad u\bigl(\alpha(t)\bigr) \le z_1\bigl(\alpha(t)\bigr) \le z_1(t), \quad \forall t \in I. \tag{2.45}$$

Differentiating $z_1(t)$ with respect to t and using (2.45), we have

$$\begin{aligned}\frac{dz_1(t)}{dt} &= \alpha'(t)f\bigl(\alpha(t)\bigr)u\bigl(\alpha(t)\bigr) + \alpha'(t)q\bigl(\alpha(t)\bigr) \\ &\quad + \alpha'(t)f\bigl(\alpha(t)\bigr)u\bigl(\alpha(t)\bigr)\left[u\bigl(\alpha(t)\bigr) + \int_0^{\alpha(t)} g(\lambda)u(\lambda)\,d\lambda\right] \\ &\le \alpha'(t)q\bigl(\alpha(t)\bigr) + \bigl[\alpha'(t)f\bigl(\alpha(t)\bigr) + \alpha'(t)f\bigl(\alpha(t)\bigr)Y_1(t)\bigr]z_1(t), \quad \forall t \in I,\end{aligned} \tag{2.46}$$

where

$$Y_1(t) := z_1(t) + \int_0^{\alpha(t)} g(\lambda)z_1(\lambda)\,d\lambda, \quad \forall t \in I. \tag{2.47}$$

Then $Y_1(t)$ is a positive and nondecreasing function on I with $Y_1(0) = z_1(0) = u_0$ and

$$z_1(t) \le Y_1(t). \tag{2.48}$$

Differentiating $Y_1(t)$ with respect to t and using (2.46), (2.47) and (2.48), we get

$$\begin{aligned}\frac{dY_1(t)}{dt} &\le \alpha'(t)q\bigl(\alpha(t)\bigr) + \bigl[\alpha'(t)f\bigl(\alpha(t)\bigr) + \alpha'(t)f\bigl(\alpha(t)\bigr)Y_1(t)\bigr]z_1(t) + \alpha'(t)g\bigl(\alpha(t)\bigr)z_1(t) \\ &\le \alpha'(t)q\bigl(\alpha(t)\bigr) + \bigl[\alpha'(t)f\bigl(\alpha(t)\bigr) + \alpha'(t)g\bigl(\alpha(t)\bigr)\bigr]Y_1(t) + \alpha'(t)f\bigl(\alpha(t)\bigr)Y_1^2(t)\end{aligned} \tag{2.49}$$

for all $t \in I$. Applying Lemma 4 to (2.49), we obtain

$$Y_1(t) \le L_1(t), \quad \forall t \in [0, T_4]. \tag{2.50}$$

From (2.46) and (2.50), we get

$$\frac{dz_1(t)}{dt} \le \alpha'(t)q\bigl(\alpha(t)\bigr) + \bigl[\alpha'(t)f\bigl(\alpha(t)\bigr) + \alpha'(t)f\bigl(\alpha(t)\bigr)L_1(t)\bigr]z_1(t), \quad \forall t \in [0, T_4]. \tag{2.51}$$

Applying Lemma 3 to (2.51) and using Remark 1, we obtain

$$z_1(t) \le \exp\left(\ln\left(u_0 + \int_0^{\alpha(t)} q(s)\,ds\right) + \int_0^{\alpha(t)} \bigl(f(s) + f(s)L_1\bigl(\alpha^{-1}(s)\bigr)\bigr)\,ds\right), \quad \forall t \in [0, T_4].$$

From (2.45), the estimation (2.43) of the unknown function in the inequality (1.1) is obtained. □

Theorem 2 *Suppose $\varphi \in C(I,I)$ and $\alpha \in C^1(I,I)$ are increasing functions with $\alpha(t) \le t$, $\alpha(0) = 0$, $\forall t \in I$. We assume that $u(t), f(t)$, $q(t)$ and $g(t)$ are nonnegative real-valued continuous functions defined on I and satisfy the inequality* (1.2). *Then*

$$u(t) \le \Phi^{-1}\biggl(\Phi\biggl(u_0 + \int_0^{\alpha(t)} q(s)\,ds\biggr) + \int_0^{\alpha(t)} \bigl(f(s) + f(s)L_2\bigl(\alpha^{-1}(s)\bigr)\bigr)\,ds\biggr), \quad \forall t \in [0, T_4], \tag{2.52}$$

where Φ is defined by (2.5),

$$L_2(t) = \exp\biggl(\Phi_1^{-1}\biggl(\Phi_1\biggl(\ln\biggl(u_0 + \int_0^{\alpha(t)} q(s)\,ds\biggr) + \int_0^{\alpha(t)} g(s)\,ds\biggr) + \int_0^{\alpha(t)} f(s)\,ds + \int_0^{\alpha(t)} \frac{f(s)\,ds}{\exp(\ln(u_0 + \int_0^{\alpha(t)} q(s)\,ds) + \int_0^{\alpha(t)} g(s)\,ds)}\biggr)\biggr), \quad \forall t \in [0, T_4], \tag{2.53}$$

Φ_1 is defined by (2.28), *T_4 is the largest number such that*

$$\begin{aligned}
&\Phi\biggl(u_0 + \int_0^{\alpha(t)} q(s)\,ds\biggr) + \int_0^{\alpha(t)} \bigl(f(s) + f(s)L_2\bigl(\alpha^{-1}(s)\bigr)\bigr)\,ds \le \int_1^{\infty} \frac{ds}{\varphi(s)}, \\
&\Phi_1\biggl(\ln\biggl(u_0 + \int_0^{\alpha(T_4)} q(s)\,ds\biggr) + \int_0^{\alpha(T_4)} g(s)\,ds\biggr) + \int_0^{\alpha(T_4)} f(s)\,ds \\
&\quad + \int_0^{\alpha(T_4)} \frac{f(s)\,ds}{\exp(\ln(u_0 + \int_0^{\alpha(T_4)} q(s)\,ds) + \int_0^{\alpha(T_4)} g(s)\,ds)} \le \int_1^{\infty} \frac{ds}{\varphi(\exp(s))}.
\end{aligned} \tag{2.54}$$

Proof Let $z_2(t)$ denote the function on the right-hand side of (1.2), which is a positive and nondecreasing function on I with $z_2(0) = u_0$. Then (1.2) is equivalent to

$$u(t) \le z_2(t), \qquad u\bigl(\alpha(t)\bigr) \le z_2\bigl(\alpha(t)\bigr) \le z_2(t), \quad \forall t \in I. \tag{2.55}$$

Differentiating $z_2(t)$ with respect to t and using (2.55), we have

$$\begin{aligned}
\frac{dz_2(t)}{dt} &= \alpha'(t)f\bigl(\alpha(t)\bigr)\varphi\bigl(u\bigl(\alpha(t)\bigr)\bigr) + \alpha'(t)q\bigl(\alpha(t)\bigr) \\
&\quad + \alpha'(t)f\bigl(\alpha(t)\bigr)\varphi\bigl(u\bigl(\alpha(t)\bigr)\bigr)\biggl[u\bigl(\alpha(t)\bigr) + \int_0^{\alpha(t)} g(\lambda)u(\lambda)\,d\lambda\biggr] \\
&\le \alpha'(t)q\bigl(\alpha(t)\bigr) + \bigl[\alpha'(t)f\bigl(\alpha(t)\bigr) + \alpha'(t)f\bigl(\alpha(t)\bigr)Y_2(t)\bigr]\varphi\bigl(z_2(t)\bigr), \quad \forall t \in I,
\end{aligned} \tag{2.56}$$

where

$$Y_2(t) := z_2(t) + \int_0^{\alpha(t)} g(\lambda)z_2(\lambda)\,d\lambda, \quad \forall t \in I. \tag{2.57}$$

Then $Y_2(t)$ is a positive and nondecreasing function on I with $Y_2(0) = z_2(0) = u_0$ and

$$z_2(t) \le Y_2(t). \tag{2.58}$$

Differentiating $Y_2(t)$ with respect to t and using (2.56), (2.57) and (2.58), we get

$$\begin{aligned} \frac{dY_2(t)}{dt} &\le \alpha'(t)q\big(\alpha(t)\big) + \big[\alpha'(t)f\big(\alpha(t)\big) + \alpha'(t)f\big(\alpha(t)\big)Y_2(t)\big]\varphi\big(z_2(t)\big) + \alpha'(t)g\big(\alpha(t)\big)z_2(t) \\ &\le \alpha'(t)q\big(\alpha(t)\big) + \alpha'(t)g\big(\alpha(t)\big)Y_2(t) + \alpha'(t)f\big(\alpha(t)\big)Y_2(t)\varphi\big(Y_2(t)\big) \\ &\quad + \alpha'(t)f\big(\alpha(t)\big)\varphi\big(Y_2(t)\big) \end{aligned} \tag{2.59}$$

for all $t \in I$. Applying Lemma 5 to (2.59), we obtain

$$\begin{aligned} Y_2(t) &\le \exp\left(\Phi_1^{-1}\left(\Phi_1\left(\ln\left(u_0 + \int_0^{\alpha(t)} q(s)\,ds\right) + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} f(s)\,ds\right.\right. \\ &\quad \left.\left. + \int_0^{\alpha(t)} \frac{f(s)\,ds}{\exp(\ln(u_0 + \int_0^{\alpha(t)} q(s)\,ds) + \int_0^{\alpha(t)} g(s)\,ds)}\right)\right) \\ &= L_2(t), \quad \forall t \in [0, T_4], \end{aligned} \tag{2.60}$$

where T_4 and L_2 are defined by (2.53) and (2.54) respectively. From (2.56) and (2.60), we get

$$\begin{aligned} \frac{dz_2(t)}{dt} &\le \alpha'(t)q\big(\alpha(t)\big) + \big[\alpha'(t)f\big(\alpha(t)\big) \\ &\quad + \alpha'(t)f\big(\alpha(t)\big)L_2(t)\big]\varphi\big(z_2(t)\big), \quad \forall t \in [0, T_4]. \end{aligned} \tag{2.61}$$

Applying Lemma 3 to (2.61), we obtain

$$z_2(t) \le \Phi^{-1}\left(\Phi\left(u_0 + \int_0^{\alpha(t)} q(s)\,ds\right) + \int_0^{\alpha(t)} \big(f(s) + f(s)L_2\big(\alpha^{-1}(s)\big)\big)\,ds\right), \quad \forall t \in [0, T_4],$$

where Φ is defined by (2.5). From (2.55), the estimation (2.52) of the unknown function in the inequality (1.2) is obtained. □

Theorem 3 *Suppose $\alpha \in C^1(I, I)$ are increasing functions with $\alpha(t) \le t$, $\alpha(0) = 0$, $\forall t \in I$. We assume that $u(t)$, $f(t)$ and $g(t)$ are nonnegative real-valued continuous functions defined on I and satisfy the inequality* (1.3). *Then*

$$u(t) \le u_0 \exp\left(\int_0^t f(s)B_2(s)\right) ds, \quad \forall t \in I, \tag{2.62}$$

where

$$B_2(t) = \frac{u_0^p \exp(p \int_0^{\alpha(t)} g(s)\,ds)}{1 - pu_0^p \int_0^{\alpha(t)} f(s) \exp(p \int_0^s g(\tau)\,d\tau)\,ds}, \tag{2.63}$$

such that $pu_0^p \int_0^{\alpha(t)} f(s) \exp(p \int_0^s g(\tau)\,d\tau)\,ds \le 1$ for all $t \in I$.

Remark 3 If $\alpha(t) = t$, then Theorem 3 reduces Lemma 1.

Proof Let $z_3(t)$ denote the function on the right-hand side of (1.3), which is a positive and nondecreasing function on I with $z_3(0) = u_0$. Then (1.3) is equivalent to

$$u(t) \le z_3(t), \qquad u\bigl(\alpha(t)\bigr) \le z_3\bigl(\alpha(t)\bigr) \le z_3(t), \quad \forall t \in I. \tag{2.64}$$

Differentiating $z_3(t)$ with respect to t and using (2.64), we have

$$\begin{aligned} \frac{dz_3(t)}{dt} &= \alpha'(t)f\bigl(\alpha(t)\bigr)u\bigl(\alpha(t)\bigr)\biggl[u\bigl(\alpha(t)\bigr) + \int_0^{\alpha(t)} g(\lambda)u(\lambda)\,d\lambda\biggr]^p \\ &\le \alpha'(t)f\bigl(\alpha(t)\bigr)z_3(t)\biggl[z_3(t) + \int_0^{\alpha(t)} g(\lambda)z_3(\lambda)\,d\lambda\biggr]^p \\ &= \alpha'(t)f\bigl(\alpha(t)\bigr)z_3(t)Y_3^p(t), \quad \forall t \in I, \end{aligned} \tag{2.65}$$

where

$$Y_3(t) := z_3(t) + \int_0^{\alpha(t)} g(\lambda)z_3(\lambda)\,d\lambda, \quad \forall t \in I. \tag{2.66}$$

Then $Y_3(t)$ is a positive and nondecreasing function on I with $Y_3(0) = z_3(0) = u_0$ and

$$z_3(t) \le Y_3(t). \tag{2.67}$$

Differentiating $Y_3(t)$ with respect to t and using (2.65), (2.66) and (2.67), we get

$$\begin{aligned} \frac{dY_3(t)}{dt} &\le \alpha'(t)f\bigl(\alpha(t)\bigr)z_3(t)Y_3^p(t) + \alpha'(t)g\bigl(\alpha(t)\bigr)z_3(t) \\ &\le \alpha'(t)f\bigl(\alpha(t)\bigr)Y_3^{1+p}(t) + \alpha'(t)g\bigl(\alpha(t)\bigr)Y_3(t), \quad \forall t \in I. \end{aligned} \tag{2.68}$$

From (2.68) we have

$$Y_3^{-(1+p)}(t)\frac{dY_3(t)}{dt} - \alpha'(t)g\bigl(\alpha(t)\bigr)Y_3^{-p}(t) \le \alpha'(t)f\bigl(\alpha(t)\bigr), \quad \forall t \in I. \tag{2.69}$$

Let $S_3(t) = Y_3^{-p}(t)$, then $S_3(0) = u_0^{-p}$. From (2.69) we obtain

$$\frac{dS_3(t)}{dt} + p\alpha'(t)g\bigl(\alpha(t)\bigr)S_3(t) \le -p\alpha'(t)f\bigl(\alpha(t)\bigr), \quad \forall t \in I. \tag{2.70}$$

Consider the ordinary differential equation

$$\begin{cases} \frac{dS_4(t)}{dt} + p\alpha'(t)g(\alpha(t))S_4(t) = -p\alpha'(t)f(\alpha(t)), \quad \forall t \in I, \\ S_4(0) = u_0^{-p}. \end{cases} \tag{2.71}$$

The solution of equation (2.71) is

$$S_4(t) = \exp\biggl(-\int_0^{\alpha(t)} pg(s)\,ds\biggr)\biggl(u_0^{-p} - \int_0^{\alpha(t)} pf(s)\exp\biggl(\int_0^s pg(\tau)\,d\tau\biggr)\,ds\biggr) \tag{2.72}$$

for all $t \in I$. By (2.70), (2.71) and (2.72), we obtain

$$Y_3^p(t) = S_3^{-1}(t) \le S_4^{-1}(t) = B_2(t), \quad \forall t \in I, \tag{2.73}$$

where $B_2(t)$ is as defined in (2.63). From (2.65) and (2.73), we have

$$\frac{dz_3(t)}{dt} \le \alpha'(t)f\big(\alpha(t)\big)z_3(t)B_2(t), \quad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating it from 0 to t, from (2.64) we get

$$u(t) \le z_3(t) \le u_0 \exp\bigg(\int_0^{\alpha(t)} f(s)B_2(s)\bigg)\, ds, \quad \forall t \in I.$$

The estimation (2.62) of the unknown function in the inequality (1.3) is obtained. □

Theorem 4 *Suppose $\varphi_1, \varphi_2, \alpha \in C^1(I, I)$ are increasing functions with $\varphi_i(t) > 0$, $\alpha(t) \le t$, $\forall t > 0$, $i = 1, 2$, $\alpha(0) = 0$. We assume that $u(t)$, $f(t)$ and $g(t)$ are nonnegative real-valued continuous functions defined on $I = [0, \infty)$ and satisfy the inequality* (1.4). *Then*

$$u(t) \le \Phi_2^{-1}\bigg[\Phi_3^{-1}\bigg(\Phi_3\bigg(\Phi_2(u_0) + \int_0^{\alpha(t)} g(s)\, ds\bigg) + \int_0^{\alpha(t)} f(s)\, ds\bigg)\bigg], \quad \forall t < T_5, \tag{2.74}$$

where

$$\Phi_2(r) := \int_1^r \frac{dt}{\varphi_2(t)}, \quad r > 0, \tag{2.75}$$

$$\Phi_3(r) := \int_1^r \frac{\varphi_2(\Phi_2^{-1}(s))\, ds}{\varphi_1(\Phi_2^{-1}(s))(\Phi_2^{-1}(s))^p}, \quad r > 0, \tag{2.76}$$

and T_5 is the largest number such that

$$\Phi_3\bigg(\Phi_2(u_0) + \int_0^{\alpha(t)} g(s)\, ds\bigg) + \int_0^{\alpha(t)} f(s)\, ds \le \int_1^{\infty} \frac{\varphi_2(\Phi_2^{-1}(s))\, ds}{\varphi_1(\Phi_2^{-1}(s))(\Phi_2^{-1}(s))^p},$$

$$\Phi_3^{-1}\bigg(\Phi_3\bigg(\Phi_2(u_0) + \int_0^{\alpha(t)} g(s)\, ds\bigg) + \int_0^{\alpha(t)} f(s)\, ds\bigg) \le \int_1^{\infty} \frac{dt}{\varphi_2(t)}.$$

Proof Let $z_4(t)$ denote the function on the right-hand side of (1.4), which is a positive and nondecreasing function on I with $z_4(0) = u_0$. Then (1.4) is equivalent to

$$u(t) \le z_4(t), \qquad u\big(\alpha(t)\big) \le z_4\big(\alpha(t)\big) \le z_4(t), \quad \forall t \in I. \tag{2.77}$$

Differentiating $z_4(t)$ with respect to t and using (2.77), we have

$$\begin{aligned}
\frac{dz_4(t)}{dt} &= \alpha'(t)f\big(\alpha(t)\big)\varphi_1\big(u\big(\alpha(t)\big)\big)\bigg[u\big(\alpha(t)\big) + \int_0^{\alpha(t)} g(\lambda)\varphi_2\big(u(\lambda)\big)\, d\lambda\bigg]^p \\
&\le \alpha'(t)f\big(\alpha(t)\big)\varphi_1\big(z_4(t)\big)\bigg[z_4(t) + \int_0^{\alpha(t)} g(\lambda)\varphi_2\big(z_4(\lambda)\big)\, d\lambda\bigg]^p \\
&= \alpha'(t)f\big(\alpha(t)\big)\varphi_1\big(z_4(t)\big)Y_4^p(t), \quad \forall t \in I,
\end{aligned} \tag{2.78}$$

where

$$Y_4(t) := z_4(t) + \int_0^{\alpha(t)} g(\lambda)\varphi_2\bigl(z_4(\lambda)\bigr)\,d\lambda, \quad \forall t \in I. \tag{2.79}$$

Then $Y_4(t)$ is a positive and nondecreasing function on I with $Y_4(0) = z_4(0) = u_0$ and

$$z_4(t) \le Y_4(t). \tag{2.80}$$

Differentiating $Y_4(t)$ with respect to t and using (2.78), (2.79) and (2.80), we get

$$\begin{aligned}\frac{dY_4(t)}{dt} &\le \alpha'(t)f\bigl(\alpha(t)\bigr)\varphi_1\bigl(z_4(t)\bigr)Y_4^p(t) + \alpha'(t)g\bigl(\alpha(t)\bigr)\varphi_2\bigl(z_4(t)\bigr)\\ &\le \alpha'(t)f\bigl(\alpha(t)\bigr)\varphi_1\bigl(Y_4(t)\bigr)Y_4^p(t) + \alpha'(t)g\bigl(\alpha(t)\bigr)\varphi_2\bigl(Y_4(t)\bigr), \quad \forall t \in I.\end{aligned} \tag{2.81}$$

Since $\varphi_2(Y_4(t)) > 0$, $\forall t > 0$, from (2.81) we have

$$\frac{dY_4(t)}{\varphi_2(Y_4(t))} \le \alpha'(t)f\bigl(\alpha(t)\bigr)\frac{\varphi_1(Y_4(t))Y_4^p(t)}{\varphi_2(Y_4(t))}\,dt + \alpha'(t)g\bigl(\alpha(t)\bigr)\,dt, \quad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating it from 0 to t, we get

$$\begin{aligned}\Phi_2\bigl(Y_4(t)\bigr) \le \Phi_2\bigl(Y_4(0)\bigr) &+ \int_0^t \alpha'(s)f\bigl(\alpha(s)\bigr)\frac{\varphi_1(Y_4(s))Y_4^p(s)}{\varphi_2(Y_4(s))}\,ds\\ &+ \int_0^t \alpha'(s)g\bigl(\alpha(s)\bigr)\,ds,\end{aligned} \tag{2.82}$$

for all $t \in I$, where Φ_2 is defined by (2.75). From (2.82) we have

$$\begin{aligned}\Phi_2\bigl(Y_4(t)\bigr) \le \Phi_2\bigl(Y_4(0)\bigr) &+ \int_0^T \alpha'(s)g\bigl(\alpha(s)\bigr)\,ds\\ &+ \int_0^t \alpha'(s)f\bigl(\alpha(s)\bigr)\frac{\varphi_1(Y_4(s))Y_4^p(s)}{\varphi_2(Y_4(s))}\,ds\end{aligned} \tag{2.83}$$

for all $t < T$, where $0 < T < T_5$ is chosen arbitrarily. Let $Y_5(t)$ denote the function on the right-hand side of (2.83), which is a positive and nondecreasing function on I with $Y_5(0) = \Phi_2(u_0) + \int_0^T \alpha'(s)g(\alpha(s))\,ds$ and

$$Y_4(t) \le \Phi_2^{-1}\bigl(Y_5(t)\bigr), \quad \forall t < T. \tag{2.84}$$

Differentiating $Y_5(t)$ with respect to t and using the hypothesis on φ_2/φ_1, from (2.84) we have

$$\begin{aligned}\frac{dY_5(t)}{dt} &\le \alpha'(t)f\bigl(\alpha(t)\bigr)\frac{\varphi_1(Y_4(t))Y_4^p(t)}{\varphi_2(Y_4(t))}\\ &\le \alpha'(t)f\bigl(\alpha(t)\bigr)\frac{\varphi_1(\Phi_2^{-1}(Y_5(t)))(\Phi_2^{-1}(Y_5(t)))^p}{\varphi_2(\Phi_2^{-1}(Y_5(t)))}, \quad \forall t < T.\end{aligned} \tag{2.85}$$

By the definition of Φ_3 in (2.76), from (2.85) we obtain

$$\begin{aligned}\Phi_3\big(Y_5(t)\big) &\leq \Phi_3\big(Y_5(0)\big) + \int_0^t \alpha'(s)f\big(\alpha(s)\big)\,ds\\ &\leq \Phi_3\left(\Phi_2(u_0) + \int_0^{\alpha(T)} g(s)\,ds\right) + \int_0^{\alpha(t)} f(s)\,ds, \quad \forall t < T.\end{aligned} \tag{2.86}$$

Let $t = T$, from (2.86) we have

$$\begin{aligned}\Phi_3\big(Y_5(T)\big) &\leq \Phi_3\left(\Phi_2(u_0) + \int_0^{\alpha(T)} g(s)\,ds\right)\\ &\quad + \int_0^{\alpha(T)} f(s)\,ds, \quad \forall t < T.\end{aligned} \tag{2.87}$$

Since $0 < T < T_5$ is chosen arbitrarily, from (2.77), (2.80), (2.84) and (2.87), we have

$$u(t) \leq \Phi_2^{-1}\left[\Phi_3^{-1}\left(\Phi_3\left(\Phi_2(u_0) + \int_0^{\alpha(t)} g(s)\,ds\right) + \int_0^{\alpha(t)} f(s)\,ds\right)\right], \quad \forall t < T_5.$$

This proves (2.74). □

3 Application

In this section we apply our Theorem 4 to the following differential-integral equation:

$$\begin{cases}\frac{dx(t)}{dt} = H(t, x(\alpha(t)), \int_0^t K(s, x(\alpha(s)))\,ds), & \forall t \in I,\\ x(0) = x_0,\end{cases} \tag{3.88}$$

where $K \in C(\mathbf{R} \times \mathbf{R}, \mathbf{R})$, $H \in C(\mathbf{R}^3, \mathbf{R})$, $|x_0| > 0$ is a constant, satisfy the following conditions:

$$\big|K\big(t, x(t)\big)\big| \leq g(t)\psi_2\big(\big|x(t)\big|\big), \tag{3.89}$$

$$\begin{aligned}&\left|H\left(t, x\big(\alpha(t)\big), \int_0^t K\big(s, x\big(\alpha(s)\big)\big)\,ds\right)\right|\\ &\quad \leq f(t)\psi_1\big(\big|\alpha(t)\big|\big)\left(\big|x(t)\big| + \int_0^t \big|K\big(s, x\big(\alpha(s)\big)\big)\big|\,ds\right)^p,\end{aligned} \tag{3.90}$$

where f, g are nonnegative real-valued continuous functions defined on I.

Corollary 1 *Consider the nonlinear system* (3.88) *and suppose that* K, H *satisfy the conditions* (3.89) *and* (3.90), *and* $\psi_1, \psi_2, \psi_2/\psi_1, \alpha \in C^1(I, I)$ *are increasing functions with* $\alpha(t) \leq t$, $\psi_i(t) > 0$, $\forall t > 0$, $i = 1, 2$, $\alpha(0) = 0$. *Then all the solutions of equation* (3.88) *exist on I and satisfy the following estimation*:

$$\begin{aligned}\big|x(t)\big| &\leq \Psi_1^{-1}\left[\Psi_2^{-1}\left(\Psi_2\left(\Psi_1\big(|x_0|\big) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right)\right.\right.\\ &\quad \left.\left. + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right)\right]\end{aligned} \tag{3.91}$$

for all $t < T_6$, *where*

$$\Psi_1(r) := \int_1^r \frac{dt}{\Psi_2(t)}, \quad r > 0,$$
$$\Psi_2(r) := \int_1^r \frac{\Psi_2(\Psi_1^{-1}(s))\,ds}{\Psi_1(\Psi_1^{-1}(s))(\Psi_1^{-1}(s))^p}, \quad r > 0,$$

and T_6 *is the largest number such that*

$$\Psi_2\left(\Psi_1\left(|x_0|\right) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right) + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds \le \int_1^\infty \frac{\varphi_2(\Phi_2^{-1}(s))\,ds}{\varphi_1(\Phi_2^{-1}(s))(\Phi_2^{-1}(s))^p},$$
$$\Psi_2^{-1}\left(\Psi_2\left(\Psi_1\left(|x_0|\right) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right) + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\,ds\right) \le \int_1^\infty \frac{dt}{\varphi_2(t)}.$$

Proof Integrating both sides of equation (3.88) from 0 to t, we get

$$x(t) = x_0 + \int_0^t H\left(s, x(\alpha(s)), \int_0^s K(\tau, x(\alpha(\tau)))\,d\tau\right) ds, \quad \forall t \in I. \tag{3.92}$$

Using the conditions (3.89) and (3.90), from (3.92) we obtain

$$\begin{aligned}
|x(t)| &\le |x_0| + \int_0^t f(s)\varphi_1\left(\left|x(\alpha(s))\right|\right)\left(\left|x(\alpha(s))\right| + \int_0^s g(\tau)\varphi_2\left(\left|x(\alpha(\tau))\right|\right) d\tau\right)^p ds \\
&\le |x_0| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\varphi_1\left(\left|x(s)\right|\right) \\
&\quad \times \left(\left|x(s)\right| + \int_0^s \frac{g(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))}\varphi_2\left(\left|x(\tau)\right|\right) d\tau\right)^p ds
\end{aligned} \tag{3.93}$$

for all $t \in I$. Applying Theorem 4 to (3.93), we get the estimation (3.91). This completes the proof of Corollary 1. □

Competing interests
The author declares that they have no competing interests.

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