Xu and Rassias *Journal of Inequalities and Applications* 2012, **2012**:204
http://www.journalofinequalitiesandapplications.com/content/2012/1/204

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# A fixed point approach to the Hyers-Ulam stability of an AQ functional equation on $\beta$-Banach modules

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**Abstract**

In this paper, we establish the general solution and investigate the generalized Hyers-Ulam stability of the following mixed additive and quadratic functional equation: $f(kx+ly)+f(kx-ly)=f(kx)+f(x)+\frac{1}{2}(k-1)[(k+2)f(x)+kf(-x)]+l^2[f(y)+f(-y)]$ $(k,l\in\mathbb{Z}\backslash\{0\})$ in $\beta$-Banach modules on a Banach algebra. In addition, we show that under some suitable conditions, an approximately mixed additive-quadratic function can be approximated by a mixed additive and quadratic mapping.

**MSC:** 39B82; 39B52; 46H25

**Keywords:** Hyers-Ulam stability; additive and quadratic equation; $\beta$-Banach module; fixed point method

## 1 Introduction and preliminaries

Starting with the article of Hyers [1] and continuing with those of Rassias [2], Găvruța [3], Czervick [4] and so on, the authors used the 'direct method' to prove the stability properties for functional equations. Namely, the exact solution of the functional equation is explicitly constructed as a limit of a sequence starting from the given approximate solution.

On the other hand, Baker [5] used the Banach fixed point theorem to give a Hyers-Ulam stability result for a nonlinear functional equation in a single variable. In 2002, Cădariu and Radu delivered a lecture entitled 'On the stability of the Cauchy functional equation: a fixed point approach' in 'The 14th European Conference on Iteration Theory - ECIT 2002, Evora, Portugal, 2002'. Their idea was to obtain, in $\beta$-normed spaces, the existence of the exact solution and the error estimations by using the fixed point alternative theorem [6]. This new method was used in two successive papers [7, 8] in 2003 to obtain the properties of generalized Hyers-Ulam stability for Jensen's functional equation. Also, the lecture from ECIT 2002 was materialized in [9]. After that, a lot of papers used the 'fixed point alternative' to obtain generalized Hyers-Ulam stability results for different functional equations in various spaces. The reader is referred to the following books and research papers which provide an extensive account of the progress made on Ulam's problem during the last seventy years (see, for instance, [10–25]).

The motivation of this paper is to present a new mixed additive and quadratic ('AQ' for short) functional equation. We obtain the general solution of the AQ-functional equation.

Moreover, we prove the generalized Hyers-Ulam stability of the AQ-functional equation in Banach modules on a Banach algebra using the fixed point method.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric bi-additive function [20]. It is natural that such an equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function $B$ such that $f(x)=B(x,x)$ for all $x$ (see [20]). The bi-additive function $B$ is given by $B(x,y)=\frac{1}{4}(f(x+y)+f(x-y))$. In [4], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation (1.1). A Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f: E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space (see [26]). Cholewa [27] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. Grabiec in [28] has generalized the above mentioned results. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces (see, for instance, [10, 17, 18, 21, 29]).

Now, we consider a mapping $f: X \to Y$ that satisfies the following general mixed additive and quadratic functional equation:

$$\begin{aligned} f(kx+ly)+f(kx-ly) &= f(kx)+f(x)+\frac{1}{2}(k-1)\big[(k+2)f(x)+kf(-x)\big] \\ &\quad + l^2\big[f(y)+f(-y)\big], \end{aligned} \tag{1.2}$$

where $k,l \in \mathbb{Z}\backslash\{0\}$. It is easy to see that the function $f(x)=ax^2+bx$ is a solution of the functional equation (1.2).

Let $\beta$ be a real number with $0<\beta \le 1$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a linear space over $\mathbb{K}$. A real-valued function $\|\cdot\|_\beta$ is called a $\beta$-norm on $X$ if and only if it satisfies

($\beta N1$) $\|x\|_\beta = 0$ if and only if $x=0$;
($\beta N2$) $\|\lambda x\|_\beta = |\lambda|^\beta \cdot \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;
($\beta N3$) $\|x+y\|_\beta \le \|x\|_\beta + \|y\|_\beta$ for all $x,y \in X$.

The pair $(X, \|\cdot\|_\beta)$ is called a $\beta$-normed space (see [30]). A $\beta$-Banach space is a complete $\beta$-normed space.

For explicit later use, we recall the following result by Diaz and Margolis [6].

**Theorem 1.1** *Let $(\Omega, d)$ be a complete generalized metric space and $J: \Omega \to \Omega$ be a strictly contractive mapping with the Lipschitz constant $L<1$, that is,*

$$d(Jx,Jy) \le Ld(x,y), \quad \forall x,y \in \Omega.$$

*Then for each given $x \in \Omega$, either*

$$d\big(J^n x, J^{n+1}x\big) = \infty, \quad \forall n \ge 0,$$

*or there exists a non-negative integer $n_0$ such that*

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* $y^*$ *of* $J$;
(3) $y^*$ *is the unique fixed point of* $J$ *in the set* $\Omega^* = \{y \in \Omega \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in \Omega^*$.

## 2 General solution

Throughout this section, $X$ and $Y$ will be real vector spaces. Before proceeding with the proof of Theorem 2.3, which is the main result in this section, we shall need the following two lemmas:

**Lemma 2.1** *If an odd mapping* $f : X \to Y$ *satisfies* (1.2) *for all* $x, y \in X$, *then* $f$ *is additive.*

*Proof* Since $f$ satisfies the functional equation (1.2), putting $x = y = 0$ in (1.2), we get $(1 - k^2 - 2l^2)f(0) = 0$, or $f(0) = 0$ if $k^2 + 2l^2 \neq 1$. Note that, in view of the oddness of $f$, we have $f(-x) = -f(x)$ for all $x \in X$. If $x = 0$ in $f(-x) = -f(x)$, one gets $f(0) = -f(0)$, or $f(0) = 0$ without assuming the condition $k^2 + 2l^2 \neq 1$. Yet, it is assumed that $f$ is odd. Hence, (1.2) implies the following equation:

$$f(kx + ly) + f(kx - ly) = f(kx) + kf(x) \tag{2.1}$$

for all $x, y \in X$. Letting $y = 0$ in (2.1), we get

$$f(kx) = kf(x) \tag{2.2}$$

for all $x \in X$. Replacing $x$ and $y$ by $lx$ and $kx$, respectively, in (2.1) and using (2.2), we have

$$f(2lx) = 2f(lx) \tag{2.3}$$

for all $x \in X$. Replacing $y$ by $\frac{x}{l}$ in (2.3), we get

$$f(2x) = 2f(x) \tag{2.4}$$

for all $x \in X$. Replacing $y$ by $\frac{ky}{l}$ in (2.1) and using (2.2), we obtain

$$f(x + y) + f(x - y) = 2f(x) \tag{2.5}$$

for all $x, y \in X$. Replacing $x$ and $y$ by $\frac{x+y}{2}$ and $\frac{x-y}{2}$, respectively, in (2.5) and using (2.4), we get

$$f(x + y) = f(x) + f(y) \tag{2.6}$$

for all $x, y \in X$. Therefore, the mapping $f : X \to Y$ is additive. □

**Lemma 2.2** *If an even mapping* $f : X \to Y$ *satisfies* (1.2) *for all* $x, y \in X$, *then* $f$ *is quadratic.*

*Proof* Since $f$ satisfies the functional equation (1.2), putting $x = y = 0$ in (1.2), we get $f(0) = 0$. Note that, in view of the evenness of $f$, we have $f(-x) = f(x)$ for all $x \in X$. Hence, (1.2) implies the following equation:

$$f(kx + ly) + f(kx - ly) = f(kx) + k^2 f(x) + 2l^2 f(y) \tag{2.7}$$

for all $x, y \in X$. Letting $y = 0$ in (2.7) and using $f(0) = 0$, we get

$$f(kx) = k^2 f(x) \tag{2.8}$$

for all $x \in X$. Replacing $x$ and $y$ by 0 and $x$, respectively, in (2.7) and using $f(0) = 0$, we have

$$f(lx) = l^2 f(x) \tag{2.9}$$

for all $x \in X$. Replacing $x$ by $lx$ in (2.7) and using (2.9), we get

$$f(kx + y) + f(kx - y) = f(kx) + k^2 f(x) + 2f(y) \tag{2.10}$$

for all $x \in X$. Replacing $y$ by $ky$ in (2.10) and using (2.10), we obtain

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.11}$$

for all $x, y \in X$. Therefore, the function $f : X \to Y$ is quadratic. □

Now, we are ready to find the general solution of (1.2).

**Theorem 2.3** *A mapping $f : X \to Y$ satisfies* (1.2) *for all $x, y \in X$ if and only if there exist a symmetric bi-additive mapping $B : X \times X \to Y$ and an additive mapping $A : X \to Y$ such that $f(x) = B(x, x) + A(x)$ for all $x \in X$.*

*Proof* If there exist a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x, x) + A(x)$ for all $x \in X$, it is easy to show that

$$\begin{aligned} f(kx + ly) + f(kx - ly) &= 2k^2 B(x, x) + 2l^2 B(y, y) + 2kA(x) \\ &= f(kx) + f(x) + \frac{1}{2}(k - 1)\big[(k + 2)f(x) + kf(-x)\big] + l^2\big[f(y) + f(-y)\big] \end{aligned}$$

for all $x, y \in X$. Therefore, the function $f : X \to Y$ satisfies (1.2).

Conversely, we decompose $f$ into the odd part and the even part by putting

$$f_o(x) = \frac{f(x) - f(-x)}{2} \quad \text{and} \quad f_e(x) = \frac{f(x) + f(-x)}{2}$$

for all $x \in X$. It is clear that $f(x) = f_o(x) + f_e(x)$ for all $x \in X$. It is easy to show that the functions $f_o$ and $f_e$ satisfy (1.2). Hence, by Lemmas 2.1 and 2.2, we achieve that the functions $f_o$ and $f_e$ are additive and quadratic, respectively. Therefore, there exists a symmetric bi-additive function $B : X \times X \to Y$ such that $f_e(x) = B(x, x)$ for all $x \in X$ (see [10]). So,

$$f(x) = B(x, x) + A(x)$$

for all $x \in X$, where $A(x) = f_o(x)$ for all $x \in X$. □

## 3 Approximate mixed additive and quadratic mappings

In this section, we prove the generalized Hyers-Ulam stability of the mixed additive and quadratic functional equation (1.2) using the fixed point method introduced by Radu in [7] (see also [8, 9, 31–35]).

Throughout this section, let $B$ be a unital Banach algebra with norm $\|\cdot\|_B$, $B_1 = \{b \in B \mid \|b\|_B = 1\}$, $X$ be a $\beta$-normed left $B$-module and $Y$ be a $\beta$-normed left Banach $B$-module, and let $k, l \in \mathbb{Z}\backslash\{0\}$ be fixed integers. For a given mapping $f : X \to Y$, we define the difference operators

$$\begin{aligned} D_b f(x,y) &= f(kbx + lby) + f(kbx - lby) - bf(kx) - bf(x) \\ &\quad - \frac{1}{2}(k-1)b\big[(k+2)f(x) + kf(-x)\big] - l^2 b\big[f(y) + f(-y)\big] \end{aligned}$$

and

$$\begin{aligned} \widetilde{D}_b f(x,y) &= f(kbx + lby) + f(kbx - lby) - b^2 f(kx) - b^2 f(x) \\ &\quad - \frac{1}{2}(k-1)b^2\big[(k+2)f(x) + kf(-x)\big] - l^2 b^2\big[f(y) + f(-y)\big] \end{aligned}$$

for all $x, y \in X$ and $b \in B_1$.

**Theorem 3.1** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{|k|^{n\beta}} \varphi\big(k^n x, k^n y\big) = 0 \tag{3.1}$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping such that*

$$\big\|D_b f(x,y)\big\|_\beta \le \varphi(x,y) \tag{3.2}$$

*for all $x, y \in X$ and all $b \in B_1$. If there exists a Lipschitz constant $0 < L < 1$ such that*

$$\varphi(kx, 0) \le |k|^\beta L\varphi(x, 0) \tag{3.3}$$

*for all $x \in X$, then there exists a unique additive mapping $A : X \to Y$ such that*

$$\big\|f(x) - A(x)\big\|_\beta \le \frac{1}{|k|^\beta(1-L)} \varphi(x, 0) \tag{3.4}$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear, i.e., $A(bx) = bA(x)$ for all $x \in X$ and all $b \in B$.*

*Proof* Letting $b = 1$ and $y = 0$ in (3.2), we get

$$\big\|f(kx) - kf(x)\big\|_\beta \le \varphi(x, 0) \tag{3.5}$$

for all $x \in X$. Consider the set $\Omega := \{g \mid g : X \to Y, g(0) = 0\}$ and introduce the generalized metric on $\Omega$:

$$d(g,h) = \inf\big\{C \in [0,\infty) \mid \big\|g(x) - h(x)\big\|_\beta \le C\varphi(x,0), \forall x \in X\big\}. \tag{3.6}$$

It is easy to show that $(\Omega, d)$ is a complete generalized metric space (see the Theorem 2.1 of [13]). We now define a function $J : \Omega \to \Omega$ by

$$(Jg)(x) = \frac{1}{k} g(kx), \quad \forall g \in \Omega, x \in X. \tag{3.7}$$

Let $g, h \in \Omega$ and $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) < C$; by the definition of $d$, it follows

$$\left\| g(x) - h(x) \right\|_\beta \le C\varphi(x, 0), \quad \forall x \in X. \tag{3.8}$$

By the given hypothesis and the last inequality, one has

$$\left\| \frac{1}{k} g(kx) - \frac{1}{k} h(kx) \right\|_\beta \le CL\varphi(x, 0), \quad \forall x \in X. \tag{3.9}$$

Hence, it holds that $d(Jg, Jh) \le Ld(g, h)$. It follows from (3.5) that $d(Jf, f) \le \frac{1}{|k|^\beta} < \infty$. Therefore, by Theorem 1.1, $J$ has a unique fixed point $A : X \to Y$ in the set $\Omega^* = \{g \in \Omega \mid d(f, g) < \infty\}$ such that

$$A(x) := \lim_{n\to\infty} \left(J^n f\right)(x) = \lim_{n\to\infty} \frac{1}{k^n} f\left(k^n x\right) \tag{3.10}$$

and $A(kx) = kA(x)$ for all $x \in X$. Also,

$$d(A, f) \le \frac{1}{1 - L} d(Jf, f) \le \frac{1}{|k|^\beta (1 - L)}. \tag{3.11}$$

This means that (3.4) holds for all $x \in X$.

Now, we show that $A$ is additive. By (3.1), (3.2) and (3.10), we have

$$\left\| D_1 A(x, y) \right\|_\beta = \lim_{n\to\infty} \frac{1}{|k|^{n\beta}} \left\| D_1 f\left(k^n x, k^n y\right) \right\|_\beta \le \lim_{n\to\infty} \frac{1}{|k|^{n\beta}} \varphi\left(k^n x, k^n y\right) = 0,$$

that is,

$$f(kx + ly) + f(kx - ly) = f(kx) + f(x) + \frac{1}{2}(k - 1)\left[(k + 2)f(x) + kf(-x)\right] + l^2\left[f(y) + f(-y)\right]$$

for all $x, y \in X$. Therefore, by Lemma 2.1, we get that the mapping $A$ is additive. To prove the uniqueness assertion, let us assume that there exists an additive function $T : X \to Y$ which satisfies (3.4). Since $d(f, T) \le 1/[|k|^\beta (1 - L)]$ and $T$ is additive, we get $T \in \Omega^*$ and $(JT)(x) = \frac{1}{k} T(kx) = T(x)$ for all $x \in X$, *i.e.*, $T$ is a fixed point of $J$. Since $A$ is the unique fixed point of $J$ in $\Omega^*$, then $T = A$.

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then, by the same reasoning as in the proof of [2], $A$ is $\mathbb{R}$-linear. Since $A$ is additive, $A(rx) = rA(x)$ for any rational number $r$. Fix $x_0 \in X$ and $\rho \in Y^*$ (the dual space of $Y$). Consider the mapping $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) := \rho(A(tx_0))$, $t \in \mathbb{R}$. Then $\varphi(t + s) = \varphi(t) + \varphi(s)$, $t, s \in \mathbb{R}$, *i.e.* $\varphi$ is a group homomorphism. Moreover, $\varphi$ is a Borel function because of the following reasoning. Let $\varphi(t) = \lim_{n\to\infty} \rho(f(k^n tx_0))/k^n$ and put $\varphi_n(t) = \rho(f(k^n tx_0))/k^n$. Then $\varphi_n(t)$ are continuous functions. But $\varphi(t)$ is the pointwise limit of continuous functions, thus $\varphi(t)$ is a Borel function. It is a known fact that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a function such that $\varphi$ is a group homomorphism, *i.e.* $\varphi(t+s) = \varphi(t) + \varphi(s)$ and $\varphi$ is a measurable function, then $\varphi$ is continuous. Therefore, $\varphi(t)$ is a continuous function. Let $a \in \mathbb{R}$. Then $a = \lim_{n\to\infty} r_n$, where $\{r_n\}$ is a sequence of rational numbers. Thus

$$\varphi(at) = \varphi\Big(t \lim_{n\to\infty} r_n\Big) = \lim_{n\to\infty} \varphi(tr_n) = \Big(\lim_{n\to\infty} r_n\Big)\varphi(t) = a\varphi(t).$$

Therefore, $\varphi(at) = a\varphi(t)$ for any $a \in \mathbb{R}$. And then, $A(ax) = aA(x)$ for any $a \in \mathbb{R}$. Hence, the additive mapping $A$ is $\mathbb{R}$-linear.

Letting $y = 0$ in (3.2), we get

$$\big\|2f(kbx) - bf(kx) - kbf(x)\big\|_\beta \le \varphi(x, 0) \tag{3.12}$$

for all $x \in X$ and all $b \in B_1$. By definition of $A$, (3.1) and (3.12), we obtain

$$\begin{aligned}\big\|2A(kbx) - bA(kx) - kbA(x)\big\|_\beta &= \lim_{n\to\infty} \frac{1}{|k|^{n\beta}} \big\|2f\big(k^{n+1}bx\big) - bf\big(k^{n+1}x\big) - kbf\big(k^n x\big)\big\|_\beta \\ &\le \lim_{n\to\infty} \frac{1}{|k|^{n\beta}} \varphi\big(k^n x, 0\big) = 0\end{aligned}$$

for all $x \in X$ and all $b \in B_1$. So, $2A(kbx) - bA(kx) - kbA(x) = 0$ for all $x \in X$ and all $b \in B_1$. Since $A$ is additive, we get $A(bx) = bA(x)$ for all $x \in X$ and all $b \in B_1 \cup \{0\}$. Now, let $a \in B\backslash\{0\}$. Since $A$ is $\mathbb{R}$-linear,

$$\begin{aligned}A(bx) &= A\left(\|b\|_B \cdot \frac{b}{\|b\|_B} x\right) = \|b\|_B \cdot A\left(\frac{b}{\|b\|_B} x\right) \\ &= \|b\|_B \cdot \frac{b}{\|b\|_B} A(x) = bA(x)\end{aligned}$$

for all $x \in X$ and all $b \in B$. This proves that $A$ is $B$-linear. □

**Corollary 3.2** *Let $0 < p < 1$, $\delta, \theta \in [0, \infty)$, and let $f : X \to Y$ be an odd mapping for which*

$$\big\|D_b f(x, y)\big\|_\beta \le \delta + \theta\big(\|x\|_\beta^p + \|y\|_\beta^p\big) \tag{3.13}$$

*for all $x, y \in X$ and $b \in B_1$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\big\|f(x) - A(x)\big\|_\beta \le \frac{1}{|k|^\beta - |k|^{\beta p}}\delta + \frac{1}{|k|^\beta - |k|^{\beta p}}\theta\|x\|_\beta^p$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear.*

*Proof* The proof follows from Theorem 3.1 by taking $\varphi(x, y) = \delta + \theta(\|x\|_\beta^p + \|y\|_\beta^p)$ for all $x, y \in X$. We can choose $L = |k|^{\beta(p-1)}$ to get the desired result. □

The generalized Hyers-Ulam stability problem for the case of $p = 1$ was excluded in Corollary 3.2. In fact, the functional equation (1.2) is not stable for $p = 1$ in (3.13) as we shall see in the following example, which is based on the example given in [36] (see also [34]).

**Example 3.3** Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(x) = \sum_{m=0}^{\infty} \alpha^{-m} \phi\big(\alpha^m x\big)$$

for all $x \in \mathbb{C}$, where $\alpha > \max\{|k|, |l|\}$. Let

$$\begin{aligned} D_b f(x,y) = {} & f(kbx + lby) + f(kbx - lby) - bf(kx) - bf(x) \\ & - \frac{1}{2}(k-1)b\big[(k+2)f(x) + kf(-x)\big] - l^2 b\big[f(y) + f(-y)\big] \end{aligned}$$

for all $x, y \in \mathbb{C}$ and $b \in \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then $f$ satisfies the functional inequality

$$\big|D_b f(x,y)\big| \leq \frac{\alpha^2(k^2 + 3 + 2l^2)}{\alpha - 1}\big(|x| + |y|\big) \tag{3.14}$$

for all $x, y \in \mathbb{C}$, but there do not exist an additive mapping $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - A(x)| \leq d|x|$ for all $x \in \mathbb{C}$.

It is clear that $f$ is bounded by $\frac{\alpha}{\alpha-1}$ on $\mathbb{C}$. If $|x| + |y| = 0$ or $|x| + |y| \geq \frac{1}{\alpha}$, then

$$\big|D_\mu f(x,y)\big| \leq \frac{\alpha^2(k^2 + 3 + 2l^2)}{\alpha - 1}\big(|x| + |y|\big).$$

Now, suppose that $0 < |x| + |y| < \frac{1}{\alpha}$. Then there exists an integer $n \geq 1$ such that

$$\frac{1}{\alpha^{n+1}} \leq |x| + |y| < \frac{1}{\alpha^n}. \tag{3.15}$$

Hence,

$$\alpha^m |kbx \pm lby| < 1, \qquad \alpha^m |kx| < 1, \qquad \alpha^m |x| < 1, \qquad \alpha^m |y| < 1$$

for all $m = 0, 1, \ldots, n-1$. From the definition of $f$ and (3.15), we obtain that

$$\begin{aligned} \big|D_b f(x,y)\big| = {} & \Bigg| \sum_{m=n}^{\infty} \alpha^{-m} \phi\big(\alpha^m (kbx + lby)\big) + \sum_{m=n}^{\infty} \alpha^{-m} \phi\big(\alpha^m (kbx - lby)\big) \\ & - b \sum_{m=n}^{\infty} \alpha^{-m} \phi\big(\alpha^m kx\big) - b \sum_{m=n}^{\infty} \alpha^{-m} \phi\big(\alpha^m x\big) \\ & - \frac{1}{2}(k-1)b \Bigg[ (k+2) \sum_{m=n}^{\infty} \alpha^{-m} \phi\big(\alpha^m x\big) + k \sum_{m=n}^{\infty} \alpha^{-m} \phi\big(-\alpha^m x\big) \Bigg] \end{aligned}$$

$$+ l^2 b\left[\sum_{m=n}^{\infty}\alpha^{-m}\phi\big(\alpha^m y\big) + \sum_{m=n}^{\infty}\alpha^{-m}\phi\big(-\alpha^m y\big)\right]\Bigg|$$
$$\leq \frac{\alpha^2(k^2+3+2l^2)}{\alpha-1}\big(|x|+|y|\big).$$

Therefore, $f$ satisfies (3.14). Now, we claim that the functional equation (1.2) is not stable for $p = 1$ in Corollary 3.2. Suppose, on the contrary, that there exist an additive mapping $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - A(x)| \leq d|x|$ for all $x \in \mathbb{C}$. Then there exists a constant $c \in \mathbb{C}$ such that $A(x) = cx$ for all rational numbers $x$. So, we obtain that

$$\big|f(x)\big| \leq \big(d + |c|\big)|x| \tag{3.16}$$

for all rational numbers $x$. Let $s \in \mathbb{N}$ with $s + 1 > d + |c|$. If $x$ is a rational number in $(0, \alpha^{-s})$, then $\alpha^m x \in (0,1)$ for all $m = 0, 1, \ldots, s$, and for this $x$, we get

$$f(x) = \sum_{m=0}^{\infty}\frac{\phi(\alpha^m x)}{\alpha^m} \geq \sum_{m=0}^{s}\frac{\phi(\alpha^m x)}{\alpha^m} = (s+1)x > \big(d+|c|\big)x,$$

which contradicts (3.16).

**Corollary 3.4** *Let $t, s > 0$ such that $\lambda := t + s < 1$ and $\delta$, $\theta$ be non-negative real numbers, and let $f : X \to Y$ be an odd mapping for which*

$$\big\|D_b f(x,y)\big\|_\beta \leq \delta + \theta\big[\|x\|_\beta^t\|y\|_\beta^s + \big(\|x\|_\beta^\lambda + \|y\|_\beta^\lambda\big)\big]$$

*for all $x, y \in X$ and $b \in B_1$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\big\|f(x) - A(x)\big\|_\beta \leq \frac{1}{|k|^\beta - |k|^{\beta\lambda}}\delta + \frac{1}{|k|^\beta - |k|^{\beta\lambda}}\theta\,\|x\|_\beta^\lambda$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear.*

*Proof* The proof follows from Theorem 3.1 by taking $\varphi(x,y) = \delta + \theta[\|x\|_\beta^r\|y\|_\beta^s + (\|x\|_\beta^\lambda + \|y\|_\beta^\lambda)]$ for all $x, y \in X$. We can choose $L = |k|^{\beta(\lambda-1)}$ to get the desired result. □

The generalized Hyers-Ulam stability problem for the case of $\lambda = 1$ was excluded in Corollary 3.4. Similar to Theorem 3.1, one can obtain the following theorem.

**Theorem 3.5** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty}|k|^{n\beta}\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping such that*

$$\big\|D_b f(x,y)\big\|_\beta \leq \varphi(x,y)$$

*for all $x, y \in X$ and all $b \in B_1$. If there exists a Lipschitz constant $0 < L < 1$ such that $\varphi(x, 0) \leq |k|^{-\beta} L\varphi(kx, 0)$ for all $x \in X$, then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_\beta \leq \frac{L}{|k|^\beta(1 - L)}\varphi(x, 0)$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear.*

As applications of Theorem 3.5, one can get the following Corollaries 3.6 and 3.7.

**Corollary 3.6** *Let $r > 1$ and $\theta$ be a non-negative real number, and let $f : X \to Y$ be an odd mapping such that*

$$\|D_b f(x, y)\|_\beta \leq \theta\left(\|x\|_\beta^r + \|y\|_\beta^r\right)$$

*for all $x, y \in X$ and $b \in B_1$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_\beta \leq \frac{1}{|k|^{\beta r} - |k|^\beta}\theta\|x\|_\beta^r$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear.*

**Corollary 3.7** *Let $t, s > 0$ such that $\lambda := t + s > 1$ and $\theta$ be a non-negative real number, and let $f : X \to Y$ be an odd mapping such that*

$$\|D_b f(x, y)\|_\beta \leq \theta\left[\|x\|_\beta^t\|y\|_\beta^s + \left(\|x\|_\beta^\lambda + \|y\|_\beta^\lambda\right)\right]$$

*for all $x, y \in X$ and $b \in B_1$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_\beta \leq \frac{1}{|k|^{\beta\lambda} - |k|^\beta}\theta\|x\|_\beta^\lambda$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear.*

**Theorem 3.8** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{|k|^{2n\beta}}\varphi\left(k^n x, k^n y\right) = 0 \tag{3.17}$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ such that*

$$\|\widetilde{D}_b f(x, y)\|_\beta \leq \varphi(x, y) \tag{3.18}$$

*for all $x, y \in X$ and all $b \in B_1$. If there exists a Lipschitz constant $0 < L < 1$ such that*

$$\varphi(kx, 0) \leq |k|^{2\beta} L\varphi(x, 0) \tag{3.19}$$

*for all $x \in X$, then there exists a unique quadratic mapping $Q: X \to Y$ such that*

$$\|f(x) - Q(x)\|_\beta \le \frac{1}{|k|^{2\beta}(1-L)}\varphi(x,0) \tag{3.20}$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $Q$ is B-quadratic, i.e., $Q(bx) = b^2Q(x)$ for all $x \in X$ and all $b \in B$.*

*Proof* Letting $b = 1$ and $y = 0$ in (3.18), we get

$$\|f(kx) - k^2 f(x)\|_\beta \le \varphi(x,0) \tag{3.21}$$

for all $x \in X$. Consider the set $\Omega := \{g \mid g: X \to Y, g(0) = 0\}$ and introduce the generalized metric on $\Omega$:

$$d(g,h) = \inf\{C \in [0,\infty) \mid \|g(x) - h(x)\|_\beta \le C\varphi(x,0), \forall x \in X\}.$$

It is easy to show that $(\Omega, d)$ is a complete generalized metric space (see Theorem 2.1 of [13]). We now define a function $J: \Omega \to \Omega$ by

$$(Jg)(x) = \frac{1}{k^2}g(kx), \quad \forall g \in \Omega, x \in X.$$

Let $g, h \in \Omega$ and $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) < C$; by the definition of $d$, it follows

$$\|g(x) - h(x)\|_\beta \le C\varphi(x,0), \quad \forall x \in X.$$

By the given hypothesis and the last inequality, one has

$$\left\|\frac{1}{k^2}g(kx) - \frac{1}{k^2}h(kx)\right\|_\beta \le CL\varphi(x,0), \quad \forall x \in X.$$

Hence, it holds that $d(Jg, Jh) \le Ld(g,h)$. It follows from (3.21) that $d(Jf, f) \le 1/|k|^{2\beta} < \infty$. Therefore, by Theorem 1.1, $J$ has a unique fixed point $Q: X \to Y$ in the set $\Omega^* = \{g \in \Omega \mid d(f,g) < \infty\}$ such that

$$Q(x) := \lim_{n\to\infty}(J^n f)(x) = \lim_{n\to\infty}\frac{1}{k^{2n}}f(k^n x) \tag{3.22}$$

and $Q(kx) = k^2Q(x)$ for all $x \in X$. Also,

$$d(Q,f) \le \frac{1}{1-L}d(Jf,f) \le \frac{1}{|k|^{2\beta}(1-L)}.$$

This means that (3.20) holds for all $x \in X$. The mapping $Q$ is quadratic because it satisfies equation (1.2) as follows:

$$\|\widetilde{D}_1 Q(x,y)\|_\beta = \lim_{n\to\infty}\frac{1}{|k|^{2n\beta}}\|\widetilde{D}_1 f(k^n x, k^n y)\|_\beta \le \lim_{n\to\infty}\frac{1}{|k|^{2n\beta}}\varphi(k^n x, k^n y) = 0$$

for all $x, y \in X$; therefore, by Lemma 2.2, it is quadratic. To prove the uniqueness assertion, let us assume that there exists a quadratic mapping $S : X \to Y$ which satisfies (3.20). Since $d(f, S) \leq 1/[|k|^{2\beta}(1-L)]$ and $S$ is quadratic, we get $S \in \Omega^*$ and $(JS)(x) = \frac{1}{k^2}S(kx) = S(x)$ for all $x \in X$, *i.e.*, $S$ is a fixed point of $J$. Since $Q$ is the unique fixed point of $J$ in $\Omega^*$, then $S = Q$.

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then by the same reasoning as in the proof of [2], $Q$ is $\mathbb{R}$-quadratic. Letting $y = 0$ in (3.18), we get

$$\left\| 2f(kbx) - b^2 f(kx) - k^2 b^2 f(x) \right\|_\beta \leq \varphi(x, 0) \tag{3.23}$$

for all $x \in X$ and all $b \in B_1$. By definition of $Q$, (3.17) and (3.23), we obtain

$$\left\| 2Q(kbx) - b^2 Q(kx) - k^2 b^2 Q(x) \right\|_\beta \leq \lim_{n\to\infty} \frac{1}{k^{2n\beta}} \varphi\left(k^n x, 0\right) = 0$$

for all $x \in X$ and all $b \in B_1$. So, $2Q(kbx) - b^2 Q(kx) - k^2 b^2 Q(x) = 0$ for all $x \in X$ and all $b \in B_1$. Since $Q(kx) = k^2 Q(x)$, we get $Q(bx) = b^2 Q(x)$ for all $x \in X$ and all $b \in B_1 \cup \{0\}$. Now, let $b \in B \backslash \{0\}$. Since $Q$ is $\mathbb{R}$-quadratic,

$$\begin{aligned} Q(bx) &= Q\left( \|b\|_B \cdot \frac{b}{\|b\|_B} x \right) = \|b\|_B^2 \cdot Q\left( \frac{b}{\|b\|_B} x \right) \\ &= \|b\|_B^2 \cdot \left( \frac{b}{\|b\|_B} \right)^2 Q(x) = b^2 Q(x) \end{aligned}$$

for all $x \in X$ and all $b \in B$. This proves that $Q$ is $B$-quadratic. □

**Corollary 3.9** *Let* $0 < p < 2$, $\delta, \theta \in [0, \infty)$, *and let* $f : X \to Y$ *be an even mapping with* $f(0) = 0$ *such that*

$$\left\| \widetilde{D}_b f(x, y) \right\|_\beta \leq \delta + \theta \left( \|x\|_\beta^p + \|y\|_\beta^p \right)$$

*for all* $x, y \in X$ *and* $b \in B_1$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\left\| f(x) - Q(x) \right\|_\beta \leq \frac{1}{|k|^{2\beta} - |k|^{\beta p}} \delta + \frac{1}{|k|^{2\beta} - |k|^{\beta p}} \theta \|x\|_\beta^p$$

*for all* $x \in X$. *Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* $Q$ *is* $B$-*quadratic.*

*Proof* The proof follows from Theorem 3.8 by taking $\varphi(x, y) = \delta + \theta(\|x\|_\beta^p + \|y\|_\beta^p)$ for all $x, y \in X$. We can choose $L = |k|^{\beta(p-2)}$ to get the desired result. □

The following example shows that the generalized Hyers-Ulam stability problem for the case of $p = 2$ was excluded in Corollary 3.9. This example is a modified version of Czerwik [4].

**Example 3.10** Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^2, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases}$$

Consider the function $f:\mathbb{C}\to\mathbb{C}$ be defined by

$$f(x)=\sum_{m=0}^{\infty}\alpha^{-2m}\phi\bigl(\alpha^m x\bigr)$$

for all $x\in\mathbb{C}$, where $\alpha>\max\{|k|,|l|\}$. Let

$$\begin{aligned}\widetilde{D}_b f(x,y)&=f(kbx+lby)+f(kbx-lby)-b^2f(kx)-b^2f(x)\\&\quad-\frac{1}{2}(k-1)b^2\bigl[(k+2)f(x)+kf(-x)\bigr]-l^2b^2\bigl[f(y)+f(-y)\bigr]\end{aligned}$$

for all $x,y\in\mathbb{C}$ and $b\in\mathbb{T}:=\{\lambda\in\mathbb{C}\mid|\lambda|=1\}$. Then $f$ satisfies the functional inequality

$$\bigl|\widetilde{D}_\mu f(x,y)\bigr|\le\frac{(k^2+3+2l^2)\alpha^4}{\alpha^2-1}\bigl(|x|^2+|y|^2\bigr)\tag{3.24}$$

for all $x,y\in\mathbb{C}$, but there do not exist a quadratic mapping $Q:\mathbb{C}\to\mathbb{C}$ and a constant $d>0$ such that $|f(x)-Q(x)|\le d|x|^2$ for all $x\in\mathbb{C}$.

It is clear that $f$ is bounded by $\frac{\alpha^2}{\alpha^2-1}$ on $\mathbb{C}$. If $|x|^2+|y|^2=0$ or $|x|^2+|y|^2\ge\frac{1}{\alpha^2}$, then

$$\bigl|\widetilde{D}_\mu f(x,y)\bigr|\le\frac{(k^2+3+2l^2)\alpha^4}{\alpha^2-1}\bigl(|x|^2+|y|^2\bigr).$$

Now, suppose that $0<|x|^2+|y|^2<\frac{1}{\alpha^2}$. Then there exists an integer $n\ge1$ such that

$$\frac{1}{\alpha^{2(n+2)}}\le|x|^2+|y|^2<\frac{1}{\alpha^{2(n+1)}}.\tag{3.25}$$

Hence,

$$\alpha^m|kbx\pm lby|<1,\qquad\alpha^m|kx|<1,\qquad\alpha^m|x|<1,\qquad\alpha^m|y|<1$$

for all $m=0,1,\ldots,n-1$. From the definition of $f$ and the inequality (3.25), we obtain that

$$\bigl|\widetilde{D}_\mu f(x,y)\bigr|\le\frac{(k^2+3+2l^2)\alpha^4}{\alpha^2-1}\bigl(|x|^2+|y|^2\bigr).$$

Now, we claim that the functional equation (1.2) is not stable for $p=2$ in Corollary 3.9. Suppose, on the contrary, that there exist a quadratic mapping $Q:\mathbb{C}\to\mathbb{C}$ and a constant $d>0$ such that $|f(x)-Q(x)|\le d|x|^2$ for all $x\in\mathbb{C}$. Then there exists a constant $c\in\mathbb{C}$ such that $Q(x)=cx^2$ for all rational numbers $x$. So, we obtain that

$$\bigl|f(x)\bigr|\le\bigl(d+|c|\bigr)|x|^2\tag{3.26}$$

for all rational numbers $x$. Let $s\in\mathbb{N}$ with $s+1>d+|c|$. If $x$ is a rational number in $(0,\alpha^{-s})$, then $\alpha^m x\in(0,1)$ for all $m=0,1,\ldots,s$, and for this $x$, we get

$$f(x)=\sum_{m=0}^{\infty}\frac{\phi(\alpha^m x)}{\alpha^{2m}}\ge\sum_{m=0}^{s}\frac{\phi(\alpha^m x)}{\alpha^{2m}}=(s+1)x^2>\bigl(d+|c|\bigr)x^2,$$

which contradicts (3.26).

Similar to Corollary 3.9, one can obtain the following corollary.

**Corollary 3.11** *Let $t, s > 0$ such that $\lambda := t + s < 2$ and $\delta$, $\theta$ be non-negative real numbers, and let $f : X \to Y$ be an even mapping with $f(0) = 0$ such that*

$$\left\|\widetilde{D}_b f(x,y)\right\|_\beta \leq \delta + \theta\left[\|x\|_\beta^t \|y\|_\beta^s + \left(\|x\|_\beta^\lambda + \|y\|_\beta^\lambda\right)\right]$$

*for all $x, y \in X$ and $b \in B_1$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\left\|f(x) - Q(x)\right\|_\beta \leq \frac{1}{|k|^{2\beta} - |k|^{\beta\lambda}}\delta + \frac{1}{|k|^{2\beta} - |k|^{\beta\lambda}}\theta\|x\|_\beta^\lambda$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $Q$ is B-quadratic.*

Similar to Theorem 3.8, one can obtain the following theorem.

**Theorem 3.12** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} k^{2n\beta}\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0 \tag{3.27}$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping such that*

$$\left\|\widetilde{D}_b f(x,y)\right\|_\beta \leq \varphi(x,y) \tag{3.28}$$

*for all $x, y \in X$ and all $b \in B_1$. If there exists a Lipschitz constant $0 < L < 1$ such that $\varphi(x, 0) \leq k^{-2\beta}L\varphi(kx, 0)$ for all $x \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\left\|f(x) - Q(x)\right\|_\beta \leq \frac{L}{k^{2\beta}(1 - L)}\varphi(x, 0) \tag{3.29}$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $Q$ is B-quadratic.*

**Remark 3.13** Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (3.28). Let $0 < L < 1$ be a constant such that $\varphi(x, 0) \leq k^{-2\beta}L\varphi(kx, 0)$ for all $x \in X$. $f(0) = 0$, since $\varphi(0, 0) = 0$.

We now prove our main theorem in this section.

**Theorem 3.14** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{|k|^{n\beta}}\varphi\left(k^n x, k^n y\right) = 0 \tag{3.30}$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\|D_b f(x,y)\right\|_\beta \leq \varphi(x,y) \quad \textit{and} \quad \left\|\widetilde{D}_b f(x,y)\right\|_\beta \leq \varphi(x,y) \tag{3.31}$$

*for all* $x, y \in X$ *and all* $b \in B_1$. *If there exists a Lipschitz constant* $0 < L < 1$ *such that*

$$\varphi(kx, 0) \le |k|^\beta L\varphi(x, 0) \tag{3.32}$$

*for all* $x \in X$, *then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\left\|f(x) - A(x) - Q(x)\right\|_\beta \le \frac{|k|^\beta + 1}{|k|^{2\beta}(1 - L)}\left[\varphi(x, 0) + \varphi(-x, 0)\right] \tag{3.33}$$

*for all* $x \in X$. *Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then A is B-linear and Q is B-quadratic.*

*Proof* If we decompose $f$ into the even and the odd parts by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2} \tag{3.34}$$

for all $x \in X$, then $f(x) = f_e(x) + f_o(x)$. Let $\psi(x, y) = [\varphi(x, y) + \varphi(-x, -y)]/2^\beta$, then by (3.30)-(3.32) and (3.34), we have

$$\lim_{n\to\infty} \frac{1}{|k|^{n\beta}}\psi\left(k^n x, k^n y\right) = 0, \qquad \psi(kx, 0) \le |k|^\beta L\psi(x, 0),$$
$$\left\|D_b f_o(x, y)\right\|_\beta \le \psi(x, y), \qquad \left\|\widetilde{D}_b f_e(x, y)\right\|_\beta \le \psi(x, y).$$

Hence, by Theorems 3.1 and 3.8, there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\left\|f_o(x) - A(x)\right\|_\beta \le \frac{1}{k^\beta(1 - L)}\psi(x, 0), \qquad \left\|f_e(x) - Q(x)\right\|_\beta \le \frac{1}{k^{2\beta}(1 - L)}\psi(x, 0)$$

for all $x \in X$. Therefore,

$$\begin{aligned}\left\|f(x) - A(x) - Q(x)\right\|_\beta &\le \left\|f_o(x) - A(x)\right\|_\beta + \left\|f_e(x) - Q(x)\right\|_\beta \\ &\le \frac{1}{|k|^\beta(1 - L)}\psi(x, 0) + \frac{1}{|k|^{2\beta}(1 - L)}\psi(x, 0) \\ &= \frac{|k|^\beta + 1}{|k|^{2\beta}(1 - L)}\left[\varphi(x, 0) + \varphi(-x, 0)\right]\end{aligned}$$

for all $x \in X$. □

**Corollary 3.15** *Let* $0 < p < 1$ *and* $\delta, \theta$ *be non-negative real numbers, and let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *such that*

$$\left\|D_b f(x, y)\right\|_\beta \le \delta + \theta\left(\|x\|_\beta^p + \|y\|_\beta^p\right) \quad \textit{and} \quad \left\|\widetilde{D}_b f(x, y)\right\|_\beta \le \delta + \theta\left(\|x\|_\beta^p + \|y\|_\beta^p\right)$$

*for all* $x, y \in X$ *and* $b \in B_1$. *Then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\left\|f(x) - A(x) - Q(x)\right\|_\beta \le \frac{2(|k|^\beta + 1)}{|k|^{2\beta} - |k|^{\beta(p+1)}}\left[\delta + \theta\|x\|_\beta^p\right]$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear and $Q$ is $B$-quadratic.*

Thanks to Remark 3.13, by a similar method to the proof of Theorem 3.14, one can obtain the following theorem.

**Theorem 3.16** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} |k|^{n\beta}\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping such that*

$$\left\|D_b f(x,y)\right\|_\beta \le \varphi(x,y) \quad \textit{and} \quad \left\|\widetilde{D}_b f(x,y)\right\|_\beta \le \varphi(x,y)$$

*for all $x, y \in X$ and all $b \in B_1$. If there exists a Lipschitz constant $0 < L < 1$ such that*

$$\varphi(x,0) \le |k|^{-2\beta} L\varphi(kx,0)$$

*for all $x \in X$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\left\|f(x) - A(x) - Q(x)\right\|_\beta \le \frac{(|k|^\beta + 1)L}{|k|^{2\beta}(1-L)}\left[\varphi(x,0) + \varphi(-x,0)\right]$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear and $Q$ is $B$-quadratic.*

**Corollary 3.17** *Let $p > 2$ and $\theta$ be a non-negative real number, and let $f : X \to Y$ be a mapping for which*

$$\left\|D_b f(x,y)\right\|_\beta \le \theta\left(\|x\|_\beta^p + \|y\|_\beta^p\right) \quad \textit{and} \quad \left\|\widetilde{D}_b f(x,y)\right\|_\beta \le \theta\left(\|x\|_\beta^p + \|y\|_\beta^p\right)$$

*for all $x, y \in X$ and $b \in B_1$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\left\|f(x) - A(x) - Q(x)\right\|_\beta \le \frac{2(|k|^\beta + 1)}{|k|^{p\beta} - |k|^{2\beta}}\theta\|x\|_\beta^p$$

*for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $A$ is $B$-linear and $Q$ is $B$-quadratic.*

**Remark 3.18** The generalized Hyers-Ulam stability problem for the cases of $p = 1$ and $p = 2$ were excluded in Corollaries 3.15 and 3.17, respectively.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**

All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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**Acknowledgements**

The authors would like to thank the referees and the editors for their help and suggestions in improving this paper. The first author TZX was supported by the National Natural Science Foundation of China (NNSFC) (Grant No. 11171022).

Received: 8 February 2012 Accepted: 4 September 2012 Published: 19 September 2012

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doi:10.1186/1029-242X-2012-204
**Cite this article as:** Xu and Rassias: A fixed point approach to the Hyers-Ulam stability of an AQ functional equation on $\beta$-Banach modules. *Journal of Inequalities and Applications* 2012 **2012**:204.