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Almost stability of the Mann type iteration method with error term involving strictly hemicontractive mappings in smooth Banach spaces

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Abstract

Let K be a nonempty closed bounded convex subset of an arbitrary smooth Banach space X and $T : K \to K$ be a continuous strictly hemicontractive mapping. Under some conditions, we obtain that the Mann iteration method with error term converges strongly to a unique fixed point of T and is almost T-stable on K. As an application of our results, we establish strong convergence of a multi-step iteration process.

Keywords: Mann iteration method with error term; strictly hemicontractive operators; strongly pseudocontractive operators; local strongly pseudocontractive operators; continuous mappings; Lipschitz mappings; smooth Banach spaces

1 Introduction

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudo-contractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Schu [2] generalized the result in [1] to both uniformly continuous strongly pseudo-contractive mappings and real smooth Banach spaces. Park [3] extended the result in [1] to both strongly pseudocontractive mappings and certain smooth Banach spaces. Rhoades [4] proved that the Mann and Ishikawa iteration methods may exhibit different behavior for different classes of nonlinear mappings. Harder and Hicks [5, 6] revealed the importance of investigating the stability of various iteration procedures for various classes of nonlinear mappings. Harder [7] established applications of stability results to first-order differential equations. Afterwords, several generalizations have been made in various directions (see, for example, [2, 4, 8–21].

Let K be a nonempty closed bounded convex subset of an arbitrary smooth Banach space X and $T : K \to K$ be a continuous strictly hemicontractive mapping. Under some conditions, we obtain that the Mann iteration method with error term converges strongly to a unique fixed point of T and is almost T-stable on K. As an application, we shall also establish strong convergence of a multi-step iteration process. The results presented here generalize the corresponding results in [2–4, 10, 11, 22].

2 Preliminaries

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. The symbols $D(T)$, $R(T)$ and $F(T)$ stand for the domain, the range and the set of fixed points of $T : X \to X$ respectively (x is called a fixed point of T iff $T(x) = x$). We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

Let T be a self-mapping of K.

Definition 1 The mapping T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in K$. If $L = 1$, then T is called *non-expansive* and if $0 \leq L < 1$, T is called *contraction*.

Definition 2 [10, 22]

1. The mapping T is said to be *pseudocontractive* if the inequality

$$\|x - y\| \leq \|x - y + t[(I - T)x - (I - T)y]\| \tag{2.1}$$

   holds for each $x, y \in K$ and for all $t > 0$.
2. T is said to be strongly pseudocontractive if there exists $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{2.2}$$

   for all $x, y \in D(T)$ and $r > 0$.
3. T is said to be local strongly pseudocontractive if for each $x \in D(T)$, there exists $t_x > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt_x(Tx - Ty)\| \tag{2.3}$$

   for all $y \in D(T)$ and $r > 0$.
4. T is said to be strictly hemicontractive if $F(T) \neq \emptyset$ and if there exists $t > 1$ such that

$$\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\| \tag{2.4}$$

   for all $x \in D(T)$, $q \in F(T)$ and $r > 0$.

Clearly, each strongly pseudocontractive operator is local strongly pseudocontractive.

Definition 3 [5–7] Let K be a nonempty convex subset of X and $T : K \to K$ be an operator. Assume that $x_o \in K$ and $x_{n+1} = f(T, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset K$. Suppose, furthermore, that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T) \neq \emptyset$. Let $\{y_n\}_{n=0}^{\infty}$ be any bounded sequence in K and put $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$.

(1) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable on K if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$.
(2) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be almost T-stable on K if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

It is easy to verify that an iteration scheme $\{x_n\}_{n=0}^{\infty}$ which is T-stable on K is almost T-stable on K.

Lemma 4 [3] *Let X be a smooth Banach space. Suppose one of the following holds*:
(1) *J is uniformly continuous on any bounded subsets of X,*
(2) *$\langle x - y, j(x) - j(y)\rangle \le \|x - y\|^2$ for all x, y in X,*
(3) *for any bounded subset D of X, there is a $c : [0, \infty) \to [0, \infty)$ such that*

$$\operatorname{Re}\langle x - y, j(x) - j(y)\rangle \le c(\|x - y\|),$$

for all $x, y \in D$, where c satisfies

$$\lim_{t\to 0^+} \frac{c(t)}{t} = 0. \tag{2.5}$$

Then for any $\epsilon > 0$ and any bounded subset K, there exists $\delta > 0$ such that

$$\|sx + (1 - s)y\|^2 \le (1 - 2s)\|y\|^2 + 2s \operatorname{Re}\langle x, j(y)\rangle + 2s\epsilon \tag{2.6}$$

for all $x, y \in K$ and $s \in [0, \delta]$.

Lemma 5 [10] *Let $T : D(T) \subseteq X \to X$ be an operator with $F(T) \neq \varphi$. Then T is strictly hemicontractive if and only if there exists $t > 1$ such that for all $x \in D(T)$ and $q \in F(T)$, there exists $j(x - q) \in J(x - q)$ satisfying*

$$\operatorname{Re}\langle x - Tx, j(x - q)\rangle \ge \left(1 - \frac{1}{t}\right)\|x - q\|^2. \tag{2.7}$$

Lemma 6 [4] *Let X be an arbitrary normed linear space and $T : D(T) \subseteq X \to X$ be an operator.*
(1) *If T is a local strongly pseudocontractive operator and $F(T) \neq \emptyset$, then $F(T)$ is a singleton and T is strictly hemicontractive.*
(2) *If T is strictly hemicontractive, then $F(T)$is a singleton.*

3 Main results

We now prove our main results.

Lemma 7 *Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be nonnegative real sequences, and let $\epsilon' > 0$ be a constant satisfying*

$$\beta_{n+1} \le (1 - \alpha_n^l)\beta_n + \epsilon'\alpha_n + \gamma_n; \quad l \ge 1, n \ge 0,$$

where $\sum_{n=0}^{\infty} \alpha_n^l = \infty$, $\alpha_n \le 1$ for all $n \ge 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then, $\lim_{n\to\infty} \sup \beta_n \le \epsilon'$.

Proof By a straightforward argument, for $n \geq k \geq 0$,

$$\beta_{n+1} \leq \beta_k \prod_{j=k}^{n} (1 - \alpha_j^l) + \epsilon' \sum_{j=k}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_i^l) + \sum_{j=k}^{n} \gamma_j \prod_{i=j+1}^{n} (1 - \alpha_i^l), \tag{3.1}$$

where we put $\prod_{i=n+1}^{n}(1 - \alpha_i^l) = 1$. Note that $\sum_{j=k}^{n} \alpha_j \prod_{i=j+1}^{n}(1 - \alpha_i^l) \leq 1$. It follows from (3.1) that

$$\beta_{n+1} \leq \exp\left(-\sum_{j=k}^{n} \alpha_j^l\right)\beta_k + \epsilon' + \sum_{j=k}^{n} \gamma_j. \tag{3.2}$$

For a given $\delta > 0$, there exists a positive integer k such that $\sum_{j=k}^{\infty} \gamma_j < \delta$. Thus (3.2) ensures that

$$\lim_{n\to\infty} \sup \beta_n \leq \epsilon' + \delta.$$

Letting $\delta \to 0^+$ yields $\lim_{n\to\infty} \sup \beta_n \leq \epsilon'$. □

Remark 8

(i) If $\gamma_n = 0$ for each $n \geq 0$, then Lemma 7 reduces to Lemma 1 of Park [3].
(ii) If $l = 1$, then Lemma 7 reduces to Lemma 2.1 of Liu *et al.* [4].

Theorem 9 *Let Xbe a smooth Banach space satisfying any one of the Axioms* (1)-(3) *of Lemma* 4. *Let* K *be a nonempty closed bounded convex subset of* X *and* $T : K \to K$ *be a continuous strictly hemicontractive mapping. Suppose that* $\{u_n\}_{n=0}^{\infty}$ *is an arbitrary sequence in* K *and* $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are any sequences in* $[0,1]$ *satisfying conditions* (i) $a'_n + b'_n + c'_n = 1$, (ii) $c'_n = o(b'_n)$, (iii) $\lim_{n\to\infty} b'_n = 0$ *and* (iv) $\sum_{n=0}^{\infty} b'_n = \infty$.

For a sequence $\{v_n\}_{n=0}^{\infty}$ *in* K, *suppose that* $\{x_n\}_{n=0}^{\infty}$ *is the sequence generated from an arbitrary* $x_0 \in K$ *by*

$$x_{n+1} = a'_n x_n + b'_n T v_n + c'_n u_n, \quad n \geq 0, \tag{3.3}$$

and satisfying $\lim_{n\to\infty} \|v_n - x_n\| = 0$.

Let $\{y_n\}_{n=0}^{\infty}$ *be any sequence in* K *and define* $\{\varepsilon_n\}_{n=0}^{\infty}$ *by*

$$\varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0, \tag{3.4}$$

where $p_n = a'_n y_n + b'_n T v_n + c'_n u_n$, *such that* $\lim_{n\to\infty} \|v_n - y_n\| = 0$.

Then

(a) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a unique fixed point* q *of* T,
(b) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that* $\{x_n\}_{n=0}^{\infty}$ *is almost* T*-stable on* K,
(c) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof From (ii), we have $c'_n = t_n b'_n$, where $t_n \to 0$ as $n \to \infty$.

It follows from Lemma 6 that $F(T)$ is a singleton. That is, $F(T) = \{q\}$ for some $q \in K$.

Set $M = 1 + \operatorname{diam} K$. For all $n \geq 0$, it is easy to verify that

$$\begin{aligned} M = {} & \sup_{n\geq 0}\|x_n - q\| + \sup_{n\geq 0}\|Tv_n - q\| + \sup_{n\geq 0}\|u_n - q\| \\ & + \sup_{n\geq 0}\bigl\{\|p_n - q\|\bigr\} + \sup_{n\geq 0}\|y_n - q\|. \end{aligned} \tag{3.5}$$

For given any $\epsilon > 0$ and the bounded subset K, there exists a $\delta > 0$ satisfying (2.6). Note that (ii), (iii), $\lim_{n\to\infty}\|v_n - x_n\| = 0$ and the continuity of T ensure that there exists an N such that

$$b'_n < \min\left\{\delta, \frac{1}{2(1-k)}\right\}, \qquad t_n \leq \frac{\epsilon}{16M^2}, \qquad \|Tv_n - Tx_n\| \leq \frac{\epsilon}{4M}, \quad n \geq N, \tag{3.6}$$

where $k = \frac{1}{t}$ and t satisfies (2.7). Using (3.3) and Lemma 4, we infer that

$$\begin{aligned} \|x_{n+1} - q\|^2 = {} & \bigl\|\bigl(1 - b'_n\bigr)(x_n - q) + b'_n(Tv_n - q) + c'_n(u_n - x_n)\bigr\|^2 \\ \leq {} & \bigl(\bigl\|\bigl(1 - b'_n\bigr)(x_n - q) + b'_n(Tv_n - q)\bigr\| + 2Mc'_n\bigr)^2 \\ \leq {} & \bigl\|\bigl(1 - b'_n\bigr)(x_n - q) + b'_n(Tv_n - q)\bigr\|^2 + 8M^2c'_n \\ \leq {} & \bigl(1 - 2b'_n\bigr)\|x_n - q\|^2 + 2b'_n \operatorname{Re}\bigl(Tv_n - q, j(x_n - q)\bigr) + 2\epsilon b'_n + 8M^2c'_n \\ = {} & \bigl(1 - 2b'_n\bigr)\|x_n - q\|^2 + 2b'_n \operatorname{Re}\bigl(Tx_n - q, j(x_n - q)\bigr) \\ & + 2b'_n \operatorname{Re}\bigl(Tv_n - Tx_n, j(x_n - q)\bigr) + 2\epsilon b'_n + 8M^2c'_n \\ \leq {} & \bigl(1 - 2b'_n\bigr)\|x_n - q\|^2 + 2kb'_n\|x_n - q\|^2 \\ & + 2b'_n\|Tv_n - Tx_n\|\|x_n - q\| + 2\epsilon b'_n + 8M^2c'_n \\ \leq {} & \bigl(1 - 2(1-k)b'_n\bigr)\|x_n - q\|^2 \\ & + 2Mb'_n\|Tv_n - Tx_n\| + 2\epsilon b'_n + 8M^2c'_n \\ \leq {} & \bigl(1 - 2(1-k)b'_n\bigr)\|x_n - q\|^2 + 3\epsilon b'_n, \end{aligned} \tag{3.7}$$

for all $n \geq N$.

Put

$$\begin{aligned} &\beta_n = \|x_n - q\|, \\ &\alpha_n = 2(1-k)b'_n, \\ &\epsilon' = \frac{3\epsilon}{2(1-k)}, \\ &\gamma_n = 0, \end{aligned}$$

we have from (3.7)

$$\beta_{n+1} \leq (1 - \alpha_n)\beta_n + \epsilon'\alpha_n + \gamma_n, \quad n \geq 0.$$

Observe that $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\alpha_n < 1$ for all $n \geq 0$. It follows from Lemma 7 that

$$\lim_{n\to\infty}\sup\|x_n - q\|^2 \leq \epsilon'.$$

Letting $\epsilon' \to 0^+$, we obtain that $\lim_{n\to\infty} \sup \|x_n - q\|^2 = 0$, which implies that $x_n \to q$ as $n \to \infty$.

On the same lines, we obtain

$$\begin{aligned}
\|p_n - q\|^2 &= \left\| \left(1 - b'_n\right)(y_n - q) + b'_n(Tv_n - q) + c'_n(u_n - y_n) \right\|^2 \\
&\le \left(\left\| \left(1 - b'_n\right)(y_n - q) + b'_n(Tv_n - q) \right\| + 2Mc'_n \right)^2 \\
&\le \left\| \left(1 - b'_n\right)(y_n - q) + b'_n(Tv_n - q) \right\|^2 + 8M^2 c'_n \\
&\le \left(1 - 2b'_n\right)\|y_n - q\|^2 + 2b'_n \operatorname{Re}\left(Tv_n - q, j(y_n - q)\right) + 2\epsilon b'_n + 8M^2 c'_n \\
&= \left(1 - 2b'_n\right)\|y_n - q\|^2 + 2b'_n \operatorname{Re}\left(Ty_n - q, j(y_n - q)\right) \\
&\quad + 2b'_n \operatorname{Re}\left(Tv_n - Ty_n, j(y_n - q)\right) + 2\epsilon b'_n + 8M^2 c'_n \\
&\le \left(1 - 2b'_n\right)\|y_n - q\|^2 + 2kb'_n\|y_n - q\|^2 \\
&\quad + 2b'_n\|Tv_n - Ty_n\|\|y_n - q\| + 2\epsilon b'_n + 8M^2 c'_n \\
&\le \left(1 - 2(1-k)b'_n\right)\|y_n - q\|^2 \\
&\quad + 2Mb'_n\|Tv_n - Ty_n\| + 2\epsilon b'_n + 8M^2 c'_n \\
&\le \left(1 - 2(1-k)b'_n\right)\|y_n - q\|^2 + 3\epsilon b'_n,
\end{aligned} \tag{3.8}$$

for all $n \ge N$.

Suppose that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. In view of (3.4) and (3.8), we infer that

$$\begin{aligned}
\|y_{n+1} - q\|^2 &\le \left(\|y_{n+1} - p_n\| + \|p_n - q\|\right)^2 \\
&\le \|p_n - q\|^2 + 3M\varepsilon_n \\
&\le \left[1 - 2b'_n(1-k)\right]\|y_n - q\|^2 + 3\epsilon b'_n + 3M\varepsilon_n,
\end{aligned} \tag{3.9}$$

for all $n \ge N$.

Now, put

$$\begin{aligned}
&\beta_n = \|y_n - q\|, \\
&\alpha_n = 2(1-k)b'_n, \\
&\epsilon' = \frac{3\epsilon}{2(1-k)}, \\
&\gamma_n = 3M\varepsilon_n,
\end{aligned}$$

and we have from (3.9)

$$\beta_{n+1} \le (1 - \alpha_n)\beta_n + \epsilon'\alpha_n + \gamma_n, \quad n \ge 0.$$

Observe that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n < 1$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$ for all $n \ge 0$. It follows from Lemma 7 that

$$\lim_{n\to\infty} \sup \|y_n - q\|^2 \le \epsilon'.$$

Letting $\epsilon' \to 0^+$, we obtain that $\lim_{n\to\infty}\sup\|y_n - q\|^2 = 0$, which implies that $y_n \to q$ as $n \to \infty$.

Conversely, suppose that $\lim_{n\to\infty} y_n = q$, then (iii) and (3.8) imply that

$$\begin{aligned}\varepsilon_n &\leq \|y_{n+1} - q\| + \|p_n - q\| \\ &\leq \|y_{n+1} - q\| + \big[\big[1 - 2(1-k)b'_n\big]\|y_n - q\|^2 + 3\epsilon b'_n\big]^{\frac{1}{2}} \\ &\to 0,\end{aligned}$$

as $n \to \infty$, that is, $\varepsilon_n \to 0$ as $n \to \infty$. □

Using the methods of the proof of Theorem 9, we can easily prove the following.

Theorem 10 *Let X, K, T and $\{u_n\}_{n=0}^{\infty}$, be as in Theorem* 9. *Suppose that $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are sequences in* $[0,1]$ *satisfying conditions* (i), (iii)-(iv) *and*

$$\sum_{n=0}^{\infty} c'_n < \infty.$$

If $\{x_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ are as in Theorem 9, *then the conclusions of Theorem* 9 *hold.*

Corollary 11 *Let X be a smooth Banach space satisfying any one of the Axioms* (1)-(3) *of Lemma* 4. *Let K be a nonempty closed bounded convex subset of X and $T : K \to K$ be a Lipschitz strictly hemicontractive mapping. Suppose that $\{u_n\}_{n=0}^{\infty}$ is an arbitrary sequence in K and $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are any sequences in* $[0,1]$ *satisfying conditions* (i) $a'_n + b'_n + c'_n = 1$, (ii) $c'_n = o(b'_n)$, (iii) $\lim_{n\to\infty} b'_n = 0$ *and* (iv) $\sum_{n=0}^{\infty} b'_n = \infty$.

For a sequence $\{v_n\}_{n=0}^{\infty}$ in K, suppose that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = a'_n x_n + b'_n T v_n + c'_n u_n, \quad n \geq 0,$$

and satisfying $\lim_{n\to\infty}\|v_n - x_n\| = 0$.

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in K and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$\varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0,$$

where $p_n = a'_n y_n + b'_n T v_n + c'_n u_n$, such that $\lim_{n\to\infty}\|v_n - y_n\| = 0$.

Then

(a) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point q of T,*
(b) $\sum_{n=0}^{\infty}\varepsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that $\{x_n\}_{n=0}^{\infty}$ is almost T-stable on K,*
(c) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty}\varepsilon_n = 0$.

Corollary 12 *Let X, K, T and $\{u_n\}_{n=0}^{\infty}$ be as in Corollary* 11. *Suppose that $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are sequences in* $[0,1]$ *satisfying conditions* (i), (iii)-(iv) *and*

$$\sum_{n=0}^{\infty} c'_n < \infty.$$

If $\{x_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$ *and* $\{\varepsilon_n\}_{n=0}^{\infty}$ *are as in Corollary* 11, *then the conclusions of Corollary* 11 *hold*.

Corollary 13 *Let* X *be a smooth Banach space satisfying any one of the Axioms* (1)-(3) *of Lemma* 4. *Let* K *be a nonempty closed bounded convex subset of* X *and* $T : K \to K$ *be a continuous strictly hemicontractive mapping. Suppose that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0,1]$ *satisfying conditions* (i) $\lim_{n\to\infty} \alpha_n = 0$ *and* (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For a sequence $\{v_n\}_{n=0}^{\infty}$ *in* K, *suppose that* $\{x_n\}_{n=0}^{\infty}$ *is the sequence generated from an arbitrary* $x_0 \in K$ *by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tv_n, \quad n \geq 0,$$

and satisfying $\lim_{n\to\infty} \|v_n - x_n\| = 0$.

Let $\{y_n\}_{n=0}^{\infty}$ *be any sequence in* K *and define* $\{\varepsilon_n\}_{n=0}^{\infty}$ *by*

$$\varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0,$$

where $p_n = \alpha_n y_n + (1 - \alpha_n)Tv_n$, *such that* $\lim_{n\to\infty} \|v_n - y_n\| = 0$.

Then

(a) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a unique fixed point* q *of* T,
(b) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that* $\{x_n\}_{n=0}^{\infty}$ *is almost* T*-stable on* K,
(c) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Corollary 14 *Let* X *be a smooth Banach space satisfying any one of the Axioms* (1)-(3) *of Lemma* 4. *Let* K *be a nonempty closed bounded convex subset of* X *and* $T : K \to K$ *be a Lipschitz strictly hemicontractive mapping. Suppose that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0,1]$ *satisfying conditions* (i) $\lim_{n\to\infty} \alpha_n = 0$ *and* (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For a sequence $\{v_n\}_{n=0}^{\infty}$ *in* K, *suppose that* $\{x_n\}_{n=0}^{\infty}$ *is the sequence generated from an arbitrary* $x_0 \in K$ *by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tv_n, \quad n \geq 0,$$

and satisfying $\lim_{n\to\infty} \|v_n - x_n\| = 0$.

Let $\{y_n\}_{n=0}^{\infty}$ *be any sequence in* K *and define* $\{\varepsilon_n\}_{n=0}^{\infty}$ *by*

$$\varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0,$$

where $p_n = \alpha_n y_n + (1 - \alpha_n)Tv_n$, *such that* $\lim_{n\to\infty} \|v_n - y_n\| = 0$.

Then

(a) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a unique fixed point* q *of* T,
(b) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that* $\{x_n\}_{n=0}^{\infty}$ *is almost* T*-stable on* K,
(c) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

4 Applications to a multi-step iteration process

Khan *et al.* [23] have introduced and studied a multi-step iteration process for a finite family of selfmappings. We now introduce a modified multi-step process as follows:

Let K be a nonempty closed convex subset of a real normed space E and $T_1, T_2, \ldots, T_p : K \to K$ ($p \geq 2$) be a family of selfmappings.

Algorithm 1 For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n\geq 0}$ by the iteration process of arbitrary fixed order $p \geq 2$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n^1, \\ y_n^i &= \left(1-\beta_n^i\right)x_n + \beta_n^i T_{i+1} y_n^{i+1}; \quad i = 1,2,\ldots,p-2, \\ y_n^{p-1} &= \left(1-\beta_n^{p-1}\right)x_n + \beta_n^{p-1} T_p x_n, \quad n \geq 0, \end{aligned} \tag{4.1}$$

which is called the modified multi-step iteration process, where $\{\alpha_n\}_{n\geq 0}, \{\beta_n^i\}_{n\geq 0} \subset [0,1]$, $i = 1,2,\ldots,p-1$.

For $p = 3$, we obtain the following three-step iteration process:

Algorithm 2 For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n\geq 0}$ by the iteration process:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n^1, \\ y_n^1 &= \left(1-\beta_n^1\right)x_n + \beta_n^1 T_2 y_n^2, \\ y_n^2 &= \left(1-\beta_n^2\right)x_n + \beta_n^2 T_3 x_n, \quad n \geq 0, \end{aligned} \tag{4.2}$$

where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n^1\}_{n\geq 0}$ and $\{\beta_n^2\}_{n\geq 0}$ are three real sequences in $[0,1]$.

For $p = 2$, we obtain the Ishikawa [24] iteration process:

Algorithm 3 For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n\geq 0}$ by the iteration process

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n^1, \\ y_n^1 &= \left(1-\beta_n^1\right)x_n + \beta_n^1 T_2 x_n, \quad n \geq 0, \end{aligned} \tag{4.3}$$

where $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n^1\}_{n\geq 0}$ are two real sequences in $[0,1]$.

If $T_1 = T$, $T_2 = I$, $\beta_n^1 = 0$ in (4.3), we obtain the Mann iteration process [14]:

Algorithm 4 For any given $x_0 \in K$, compute the sequence $\{x_n\}_{n\geq 0}$ by the iteration process

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{4.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Theorem 15 *Let K be a nonempty closed bounded convex subset of a smooth Banach space X and $T_1, T_2, \ldots, T_p$ $(p \geq 2)$ be selfmappings of K. Let T_1 be a continuous strictly hemicontractive mapping. Let $\{\alpha_n\}_{n\geq 0}, \{\beta_n^i\}_{n\geq 0} \subset [0,1]$, $i = 1,2,\ldots,p-1$ be real sequences in $[0,1]$ satisfying $\sum_{n\geq 0} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n^1 = 0$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n\geq 0}$ by (4.1). Then $\{x_n\}_{n\geq 0}$ converges strongly to a point in $\bigcap_{i=1}^p F(T_i) \neq \emptyset$.*

Proof By applying Corollary 13 under assumption that T_1 is continuous strictly hemicontractive mapping, we obtain Theorem 15 which proves strong convergence of the iteration process defined by (4.1). We will check only the condition $\lim_{n\to\infty}\|v_n - x_n\| = 0$ by taking $T_1 = T$ and $v_n = y_n^1$,

$$\begin{aligned}\|v_n - x_n\| &= \|y_n^1 - x_n\| \\ &= \|(1-\beta_n^1)x_n + \beta_n^1 T_2 y_n^2 - x_n\| \\ &= \beta_n^1 \|T_2 y_n^2 - x_n\| \\ &\le 2M\beta_n^1.\end{aligned}$$

Now, from the condition $\lim_{n\to\infty}\beta_n^1 = 0$, it can be easily seen that $\lim_{n\to\infty}\|v_n - x_n\| = 0$. □

Corollary 16 *Let K be a nonempty closed bounded convex subset of a smooth Banach space X and $T_1, T_2, \dots, T_p$ $(p \ge 2)$ be selfmappings of K. Let T_1 be a Lipschitz strictly hemicontractive mapping. Let $\{\alpha_n\}_{n\ge 0}, \{\beta_n^i\}_{n\ge 0} \subset [0,1]$, $i = 1, 2, \dots, p-1$ be real sequences in $[0,1]$ satisfying $\sum_{n\ge 0}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n^1 = 0$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n\ge 0}$ by (4.1). Then $\{x_n\}_{n\ge 0}$ converges strongly to a point in $\bigcap_{i=1}^p F(T_i) \neq \emptyset$.*

Remark 17 Similar results can be found for the iteration processes with error terms, we omit the details.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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