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The Hermite-Hadamard inequality for *r*-convex functions

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Abstract

In this paper, we establish the Hermite-Hadamard inequality for r-convex functions. We prove that r-convexity implies s-convexity ($0 \le r \le s$). As a result, we obtain a refinement of the Hermite-Hadamard inequality for an r-convex function ($0 \le r \le 1$). We also investigate the Hermite-Hadamard inequality for the product of an r-convex function f and an s-convex function g.

MSC: 26D15; 26D10

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1 Introduction

Let $f : [a, b] \longrightarrow R$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known as the Hermite-Hadamard inequality (see [1] for more information). Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (*e.g.*, [2, 3] and [4]). In [5], the first author obtained a new refinement of the Hermite-Hadamard inequality for convex functions. The Hermite-Hadamard inequality was generalized in [6] to an r-convex positive function which is defined on an interval $[a, b]$. A positive function f is called r-convex on $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f\big(tx + (1-t)y\big) \le \begin{cases} [tf^r(x) + (1-t)f^r(y)]^{\frac{1}{r}}, & r \neq 0, \\ [f(x)]^t [f(y)]^{1-t}, & r = 0. \end{cases}$$

It is obvious 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. One should note that if f is r-convex in $[a, b]$, then f^r is a convex function ($r > 0$).

Some refinements of the Hadamard inequality for r-convex functions could be found in [7] and [8]. In [9], Bessenyei studied Hermite-Hadamard-type inequalities for generalized 3-convex functions. In [7], the authors showed that if f is r-convex in $[a, b]$ and $0 < r \le 1$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{r}{r+1}\big[f^r(a) + f^r(b)\big]^{\frac{1}{r}}. \tag{2}$$

In this paper, first we show that if f is r-convex in $[a,b]$ and $r \geq 1$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \left[\frac{1}{2}\bigl(f^r(a)+f^r(b)\bigr)\right]^{\frac{1}{r}}. \tag{3}$$

In Theorem 2.3, we prove the following inequality for r-convex functions:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{r}{r+1}\cdot\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)} \quad (r>0). \tag{4}$$

The inequality (4) is an extension and refinement of (2) and (3). In Theorem 2.4, we show that r-convexity implies s-convexity ($0 \leq r \leq s$). We employ this result in Theorem 2.6 and Corollary 2.7 to refine the Hermite-Hadamard inequality by r-convexity ($0 \leq r \leq 1$). Finally, we generalize some results in [7] without using Minkowski's inequality. Indeed, we obtain refinements for the product of an r-convex function f and an s-convex function g ($r,s \geq 0$).

2 Main results

Theorem 2.1 *Let $f:[a,b]\longrightarrow(0,\infty)$ be r-convex and $r \geq 1$. Then the following inequality holds*:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}\bigl[\bigl(f^r(a)+f^r(b)\bigr)\bigr]^{\frac{1}{r}}.$$

Proof Since $r \geq 1$, by Jensen's inequality, we have

$$\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^r \leq \frac{1}{b-a}\int_a^b f^r(x)\,dx.$$

By convexity of f^r and the right side of (1), we obtain

$$\frac{1}{b-a}\int_a^b f^r(x)\,dx \leq \frac{1}{2}\bigl(f^r(a)+f^r(b)\bigr).$$

Thus,

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \left[\frac{1}{2}\bigl(f^r(a)+f^r(b)\bigr)\right]^r.$$

□

Corollary 2.2 *Let $f:[a,b]\longrightarrow(0,\infty)$ be a 1-convex function. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}\bigl(f(b)+f(a)\bigr). \tag{5}$$

Theorem 2.3 *Let $f:[a,b]\longrightarrow(0,\infty)$ be r-convex and $r \geq 0$. Then the following inequalities hold*:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \begin{cases} \frac{r}{r+1}\left(\frac{f^{r+1}(b)-f^{r+1}(a)}{f^r(b)-f^r(a)}\right), & r \neq 0, \\ [f(b)-f(a)]\ln\frac{f(b)}{f(a)}, & r=0. \end{cases}$$

Proof First, let $r > 0$. Since f is r-convex, for all $t \in [0,1]$, we have

$$f\bigl(ta + (1-t)b\bigr) \le \bigl[tf^r(a) + (1-t)f^r(b)\bigr]^{\frac{1}{r}}.$$

It is easy to observe that

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \int_0^1 f\bigl(ta + (1-t)b\bigr)\,dt\\ &\le \int_0^1 \bigl[tf^r(a) + (1-t)f^r(b)\bigr]^{\frac{1}{r}}\,dt\\ &= \int_0^1 \bigl[t\bigl(f^r(a) - f^r(b)\bigr) + f^r(b)\bigr]^{\frac{1}{r}}\,dt.\end{aligned}$$

By substitution $t(f^r(a) - f^r(b)) + f^r(b) = z$, we obtain

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &\le \frac{1}{f^r(b) - f^r(a)}\int_{f^r(a)}^{f^r(b)} z^{\frac{1}{r}}\,dz\\ &= \frac{1}{f^r(b) - f^r(a)} \cdot \frac{1}{1+\frac{1}{r}}\bigl[z^{1+\frac{1}{r}}\bigr]_{f^r(a)}^{f^r(b)}\\ &= \frac{r}{r+1}\left(\frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)}\right).\end{aligned}$$

For $r = 0$, we have

$$f\bigl(tx + (1-t)y\bigr) \le \bigl[f(x)\bigr]^t\bigl[f(y)\bigr]^{1-t}.$$

So,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \int_0^1 f\bigl(ta + (1-t)b\bigr)\,dt\\ &\le \int_0^1 \bigl[f(a)\bigr]^t\bigl[f(b)\bigr]^{1-t}\,dt\\ &= f(b)\int_0^1 \left[\frac{f(a)}{f(b)}\right]^t\,dt\\ &= f(b)\frac{\left[\frac{f(a)}{f(b)}\right]^t}{\ln\frac{f(a)}{f(b)}}\Bigg|_0^1\\ &= \frac{\bigl[f(b) - f(a)\bigr]}{\ln\frac{f(b)}{f(a)}}.\end{aligned}$$

The proof is completed. □

With the hypotheses of Theorem 2.3, if $f(a) = f(b)$, its proving process shows that $\frac{1}{b-a}\int_a^b f(x)\,dx$ can be dominated by $f(a)$ where $r \ge 0$.

Note that if we put $r = 1$ in Theorem 2.3, we can obtain again the inequality (5).

Theorem 2.4 *Let $f : [a,b] \longrightarrow (0,\infty)$ be r-convex on $[a,b]$ and $0 \le r \le s$. Then f is s-convex. In particular, if f is r-convex and $0 \le r \le 1$, then f is convex.*

In order to prove the above theorem, we need the following lemma.

Lemma 2.5 *If $0 \le \alpha \le 1$ and $0 \le r \le s$, then the following inequalities hold for every pair of non-negative real numbers x and y:*

$$x^{\alpha} y^{1-\alpha} \le \left(\alpha x^r + (1-\alpha) y^r\right)^{\frac{1}{r}} \le \left(\alpha x^s + (1-\alpha) y^s\right)^{\frac{1}{s}}. \tag{6}$$

Proof The left side of the inequality is clear by Young's inequality. The right side is obvious if either x or y equals zero. So, let $x > 0$ and $y > 0$. Consider $f : [0,\infty) \longrightarrow R$ defined by

$$f(t) = \left(\alpha t^r + 1 - \alpha\right)^s - \left(\alpha t^s + 1 - \alpha\right)^r.$$

Then $f'(t) = rs\alpha[t^{r-1}(\alpha t^r + 1 - \alpha)^{s-1} - t^{s-1}(\alpha t^s + 1 - \alpha)^{r-1}]$. So, $t = 1$ is a critical point of f. By an easy calculation, we see that $f''(1) = rs\alpha(1-\alpha)(r-s) \le 0$. It follows that f attains its maximum at $t = 1$. Thus, $f(t) \le f(1) = 0$. This shows that

$$\left(\alpha t^r + 1 - \alpha\right)^s \le \left(\alpha t^s + 1 - \alpha\right)^r.$$

Now, if we put $t = \frac{x}{y}$ in the above inequality, we get

$$\left(\alpha x^r + (1-\alpha) y^r\right)^s \le \left(\alpha x^s + (1-\alpha) y^s\right)^r.$$

Therefore, we can deduce the right side of (6) by taking rsth root. □

Proof of Theorem 2.4 Since f is r-convex, by Lemma 2.5 for all $x, y \in [a,b]$ and $t \in [0,1]$, we have

$$f\left(tx + (1-t)y\right) \le \begin{cases} t[f^r(x) + (1-t)f^r(y)]^{\frac{1}{r}} \le [tf^s(x) + (1-t)f^s(y)]^{\frac{1}{s}}, & 0 < r \le s, \\ [f(x)]^t [f(y)]^{1-t} \le [tf^s(x) + (1-t)f^s(y)]^{\frac{1}{s}}, & 0 = r \le s. \end{cases}$$

Hence, f is s-convex. □

Theorem 2.6 *Let $f : [a,b] \longrightarrow (0,\infty)$ be r-convex on $[a,b]$ and $0 \le r \le s$. Then the following inequalities hold:*

$$\frac{1}{b-a} \int_a^b f(x)\,dx \le \begin{cases} \frac{r}{r+1} \frac{f^{r+1}(b) - f^{r+1}(a)}{f^r(b) - f^r(a)} \le \frac{s}{s+1} (\frac{f^{s+1}(b) - f^{s+1}(a)}{f^s(b) - f^s(a)}), & 0 < r \le s, \\ [f(b) - f(a)] \ln \frac{f(b)}{f(a)} \le \frac{s}{s+1} (\frac{f^{s+1}(b) - f^{s+1}(a)}{f^s(b) - f^s(a)}), & 0 = r < s. \end{cases}$$

Proof The left side of the inequalities is clear by Theorem 2.3. For the right side, by the inequality in (6), we have

$$\left[tf^r(x) + (1-t)f^r(b)\right]^{\frac{1}{r}} \le \left[tf^s(a) + (1-t)f^s(b)\right]^{\frac{1}{s}}.$$

By integrating it on $[0,1]$, we obtain

$$\int_0^1 \left[tf^r(a) + (1-t)f^r(b)\right] dt \le \int_0^1 \left[tf^s(a) + (1-t)f^s(b)\right]^{\frac{1}{t}} dt.$$

Thus,

$$\frac{r}{r+1}\left(\frac{f^{r+1}(b)-f^{r+1}(a)}{f(b)-f(a)}\right)\leq\frac{s}{s+1}\left(\frac{f^{s+1}(b)-f^{s+1}(a)}{f^{s}(b)-f^{s}(a)}\right).$$

Also, another inequality can be deduced by integrating the inequalities in (6) if we replace x and y by $f(x)$ and $f(y)$, respectively. □

Corollary 2.7 *Let $f:[a,b]\longrightarrow(0,\infty)$ be r-convex and $0\leq r\leq 1$. Then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\big[f(b)-f(a)\big]\ln\frac{f(b)}{f(a)}\\&\leq\frac{r}{r+1}\left(\frac{f^{r+1}(b)-f^{r+1}(a)}{f^{r}(b)-f^{r}(a)}\right)\leq\frac{f(a)+f(b)}{2}.\end{aligned}$$

In other words, when f is r-convex and $0\leq r\leq 1$, we can refine the Hermite-Hadamard inequalities through Theorem 2.6.

Theorem 2.8 *Let $f,g:[a,b]\longrightarrow(0,\infty)$ be r-convex and s-convex functions respectively on $[a,b]$ and $r,s>0$. Then the following inequality holds*:

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\leq{}&\frac{1}{2}\left(\frac{r}{r+2}\right)\left(\frac{f^{r+2}(b)-f^{r+2}(a)}{f^{r}(b)-f^{r}(a)}\right)\\&+\frac{1}{2}\left(\frac{s}{s+2}\right)\left(\frac{g^{s+2}(b)-g^{s+2}(a)}{g^{s}(b)-g^{s}(a)}\right)\\&\big(f(a)\neq f(b),g(a)\neq g(b)\big).\end{aligned}$$

Proof Since f is r-convex and g is s-convex, for all $t\in[0,1]$, we have

$$\begin{aligned}&f\big(ta+(1-t)b\big)\leq\big[tf^{r}(a)+(1-t)f^{r}(b)\big]^{\frac{1}{r}},\\&g\big(ta+(1-t)b\big)\leq\big[tg^{s}(a)+(1-t)g^{s}(b)\big]^{\frac{1}{s}}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\,dx&=\int_0^1 f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)\,dt\\&\leq\int_0^1\big[tf^{r}(a)+(1-t)f^{r}(b)\big]^{\frac{1}{r}}\big[tg^{s}(a)+(1-t)g^{s}(b)\big]^{\frac{1}{s}}\,dt\\&=\int_0^1\big[t\big(f^{r}(a)-f^{r}(b)\big)+f^{r}(b)\big]^{\frac{1}{r}}\big[t\big(g^{s}(a)-g^{s}(b)\big)+g^{s}(b)\big]^{\frac{1}{s}}\,dt.\end{aligned}$$

Applying Cauchy's inequality, we get

$$\begin{aligned}&\int_0^1\big[t\big(f^{r}(a)-f^{r}(b)\big)+f^{r}(b)\big]^{\frac{1}{r}}\big[t\big(g^{s}(a)-g^{s}(b)\big)+g^{s}(b)\big]^{\frac{1}{s}}\,dt\\&\quad\leq\frac{1}{2}\int_0^1\big[t\big(f^{r}(a)-f^{r}(b)\big)+f^{r}(b)\big]^{\frac{2}{r}}\,dt+\frac{1}{2}\int_0^1\big[t\big(g^{s}(a)-g^{s}(b)\big)+g^{s}(b)\big]^{\frac{2}{s}}\,dt.\end{aligned}\tag{7}$$

Similar to the proof of Theorem 2.3 and by substitution $t(f^r(a) - f^r(b)) + f^r(b) = z$, we obtain

$$\int_0^1 \left[t\left(f^r(a) - f^r(b)\right) + f^r(b)\right]^{\frac{2}{r}} dt = \frac{r}{r+2}\left(\frac{f^{r+2}(b) - f^{r+2}(a)}{f^r(b) - f^r(a)}\right). \tag{8}$$

Similarly,

$$\int_0^1 \left[t\left(g^s(a) - g^s(b)\right) + g^s(b)\right]^{\frac{1}{s}} dt = \frac{s}{s+2}\left(\frac{g^{s+2}(b) - g^{s+2}(a)}{g^s(b) - g^s(a)}\right). \tag{9}$$

Using (7), (8) and (9), we can obtain the desired result. □

Remark 2.9 If the conditions of Theorem 2.8 hold, and $r \leq s$, by Theorem 2.4, f is s-convex. Thus, the result of Theorem 2.8 could be as follows:

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leq \frac{1}{2}\left(\frac{s}{s+2}\right)\left[\frac{f^{s+2}(b) - f^{s+2}(a)}{f^s(b) - f^s(a)} + \frac{g^{s+2}(b) - g^{s+2}(a)}{g^s(b) - g^s(a)}\right].$$

If $f = g$, we have

$$\frac{1}{b-a}\int_a^b f^2(x)\,dx \leq \frac{s}{s+2}\left(\frac{f^{s+2}(b) - f^{s+2}(a)}{f^s(b) - f^s(a)}\right).$$

Now, if $f = g$ and $r = s = 2$ in Theorem 2.8, we have

$$\frac{1}{b-a}\int_a^b f^2(x)\,dx \leq \frac{1}{2}\left(f^2(b) + f^2(a)\right),$$

which is the same result as in [7, Corollary 2.5]. This shows that Theorem 2.8 is a generalization of [7, Theorem 2.3]. In fact, the condition $r, s \leq 2$ is redundant.

Theorem 2.10 *Let* $f, g : [a,b] \longrightarrow (0,\infty)$ *be* 0*-convex on* $[a,b]$. *Then the following inequality holds*:

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leq \left[f(b)g(b) - f(a)g(a)\right]\ln\frac{f(b)g(b)}{f(a)g(a)}.$$

Proof Since f and g are 0-convex, for all $t \in [0,1]$, we have

$$f\left(tx + (1-t)y\right) \leq \left[f(x)\right]^t\left[f(y)\right]^{1-t},$$
$$g\left(tx + (1-t)y\right) \leq \left[g(x)\right]^t\left[g(y)\right]^{1-t}.$$

For all $x, y \in [0,1]$, and thus

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\,dx &= \int_0^1 \left[f\left(ta + (1-t)b\right)\right]\left[g\left(tx + (1-t)y\right)\right] dt \\ &\leq \int_0^1 \left[f(a)\right]^t\left[f(b)\right]^{1-t}\left[g(a)\right]^t\left[g(b)\right]^{1-t} dt\end{aligned}$$

$$= f(b)g(b)\int_0^1 \left[\frac{f(a)g(a)}{f(b)g(b)}\right]^t dt$$
$$= \left[f(b)g(b) - f(a)g(a)\right]\ln\frac{f(b)g(b)}{f(a)g(a)}. \qquad \square$$

Corollary 2.11 *With the hypotheses of the above theorem and $f = g$, we have*

$$\frac{1}{b-a}\int_a^b f^2(x)\,dx \le \left[f^2(b) - f^2(a)\right]\ln\frac{f^2(b)}{f^2(a)}.$$

Theorem 2.12 *Let $f, g : [a,b] \longrightarrow (0,\infty)$ be r-convex and 0-convex functions respectively on $[a,b]$ and $r > 0$. Then the following inequality holds*:

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \le \frac{1}{2}\left(\frac{r}{r+2}\right)\left(\frac{f^{r+2}(b) - f^{r+2}(a)}{f^r(b) - f^r(a)}\right) + \frac{1}{4}\left[g(b)^2 - g(a)^2\right]\ln\frac{g(b)}{g(a)}$$
$$\big(f(a) \neq f(b)\big).$$

Proof Since f is r-convex and g is 0-convex, for all $t \in [0,1]$, we have

$$f\big(ta + (1-t)b\big) \le \left[tf^r(a) + (1-t)f^r(b)\right]^{\frac{1}{r}},$$
$$g\big(tx + (1-t)y\big) \le \left[g(x)\right]^t\left[g(y)\right]^{1-t}.$$

Thus,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\,dx &= \int_0^1 f\big(ta + (1-t)b\big)g\big(ta + (1-t)b\big)\,dt \\ &\le \int_0^1 \left[tf^r(a) + (1-t)f^r(b)\right]^{\frac{1}{r}}\left[g(a)\right]^t\left[g(b)\right]^{1-t}\,dt \\ &= \int_0^1 \left[t\big(f^r(a) - f^r(b)\big) + f^r(b)\right]^{\frac{1}{r}}\left[g(a)\right]^t\left[g(b)\right]^{1-t}\,dt.\end{aligned}$$

Again, Cauchy's inequality shows that

$$\begin{aligned}&\int_0^1 \left[t\big(f^r(a) - f^r(b)\big) + f^r(b)\right]^{\frac{1}{r}}\left[g(a)\right]^t\left[g(b)\right]^{1-t}\,dt \\ &\quad\le \frac{1}{2}\int_0^1 \left[t\big(f^r(a) - f^r(b)\big) + f^r(b)\right]^{\frac{2}{r}}\,dt + \frac{1}{2}\int_0^1 \left[g(a)\right]^{2t}\left[g(b)\right]^{2-2t}\,dt. \qquad (10)\end{aligned}$$

We have

$$\int_0^1 \left[t\big(f^r(a) - f^r(b)\big) + f^r(b)\right]^{\frac{2}{r}}\,dt = \frac{r}{r+2}\left(\frac{f^{r+2}(b) - f^{r+2}(a)}{f^r(b) - f^r(a)}\right). \qquad (11)$$

Similar to the proof of Theorem 2.10, we can show that

$$\int_0^1 \left[g(a)\right]^{2t}\left[g(b)\right]^{2-2t}\,dt = \frac{1}{2}\left[g(b)^2 - g(a)^2\right]\ln\frac{g(b)}{g(a)}. \qquad (12)$$

Using (10), (11) and (12), we can obtain the desired result. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final draft.

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