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Hankel determinant problem of a subclass of analytic functions

Muhammad Arif[1*], Khalida Inayat Noor[2] and Mohsan Raza[2]

* Correspondence: marifmaths@yahoo.com
[1]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan
Full list of author information is available at the end of the article

Abstract
In this article, we study the Hankel determinant problem of a subclass of analytic functions introduced recently by Arif et al.
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1 Introduction

Let $\mathcal{A}$ be the class of analytic function satisfying the condition $f(0) = 0, f'(0) - 1 = 0$ in the open unit disc $\mathcal{E} = \{z :| z |< 1\}$. By $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{K}$ we means the well-known subclasses of $\mathcal{A}$ which consist of univalent, starlike, convex, and close-to-convex functions, respectively.

Let $\mathcal{V}_k^\lambda(\sigma)$, $k \geq 2, 0 \leq \sigma < 1$, λ real, $|\lambda| < \frac{\pi}{2}$, denote the class of functions $f_1(z)$ analytic and locally univalent in $\mathscr{E}$, $f_1(0) = 0$, $f_1'(0) = 1$ and satisfying

$$\int_0^{2\pi} \left| \left\{ \text{Re } e^{i\lambda} \frac{(zf_1'(z))'}{f_1'(z)} - \sigma \cos\lambda \right\} \Big/ (1 - \sigma) \right| d\theta \leq k\pi \cos\lambda, \; z = re^{i\theta}. \tag{1.1}$$

This class was introduced and studied in details by Moulis [1]. For $\lambda = 0$, we obtain the class $\mathcal{V}_k(\sigma)$ of analytic functions with bounded boundary rotations of order σ studied by Padmanabhan et al. [2] and when $\sigma = 0$ and $\lambda = 0$, we get the class $\mathcal{V}_k$ discussed by Paatero [3], see also [4-8]. Also it can easily be shown that $f_1(z) \in \mathcal{V}_k^\lambda(\sigma)$ if and only if there exists $f_2(z) \in \mathcal{V}_k$ such that

$$f_1'(z) = (f_2'(z))^{(1-\sigma)^{e^{-i\lambda}\cos\lambda}}. \tag{1.2}$$

We now consider a class of analytic functions defined by Arif et al. [9] as follows:

Definition 1.1. Let $f(z) \in \mathcal{A}$ in E. Then $f(z) \in \tilde{\mathcal{B}}_k(\lambda, \sigma, \beta, \gamma)$, if for $k \geq 2$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, λ is real with $|\lambda| < \frac{\pi}{2}$ there exists a function $f_1(z) \in \mathcal{V}_k^\lambda(\sigma)$, $0 \leq \sigma < 1$, such that

$$\left| \arg \left\{ \frac{z^{1-\gamma} f'(z)}{f(z)} \left(\frac{f(z)}{f_1'(z)} \right)^\gamma \right\} \right| \leq \frac{\beta\pi}{2}, \; z \in E.$$

By giving specific values to the parameters k, σ, λ, β, and γ in $\tilde{\mathcal{B}}_k(\lambda, \sigma, \beta, \gamma)$, we obtain many important subclasses studied by various authors in earlier articles, see [10-16].

Using (1.1) and (1.2), we have

$$zf'(z) = z^{\gamma}(f(z))^{1-\gamma}(f_1'(z))^{\gamma}p^{\beta}(z), \tag{1.3}$$

where $f_1(z) \in \mathcal{V}_k^{\lambda}(\sigma)$ and $p(z)$ belongs to the class $\mathcal{P}$ of functions whose real part is positive.

Throughout in this article, we shall assume, unless otherwise stated, that $k \geq 2$, $0 \leq \beta \leq 1$, $0 < \gamma \leq 1$, λ is real with $|\lambda| < \frac{\pi}{2}$, $0 \leq \sigma < 1$.

In [17], the qth Hankel determinant $\mathcal{H}_q(n)$, $q \geq 1$, $n \geq 1$, for a function $f(z) \in \mathcal{A}$ is stated by Noonan and Thomas as:

Definition 1.2. Let $f(z) \in \mathcal{A}$. Then the qth Hankel determinant of $f(z)$ is defined for $q \geq 1$, $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.4}$$

The Hankel determinant plays an important role, for instance, in the study of the singularities by Hadamard, see [[18], p. 329], Edrei [19] and in the study of power series with integral coefficients by Polya [[20], p. 323], Cantor [21], and many others.

In this article, we shall determine the rate of growth of the Hankel determinant $\mathcal{H}_q(n)$ for $f(z) \in \tilde{\mathcal{B}}_k(\lambda, \sigma, \beta, \gamma)$ with $0 < \beta < 2$, as $n \to \infty$. This determinant has been considered by several authors. That is, Noor [22] determined the rate of growth of $\mathcal{H}_q(n)$ as $n \to \infty$ for a function $f(z)$ belongs to the class $\mathcal{V}_k$. Pommerenke in [23] studied the Hankel determinant for starlike functions. The Hankel determinant problem for other interesting classes of analytic functions were discussed by Noor [11,12,24].

Lemma 1.1. Let $f(z) \in \mathcal{A}$. Let the qth Hankel determinant of $f(z)$ for $q \geq 1$, $n \geq 1$ be defined by (1.4). Then, writting $\Delta_j(n) = \Delta_j(n, z_1, f(z))$, we have

$$\mathcal{H}_q(n) = \begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \cdots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \cdots & \Delta_{q-2}(n+q-2) \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q-2) & \cdots & \Delta_q(n+2q-2) \end{vmatrix},$$

where with $\Delta_0(n) = a_n$, we define for $j \geq 1$,

$$\Delta_j(n, z_1, f(z)) = \Delta_{j-1}(n, z_1, f(z)) - \Delta_{j-1}(n+1, z_1, f(z)). \tag{1.5}$$

Lemma 1.2. With $z_1 = \frac{n}{n+1}\gamma$ and $\nu \geq 0$ any integer,

$$\Delta_j(n+\nu, z_1, zf'(z)) = \sum_{m=0}^{j} \binom{j}{m} \frac{\gamma^m(\nu - (m-1)n)}{(n+1)^m} \Delta_{j-m}(n+m+\nu, f(z)).$$

Lemmas 1.1 and 1.2 are due to Noonan and Thomas [17].

Lemma 1.3. Let $h_1(z)$ be starlike univalent function in $\mathcal{E}$. Then

(i) there exists a z_1 with $|z_1| = r$ such that for all z, $|z| = r$

$$|z - z_1|\,|h_1(z)| \leq \frac{2r^2}{1 - r^2},$$

see [25]

(ii)

$$\frac{r}{(1+r)^2} \leq |h_1(z)| \leq \frac{r}{(1+r)^2},$$

see[26].

2 Hankel determinant problem

Theorem 2.1. Let $f(z) \in \tilde{\mathcal{B}}_k(\lambda, \sigma, \beta, \gamma)$ with $0 < \beta < 2$ and let the qth Hankel determinant $\mathcal{H}_q(n)$ of $f(z)$ be defined as in (1.4). Then

$$\mathcal{H}_q(n) = \mathrm{O}(1)\,(M(r))^{1-\gamma} \begin{cases} n^{\gamma\left(\frac{k}{2}+1\right)(1-\sigma)\cos^2\lambda+\beta-2}, & q = 1 \\ n^{\left\{\gamma\left(\frac{k}{2}+1\right)(1-\sigma)\cos^2\lambda+\beta-1\right\}q-q^2}, & q \geq 2,\ k \geq \dfrac{8(q-1)}{(1-\sigma)\gamma\cos^2\lambda} - 2, \end{cases}$$

where $k > \frac{4j+2-\beta}{(1-\sigma)\gamma\cos^2\lambda} - 2$ and O(1) is a constant depending on k, λ, β, σ, γ, and j only.

Proof. It is well known [1] that for starlike functions $h_1(z)$ and $h_2(z)$

$$f_1'(z) = \left[\frac{(h_1(z)/z)^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{(h_2(z)/z)^{\left(\frac{k}{4}-\frac{1}{2}\right)}}\right]^{(1-\sigma)\gamma e^{-i\lambda}\cos\lambda}. \tag{2.1}$$

Using (2.1) in (1.3), we have

$$zf'(z) = z^{\gamma}(f(z))^{1-\gamma}\left[\frac{(h_1(z)/z)^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{(h_2(z)/z)^{\left(\frac{k}{4}-\frac{1}{2}\right)}}\right]^{(1-\sigma)\gamma e^{-i\lambda}\cos\lambda} p^{\beta}(z). \tag{2.2}$$

where $p(z) \in \mathcal{P}$.

Let $F(z) = zf'(z)$. Then for $j \geq 1$, z_1 any non-zero complex and $z = re^{i\theta}$, consider $\Delta_j(n, z_1, F(z))$ as defined by (1.5). Then

$$|\Delta_j(n, z_1, F(z))| = \frac{1}{2\pi r^{n+j}}\left|\int_0^{2\pi} (z - z_1)^j F(z) e^{i(n+j)\theta}\, d\theta\right|,$$

and by using (2.2), we have

$$
\begin{aligned}
\mid \Delta_j(n, z_1, F(z)) \mid &\le \frac{1}{2\pi r^{n+j}} \int_0^{2\pi} \mid z - z_1 \mid^j \mid z \mid^{\gamma} \mid f(z) \mid^{1-\gamma} \left| \left[\frac{(h_1(z)/z)^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{(h_2(z)/z)^{\left(\frac{k}{4}-\frac{1}{2}\right)}} \right]^{(1-\sigma)\gamma e^{-i\lambda}\cos\lambda} \right| \mid p(z) \mid^{\beta} d\theta \\
&\le \frac{(M(r))^{1-\gamma}}{2\pi r^{n+j}} \int_0^{2\pi} \mid z - z_1 \mid^j \left| \left[\frac{(h_1(z)/z)^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{(h_2(z)/z)^{\left(\frac{k}{4}-\frac{1}{2}\right)}} \right]^{(1-\sigma)\gamma e^{-i\lambda}\cos\lambda} \right| \mid p(z) \mid^{\beta} d\theta.
\end{aligned}
\tag{2.3}
$$

Since

$$
\begin{aligned}
\left| \left[\frac{(h_1(z)/z)^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{(h_2(z)/z)^{\left(\frac{k}{4}-\frac{1}{2}\right)}} \right]^{(1-\sigma)\gamma e^{-i\lambda}\cos\lambda} \right| &\le \left[\frac{\mid (h_1(z)/z) \mid^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{\mid (h_2(z)/z) \mid^{\left(\frac{k}{4}-\frac{1}{2}\right)}} \right]^{\gamma(1-\sigma)\cos^2\lambda} e^{\gamma(1-\sigma)\gamma\frac{\pi}{2}\frac{\sin 2\lambda}{2}} \\
&= \frac{\mid (h_1(z)/z) \mid^{\left(\frac{k}{4}+\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda}}{\mid (h_2(z)/z) \mid^{\left(\frac{k}{4}-\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda}} c_1, \text{ say}
\end{aligned}
$$

therefore (2.3) becomes

$$
\mid \Delta_j(n,\ z_1,\ F(z)) \mid \le \frac{c_1(M(r))^{1-\gamma}}{2\pi r^{n+j}} \int_0^{2\pi} \mid z - z_1 \mid^j \frac{\mid (h_1(z)/z) \mid^{\left(\frac{k}{4}+\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda}}{\mid (h_2(z)/z) \mid^{\left(\frac{k}{4}-\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda}} \mid p(z) \mid^{\beta} d\theta.
$$

Now using Lemma 1.3, we have

$$
\begin{aligned}
\mid \Delta_j(n,\ z_1,\ F(z)) \mid \le{}& \frac{c_1(M(r))^{1-\gamma}}{2\pi r^{n+j}} \left(\frac{(1+r)^2}{r} \right)^{\left(\frac{k}{4}-\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda} \\
& \frac{1}{r^{\gamma(1-\sigma)\cos^2\lambda}} \left(\frac{2r^2}{1-r^2} \right)^j \int_0^{2\pi} \mid (h_1(z)) \mid^{\left(\frac{k}{4}+\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda - j} \mid p(z) \mid^{\beta} d\theta.
\end{aligned}
$$

The well-known Holder's inequality will give us

$$
\begin{aligned}
\mid \Delta_j(n,\ z_1,\ F(z)) \mid \le{}& \frac{c_1(M(r))^{1-\gamma} 2^{\left(\frac{k}{4}-\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda + j}}{r^{n-j+\left(\frac{k}{4}+\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda}} \left(\frac{1}{1-r} \right)^j \left(\frac{1}{2\pi} \int_0^{2\pi} \mid p(z) \mid^2 d\theta \right)^{\frac{\beta}{2}} \\
& \left(\frac{1}{2\pi} \int_0^{2\pi} \mid (h_1(z)) \mid^{\frac{\left(\frac{k}{2}+1\right)\gamma(1-\sigma)\cos^2\lambda - 2j}{2-\beta}} d\theta \right)^{\frac{2-\beta}{2}}.
\end{aligned}
\tag{2.4}
$$

Also, it is known [15] that, for $p(z) \in \mathcal{P}$, $z \in E$,

$$
\frac{1}{2\pi} \int_0^{2\pi} \mid p(z) \mid^2 d\theta \le \frac{1+3r^2}{1-r^2}. \tag{2.5}
$$

Using (2.5) in (2.4), we obtain

$$|\Delta_j(n,\ z_1,\ F(z))| \leq \frac{c_1(M(r))^{1-\gamma}2^{\left(\frac{k}{4}-\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda+j}}{r^{n-j+\left(\frac{k}{4}+\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda}}\left(\frac{1}{1-r}\right)^j$$

$$\left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\beta}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|(h_1(z))|\frac{\left(\frac{k}{4}+1\right)\gamma(1-\sigma)\cos^2\lambda-2j}{2-\beta}d\theta\right)^{\frac{2-\beta}{2}}.$$

Therefore, we can write

$$|\Delta_j(n,\ z_1,\ F(z))| \leq \frac{c_1(M(r))^{1-\gamma}2^{\left(\frac{k}{4}-\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda+\beta+j}}{r^{n-j+\left(\frac{k}{4}+\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda}}\left(\frac{1}{1-r}\right)^{j+\frac{\beta}{2}}$$

$$\left[r^{\left(\frac{k}{4}+\frac{1}{2}\right)\gamma(1-\sigma)\cos^2\lambda-j}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|\dfrac{(k+2)\gamma(1-\sigma)\cos^2\lambda-4j}{2-\beta}}\right]^{\frac{2-\beta}{2}}.$$

Now using a subordination result for starlike functions, we have

$$|\Delta_j(n,\ z_1,\ F(z))| \leq \frac{c_2(M(r))^{1-\gamma}}{r^n}\left(\frac{1}{1-r}\right)^{j+\frac{\beta}{2}}\left[\left(\frac{1}{1-r}\right)^{\frac{\gamma(k+2)(1-\sigma)\cos^2\lambda-4j}{2-\beta}-1}\right]^{\frac{2-\beta}{2}},$$

$$= \frac{c_2(M(r))^{1-\gamma}}{r^n}\left(\frac{1}{1-r}\right)^{\gamma\left(\frac{k}{2}+1\right)(1-\sigma)\cos^2\lambda+\beta-1-j},$$

where c_2 is a constant depending on k, λ, β, σ, γ, j only and $\gamma(k + 2)(1 - \sigma)\cos^2\lambda - 4j > 2 - \beta$.

Applying Lemma 1.2 and putting $z_1 = \left(\frac{n}{n+1}\right)e^{i\theta_n}$, $(n \to \infty)$, $r = 1 - \frac{1}{n}$, we have for

$$|\Delta_j\ \left(n,\ e^{i\theta_n},\ f(z)\right)| = \mathrm{O}(1)\,(M(r))^{1-\gamma}n^{\gamma\left(\frac{k}{2}+1\right)(1-\sigma)\cos^2\lambda+\beta-j-2},'$$

$$|\Delta_j\ \left(n,\ e^{i\theta_n},\ f(z)\right)| = \mathrm{O}(1)\,(M(r))^{1-\gamma}n^{\gamma\left(\frac{k}{2}+1\right)(1-\sigma)\cos^2\lambda+\beta-j-2},$$

where O(1) is a constant depending on k, λ, β, σ, γ, and j only.

We now estimate the rate of growth of $\mathcal{H}_q(n)$.

For $q = 1$, $\mathcal{H}_q(n) = a_n = \Delta_0(n)$ and

$$H_1(n) = a_n = \mathrm{O}(1)\,(M(r))^{1-\gamma}n^{\gamma\left(\frac{k}{2}+1\right)(1-\sigma)\cos^2\lambda+\beta-2}.$$

For $q \geq 2$, we use similar argument due to Noonan and Thomas [17] together with Lemma 1.1 to have

$$\mathcal{H}_q(n) = \mathrm{O}(1)\,(M(r))^{1-\gamma}n^{\left[\gamma\left(\frac{k}{2}+1\right)(1-\sigma)\cos^2\lambda+\beta-1\right]q-q^2},\ k \geq \frac{8(q-1)}{(1-\sigma)\gamma\cos^2\lambda}-2$$

and O(1) depends only on k, λ, β, σ, γ, and j.

For choosing different values of λ, β, σ, and γ in Theorem 2.1, we obtain the following results discussed by Noor [10,11] and Noor et al. [14].

Corollary 2.1. For $\lambda = 0,\ \beta = 1,\ \sigma = 0,\ f(z) \in \mathcal{B}_k(\gamma)$, where the class $\mathcal{B}_k(\gamma)$ was introduced by Noor et al. [14] and

$$\mathcal{H}_q(n) = \mathrm{O}(1)(M(r))^{1-\gamma} \begin{cases} n^{\gamma\left(\frac{k}{2}+1\right)-1}, & q = 1 \\ n^{\gamma\left(\frac{k}{2}+1\right)q-q^2}, & q \geq 2,\ k \geq \dfrac{8(q-1)}{\gamma} - 2 \end{cases}$$

where O(1) is a constant depending only on k and γ.

Corollary 2.2. For $\lambda = 0,\ \gamma = 1,\ f(z) \in \tilde{\mathcal{T}}_k(\beta, \sigma)$ and

$$\mathcal{H}_q(n) = \mathrm{O}(1) \begin{cases} n^{(1-\sigma)\left(\frac{k}{2}+1\right)+\beta-2}, & q = 1 \\ n^{\left[(1-\sigma)\left(\frac{k}{2}+1\right)+\beta-1\right]q-q^2}, & q \geq 2,\ k \geq \dfrac{8(q-1)}{(1-\sigma)} - 2 \end{cases}$$

where O(1) is a constant depends on k, σ, and β only.

Author details
[1]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan [2]COMSATS Institute of Information Technology, Islamabad, Pakistan

Authors' contributions
MA, KIN, and MR jointly discussed and presented the ideas of this article. MA made the text file and all the communications regarding the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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