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Inequalities for convex and *s*-convex functions on $\Delta = [a, b] \times [c, d]$

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Abstract

In this article, two new lemmas are proved and inequalities are established for co-ordinated convex functions and co-ordinated *s*-convex functions.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

In [1], Orlicz defined s-convex function in the second sense as following:

Definition 1. *A function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be s-convex in the second sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. We denote by K_s^2 the class of all s-convex functions.

Obviously one can see that if we choose $s = 1$, the above definition reduces to ordinary concept of convexity.

For several results related to above definition we refer readers to [2-10].

In [11], Dragomir defined convex functions on the co-ordinates as following:

Definition 2. *Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the coordinates if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex* *where defined for all* $y \in [c, d]$ *and* $x \in [a, b]$. *Recall that the mapping* $f : \Delta \to \mathbb{R}$ *is convex on* Δ *if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w)$$

for all $(x, y), (z, w) \in \Delta$ *and* $\lambda \in [0, 1]$.

In [11], Dragomir established the following inequalities of Hadamard-type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1. *Suppose that* $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ *is convex on the co-ordinates on* Δ. *Then one has the inequalities;*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)dxdy \\ & \leq \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x, c)dx + \frac{1}{(b-a)}\int_a^b f(x, d)dx \right. \\ & \left. + \frac{1}{(d-c)}\int_c^d f(a, y)dy + \frac{1}{(d-c)}\int_c^d f(b, y)dy\right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \tag{1.1}$$

The above inequalities are sharp.

Similar results can be found in [12-14].

In [13], Alomari and Darus defined co-ordinated *s*-convex functions and proved some inequalities based on this definition. Another definition for co-ordinated *s*-convex functions of second sense can be found in [15].

Definition 3. *Consider the bidimensional interval* $\Delta = [a, b] \times [c, d]$ *in* $[0, \infty)^2$ *with* $a < b$ *and* $c < d$. *The mapping* $f : \Delta \to \mathbb{R}$ *is s-convex on* Δ *if*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x, y) + (1-\lambda)^s f(z, w)$$

holds for all $(x, y), (z, w) \in \Delta$ *with* $\lambda \in [0, 1]$ *and for some fixed* $s \in (0, 1]$.

In [16], Sarıkaya et al. proved some Hadamard-type inequalities for co-ordinated convex functions as following:

Theorem 2. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$\begin{aligned} |J| & \leq \frac{(b-a)(d-c)}{16} \\ & \times \frac{\left|\frac{\partial^2 f}{\partial t \partial s}\right|(a, c) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|(a, d) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|(b, c) + \left|\frac{\partial^2 f}{\partial t \partial s}\right|(b, d)}{4} \end{aligned} \tag{1.2}$$

where

$$J = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy - A$$

and

$$A = \frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b [f(x,c)+f(x,d)]dx + \frac{1}{(d-c)}\int_c^d [f(a,y)+f(b,y)]dy\right].$$

Theorem 3. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differ entiable mapping on* $\Delta := [a, b] \times [c, d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q > 1$, *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$|J| \le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right]^{\frac{1}{q}} \tag{1.3}$$

where A, J are as in Theorem 2 and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q \ge 1$, *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$|J| \le \frac{(b-a)(d-c)}{16} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right]^{\frac{1}{q}} \tag{1.4}$$

where A, J are as in Theorem 2.

In [17], Barnett and Dragomir proved an Ostrowski-type inequality for double integrals as following:

Theorem 5. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be continuous on* $[a, b] \times [c, d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a, b) \times (c, d)$ *and is bounded, that is*

$$\left\|f''_{x,y}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| < \infty,$$

then we have the inequality;

$$\left| \int_a^b \int_c^d f(s,t)dtds - (b-a)\int_c^d f(x,t)dt - (d-c)\int_a^b f(s,y)ds - (b-a)(d-c)f(x,y) \right| \\ \leq \left[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{(d-c)^2}{4} + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{x,y} \right\|_\infty \tag{1.5}$$

for all $(x, y) \in [a, b] \times [c, d]$.

In [18], Sarıkaya proved an Ostrowski-type inequality for double integrals and gave a corollary as following:

Theorem 6. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order 2 exists and is bounded, i.e.,*

$$\left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| < \infty$$

for all $(t, s) \in [a, b] \times [c, d]$. *Then we have,*

$$\Bigg| (\beta_1 - \alpha_1)(\beta_2 - \alpha_2) f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) + H(\alpha_1, \alpha_2, \beta_1, \beta_2) + G(\alpha_1, \alpha_2, \beta_1, \beta_2) \\ - (\beta_2 - \alpha_2) \int_a^b f\left(t, \frac{c+d}{2} \right) dt - (\beta_1 - \alpha_1) \int_c^d f\left(\frac{a+b}{2}, s \right) ds \\ - \int_a^b [(\alpha_2 - c) f(t,c) + (d - \beta_2) f(t,d)] dt \\ - \int_c^d [(\alpha_1 - a) f(a,s) + (b - \beta_1) f(b,s)] ds + \int_a^b \int_c^d f(t,s) dsdt \Bigg| \\ \leq \left[\frac{(\alpha_1 - a)^2 + (b - \beta_1)^2}{2} + \frac{(a+b-2\alpha_1)^2 + (a+b-2\beta_1)^2}{8} \right] \\ \times \left[\frac{(\alpha_2 - c)^2 + (d - \beta_2)^2}{2} + \frac{(c+d-2\alpha_2)^2 + (c+d-2\beta_2)^2}{8} \right] \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \tag{1.6}$$

for all $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in [a, b] \times [c, d]$ *with* $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ *where*

$$H(\alpha_1, \alpha_2, \beta_1, \beta_2) \\ = (\alpha_1 - a)[(\alpha_2 - c)f(a,c) + (d - \beta_2)f(a,d)] \\ + (b - \beta_1)[(\alpha_2 - c)f(b,c) + (d - \beta_2)f(b,d)]$$

and

$$G(\alpha_1, \alpha_2, \beta_1, \beta_2) \\ = (\beta_1 - \alpha_1) \left[(\alpha_2 - c) f\left(\frac{a+b}{2}, c \right) + (d - \beta_2) f\left(\frac{a+b}{2}, d \right) \right] \\ + (\beta_2 - \alpha_2) \left[(\alpha_1 - a) f\left(a, \frac{c+d}{2} \right) + (b - \beta_1) f\left(b, \frac{c+d}{2} \right) \right].$$

Corollary 1. *Under the assumptions of Theorem 6, we have*

$$\Bigg| (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) + \int_a^b \int_c^d f(t,s) dsdt \\ - (d-c) \int_a^b f\left(t, \frac{c+d}{2} \right) dt - (b-a) \left| \int_c^d f\left(\frac{a+b}{2}, s \right) ds \right| \\ \leq \frac{1}{16} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty (b-a)^2 (d-c)^2. \tag{1.7}$$

In [19], Pachpatte established a new Ostrowski type inequality similar to inequality (1.5) by using elementary analysis.

The main purpose of this article is to establish inequalities of Hadamard-type for co-ordinated convex functions by using Lemma 1 and to establish some new Hadamard-type inequalities for co-ordinated s-convex functions by using Lemma 2.

2. Inequalities for co-ordinated convex functions

To prove our main results, we need the following lemma which contains kernels similar to Barnett and Dragomir's kernels in [17], (see the article [17, proof of Theorem 2.1]).

Lemma 1. *Let* $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta = [a, b] \times [c, d]$. *If* $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, *then the following equality holds:*

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&- \frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2}, y\right)dy - \frac{1}{(b-a)}\int_a^b f\left(x, \frac{c+d}{2}\right)dx \\
&+ \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y)dydx \\
= &\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d p(x, t)q(y, s)\frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)dsdt
\end{aligned}$$

where

$$p(x, t) = \begin{cases} (t-a), & t \in \left[a, \frac{a+b}{2}\right] \\ (t-b), & t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

and

$$q(y, s) = \begin{cases} (s-c), & s \in \left[c, \frac{c+d}{2}\right] \\ (s-d), & s \in \left(\frac{c+d}{2}, d\right] \end{cases}$$

for each $x \in [a, b]$ *and* $y \in [c, d]$.

Proof. We note that

$$B = \int_a^b \int_c^d p(x, t)q(y, s)\frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)dsdt.$$

Integration by parts, we can write

$$\begin{aligned}
B &= \int_c^d q(y,s)\left[\int_a^{\frac{a+b}{2}} (t-a)\frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)dt\right.\\
&\quad\left.+\int_{\frac{a+b}{2}}^b (t-b)\frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)dt\right]ds\\
&= \int_c^d q(y,s)\left\{\left[(t-a)\frac{\partial f}{\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right]_a^{\frac{a+b}{2}}\right.\\
&\quad-\int_a^{\frac{a+b}{2}} \frac{\partial f}{\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)dt\\
&\quad+\left[(t-b)\frac{\partial f}{\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right]_{\frac{a+b}{2}}^b\\
&\quad\left.-\int_{\frac{a+b}{2}}^b \frac{\partial f}{\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)dt\right\}ds\\
&= (b-a)\int_c^d q(y,s)\left\{\frac{\partial f}{\partial s}\left(\frac{a+b}{2}, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right.\\
&\quad\left.-\int_a^b \frac{\partial f}{\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)dt\right\}ds\\
&= (b-a)\left\{\int_c^{\frac{c+d}{2}} (s-c)\frac{\partial f}{\partial s}\left(\frac{a+b}{2}, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)ds\right.\\
&\quad+\int_{\frac{c+d}{2}}^d (s-d)\frac{\partial f}{\partial s}\left(\frac{a+b}{2}, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)ds\\
&\quad-\int_a^b\left[\int_c^{\frac{c+d}{2}} (s-c)\frac{\partial f}{\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)ds\right.\\
&\quad\left.\left.+\int_{\frac{c+d}{2}}^d (s-d)\frac{\partial f}{\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)ds\right]dt\right\}.
\end{aligned}$$

By calculating the above integrals, we have

$$\begin{aligned}
B &= (b-a)(d-c)f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&-(b-a)\int_c^d f\left(\frac{a+b}{2}, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)ds\\
&-(d-c)\int_a^b f\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{c+d}{2}\right)dt\\
&\int_a^b\int_c^d f\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)dsdt.
\end{aligned}$$

Using the change of the variable $x = \frac{b-t}{b-a}a + \frac{t-a}{b-a}b$ and $y = \frac{d-s}{d-c}c + \frac{s-c}{d-c}d$, then dividing both sides with $(b - a) \times (d - c)$, this completes the proof.

Theorem 7. *Let* $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta = [a, b] \times [c, d]$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2}, y\right)dy - \frac{1}{(b-a)}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \right|$$
$$\leq \frac{(b-a)(d-c)}{64}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].$$

Proof. We note that

$$C = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2}, y\right)dy - \frac{1}{(b-a)}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx.$$

From Lemma 1 and using the property of modulus, we have

$$|C| \leq \frac{1}{(b-a)(d-c)} \times \int_a^b\int_c^d |p(x,t)q(y,s)| \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right| dsdt$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is co-ordinated convex, we can write

$$|C| \leq \frac{1}{(b-a)(d-c)} \times \int_c^d |q(y,s)| \left\{ \int_a^{\frac{a+b}{2}} (t-a)\left[\frac{b-t}{b-a}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|\right]dt \right.$$
$$+ \int_a^{\frac{a+b}{2}} (t-a)\left[\frac{t-a}{b-a}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|\right]dt$$
$$+ \int_{\frac{a+b}{2}}^b (b-t)\left[\frac{b-t}{b-a}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|\right]dt$$
$$\left. + \int_{\frac{a+b}{2}}^b (b-t)\left[\frac{t-a}{b-a}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|\right]dt \right\} ds.$$

By computing these integrals, we obtain

$$|C| \leq \frac{(b-a)}{8(d-c)}\left[\int_c^d |q(\gamma,s)| \left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right| + \int_c^d |q(\gamma,s)| \left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|\right] ds.$$

Using co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ again, we get

$$\begin{aligned}|C| \leq \frac{(b-a}{8(d-c)} &\times \left\{\int_c^{\frac{c+d}{2}} (s-c)\left[\frac{d-s}{d-c}\left|\frac{\partial^2 f}{\partial r \partial s}(a,c)\right|\right] ds + \int_c^{\frac{c+d}{2}} (s-c)\left[\frac{s-c}{d-c}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|\right] ds\right.\\ &+ \int_{\frac{c+d}{2}}^d (d-s)\left[\frac{d-s}{d-c}\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|\right] ds + \int_{\frac{c+d}{2}}^d (d-s)\left[\frac{s-c}{d-c}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|\right] ds\\ &+ \int_c^{\frac{c+d}{2}} (s-c)\left[\frac{d-s}{d-c}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|\right] ds + \int_d^{\frac{c+d}{2}} (s-c)\left[\frac{s-c}{d-c}\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|\right] ds\\ &\left.+ \int_{\frac{c+d}{2}}^d (d-s)\left[\frac{d-s}{d-c}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|\right] ds + \int_{\frac{c+d}{2}}^d (d-s)\left[\frac{s-c}{d-c}\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|\right] ds\right\}.\end{aligned}$$

By a simple computation, we get the required result.

Remark 1. *Suppose that all the assumptions of Theorem 7 are satisfied. If we choose* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ *is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty = \sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right| < \infty,$$

we get

$$|C| \leq \frac{(b-a)(d-c)}{16}\left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty \tag{2.1}$$

which is the inequality in (1.7).

Theorem 8. *Let* $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ *bea partial differentiable mapping on* $\Delta = [a, b] \times [c, d]$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q > 1$, *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$|C| \leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{4}\right]^{\frac{1}{q}} \tag{2.2}$$

where C is in the proof of Theorem 7.

Proof. From Lemma 1, we have

$$|C| \le \frac{1}{(b-a)(d-c)} x \int_a^b \int_c^d |p(x,t)q(y,s)| \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right| dsdt.$$

By applying the well-known Hölder inequality for double integrals, then one has

$$\begin{aligned} |C| \le \frac{1}{(b-a)(d-c)} &\left\{ \left(\int_a^b \int_c^d |p(x,t)q(y,s)|^p dtds \right)^{\frac{1}{p}} \right. \\ &\times \left(\int_a^b \int_c^d \left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|^q dsdt \right)^{\frac{1}{q}}. \end{aligned} \tag{2.3}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is co-ordinated convex function on Δ, we can write

$$\begin{aligned} &\left| \frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|^q \\ &\le \left(\frac{b-t}{b-a}\right)\left(\frac{d-s}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q \\ &\quad + \left(\frac{b-t}{b-a}\right)\left(\frac{s-c}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\ &\quad + \left(\frac{t-a}{b-a}\right)\left(\frac{d-s}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q \\ &\quad + \left(\frac{t-a}{b-a}\right)\left(\frac{s-c}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q. \end{aligned} \tag{2.4}$$

Using the inequality (2.4) in (2.3), we get

$$|C| \le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{4} \right]^{\frac{1}{q}}$$

where we have used the fact that

$$\left(\int_a^b \int_c^d |p(x,t)q(y,s)|^p dtds \right)^{\frac{1}{p}} = \frac{[(b-a)(d-c)]^{1+\frac{1}{p}}}{4(p+1)^{\frac{2}{p}}}.$$

This completes the proof.

Remark 2. *Suppose that all the assumptions of Theorem 8 are satisfied. If we choose* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ *is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty = \sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right| < \infty,$$

we get

$$|C| \le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty \tag{2.5}$$

which is the inequality in (1.3) with $\left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty$.

Theorem 9. *Let* $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ *bea partial differentiable mapping on* $\Delta = [a, b] \times [c, d]$. *If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q > 1$, *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$|C| \le \frac{(b-a)(d-c)}{16} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{4}\right]^{\frac{1}{q}} \tag{2.6}$$

where C is in the proof of Theorem 7.

Proof. From Lemma 1 and applying the well-known Power mean inequality for double integrals, then one has

$$\begin{aligned}|C| &\le \frac{1}{(b-a)(d-c)} \times \int_a^b \int_c^d |p(x,t)q(y,s)| \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right| dsdt \\ &\le \frac{1}{(b-a)(d-c)}\left(\int_a^b \int_c^d |p(x,t)q(y,s)|\, dsdt\right)^{1-\frac{1}{q}} \\ &\quad \times \left[\int_a^b \int_c^d |p(x,t)q(y,s)| \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|^q dsdt\right]^{\frac{1}{q}}.\end{aligned} \tag{2.7}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is co-ordinated convex function on Δ, we can write

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d\right)\right|^q \\
&\leq \left(\frac{b-t}{b-a}\right)\left(\frac{d-s}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q \\
&\quad + \left(\frac{b-t}{b-a}\right)\left(\frac{s-c}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\
&\quad + \left(\frac{t-a}{b-a}\right)\left(\frac{d-s}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q \\
&\quad + \left(\frac{t-a}{b-a}\right)\left(\frac{s-c}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q .
\end{aligned} \tag{2.8}$$

If we use (2.8) in (2.7), we get

$$\begin{aligned}
|C| \leq \frac{1}{(b-a)(d-c)} &\left\{\left(\int_a^b \int_c^d |p(x,t)q(y,s)|\, dsdt\right)^{1-\frac{1}{q}}\right. \\
&\times \int_a^b \int_c^d |p(x,t)q(y,s)| \left[\left(\frac{b-t}{b-a}\right)\left(\frac{d-s}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left(\frac{b-t}{b-a}\right)\left(\frac{s-c}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q\right) \\
&\left. + \left(\frac{t-a}{b-a}\right)\left(\frac{d-s}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left(\frac{t-a}{b-a}\right)\left(\frac{s-c}{d-c}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q\right]\Bigg)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Computing the above integrals and using the fact that

$$\left(\int_a^b \int_c^d |p(x,t)q(y,s)|\, dtds\right)^{1-\frac{1}{q}} = \left(\frac{(b-a)^2(d-c)^2}{16}\right)^{1-\frac{1}{q}},$$

we obtained the desired result.

3. Inequalities for co-ordinated *s*-convex functions

To prove our main results we need the following lemma:

Lemma 2. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be an absolutely continuous function on* Δ *where* $a < b$, $c < d$ *and* $t, \lambda \in [0, 1]$, *if* $\frac{\partial^2 f}{\partial t \partial \lambda} \in L(\Delta)$, *then the following equality holds:*

$$D = \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)} \times E$$

where

$$\begin{aligned}
D &= \frac{f(a,c) + r_2 f(a,d) + r_1 f(b,c) + r_1 r_2 f(b,d)}{(r_1 + 1(r_2+1)} \\
&\quad + \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)dxdy \\
&\quad - \left(\frac{r_1}{r_1+1}\right)\frac{1}{d-c}\int_c^d f(b,y)dy - \left(\frac{1}{r_1+1}\right)\frac{1}{d-c}\int_c^d f(a,y)dy \\
&\quad - \left(\frac{r_2}{r_2+1}\right)\frac{1}{b-a}\int_a^b f(x,d)dx - \left(\frac{1}{r_2+1}\right)\frac{1}{b-a}\int_a^b f(x,c)dx
\end{aligned}$$

and

$$E = \int_0^1 \int_0^1 ((r_1+1)t-1)((r_2+1)\lambda-1)\frac{\partial^2 f}{\partial t \partial \lambda}(tb+(1-t)a, \lambda d+(1-\lambda)c)dtd\lambda$$

for some fixed r_1, $r_2 \in$ [0, 1].

Proof. Integration by parts, we get

$$\begin{aligned}
E &= \int_0^1 ((r_2+1)\lambda-1) \\
&\quad \times \left[\int_0^1 ((r_1+1)t-1)\frac{\partial^2 f}{\partial t\partial\lambda}(tb+(1-t)a, \lambda d+(1-\lambda)c)dt\right]d\lambda \\
&= \int_0^1 ((r_2+1)\lambda-1)\left[\frac{((r_1+1)t-1)}{(b-a)}\ \frac{\partial f}{\partial \lambda}(tb+(1-t)a, \lambda d+(1-\lambda)c)|_0^1\right. \\
&\quad \left. -\frac{r_1+1}{b-a}\int_0^1 \frac{\partial f}{\partial\lambda}(tb+(1-t)a, \lambda d+(1-\lambda)c)dt\right]d\lambda \\
&= \int_0^1 ((r_2+1)\lambda-1\left[\frac{r_1}{b-a}\frac{\partial f}{\partial\lambda}\ (b, \lambda d+(1-\lambda)c)+\frac{1}{b-a}\frac{\partial f}{\partial\lambda}(a, \lambda d+(1-\lambda)c)\right. \\
&\quad \left. -\frac{r_1+1}{b-a}\int_0^1 \frac{\partial f}{\partial\lambda}(tb+(1-t)a, \lambda d+(1-\lambda)c)dt\right]d\lambda \\
&= \frac{r_1}{b-a}\frac{((r_2+1)\lambda-1}{d-c}f(b, \lambda d+(1-\lambda)c)|_0^1 - \frac{r_1(r_2+1)}{(b-a)(d-c)}\int_0^1 f(b, \lambda d+(1-\lambda)c)d\lambda \\
&\quad + \frac{1}{b-a}\frac{((r_2+1)\lambda-1}{d-c}f(a, \lambda d+(1-\lambda)c)|_0^1 - \frac{(r_2+1)}{(b-a)(d-c)}\int_0^1 f(a, \lambda d+(1-\lambda)c)d\lambda \\
&\quad - \frac{r_1+1}{b-a}\int_0^1\left[\int_0^1 ((r_2+1)\lambda-1)\frac{\partial f}{\partial\lambda}(tb+(1-t)a, \lambda d+(1-\lambda)c)d\lambda\right]dt.
\end{aligned}$$

Computing these integrals, we obtain

$$\begin{aligned}
E = \frac{1}{(b-a)(d-c)}&[f(a,c)+r_2 f(a,d)+r_1 f(b,c)+r_1 r_2 f(b,d) \\
&- r_1(r_2+1)\int_0^1 f(b, \lambda d+(1-\lambda)c)d\lambda - (r_2+1)\int_0^1 f(a, \lambda d+(1-\lambda)c)d\lambda \\
&- r_2(r_1+1)\int_0^1 f(tb+(1-t)a, d)dt - (r_1+1)\int_0^1 f(tb+(1-t)a, c)dt \\
&+(r_1+1)(r_2+1)\int_0^1\int_0^1 f(tb+(1-t)a, \lambda d+(1-\lambda)c)dtd\lambda\Bigg].
\end{aligned}$$

Using the change of the variable $x = tb + (1 - t)\,a$ and $y = \lambda d + (1 - \lambda)\,c$ for $t, \lambda \in$ [0, 1] and multiplying the both sides by $\frac{(b-a)(d-c)}{(r_1+1(r_2+1)}$, we get the required result.

Theorem 10. *Let* $f : \Delta = [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty)$ *be an absolutely continuous function on* Δ. *If* $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$ *is s-convex function on the co-ordinates on* Δ, *then one has the inequality:*

$$\begin{aligned}
D| &\le \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)(s+1)^2(s+2)^2} \\
&\times\left[MS\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| + ML\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\right. \qquad (3.1) \\
&\left.+KR\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right| + KN\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right]
\end{aligned}$$

where

$$M = \left(s + 1 + 2(r_1 + 1)\left(\frac{r_1}{r_1 + 1}\right)^{s+2} - r_1\right)$$

$$N = \frac{1}{(r_2 + 1)^{s+1}}$$

$$L = r_2(s + 1) + \frac{1}{(r_2 + 1)^{s+1}} - 1$$

$$R = s + 1 + r_2\left(\frac{r_2}{r_2 + 1}\right)^{s+1} - r_2$$

$$S = r_2\left(\frac{r_2}{r_2 + 1}\right)^{s+1}$$

Proof. From Lemma 2 and by using co-ordinated s-convexity of $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$, we have;

$$\begin{aligned} |D| &\leq \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)} \\ &\quad \times \int_0^1 \int_0^1 \left|((r_1+1)t-1)((r_2+1)\lambda-1)\right| \left|\frac{\partial^2 f}{\partial t \partial \lambda}(tb + (1-t)a, \lambda d + (1-\lambda)c)\right| dt d\lambda \\ &\leq \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)} \\ &\quad \times \int_0^1 \left[\int_0^1 \left|((r_1+1)t-1)((r_2+1)\lambda-1)\right| \right. \\ &\quad \left. \left\{t^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b, \lambda d + (1-\lambda)c)\right| + (1-t)^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a, \lambda d + (1-\lambda)c)\right|\right\} dt\right] d\lambda. \end{aligned}$$

By calculating the above integrals, we have

$$\begin{aligned} &\int_0^1 \left|((r_1+1)t-1)\right| \left\{t^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b, \lambda d + (1-\lambda)c)\right| \right. \\ &\left. +(1-t)^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a, \lambda d + (1-\lambda)c)\right|\right\} dt \\ &= \int_0^{\frac{1}{r_1+1}} (1-(r_1+1)t) \left\{t^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b, \lambda d + (1-\lambda)c)\right| \right. \\ &\left. +(1-t)^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a, \lambda d + (1-\lambda)c)\right|\right\} dt \\ &+ \int_{\frac{1}{r_1+1}}^1 ((r_1+1)t-1) \left\{t^2 \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b, \lambda d + (1-\lambda)c)\right| \right. \\ &\left. +(1-t)^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a, \lambda d + (1-\lambda)c)\right|\right\} dt \\ &= \frac{1}{(s+1)(s+2)} \left[\left(r_1(s+1) + 2\left(\frac{1}{r_1+1}\right)^{s+1} - 1\right) \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b, \lambda d + (1-\lambda)c)\right| \right. \\ &\left. + \left(s+1+2(r_1+1)\left(\frac{r_1}{r_1+1}\right)^{s+2} - r_1\right) \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a, \lambda d + (1-\lambda)c)\right|\right]. \end{aligned} \tag{3.2}$$

By a similar argument for other integrals, by using co-ordinated s-convexity of $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$, we get

$$
\begin{aligned}
&\int_0^1 \left|((r_2+1)\lambda - 1)\right| \left\{\left|\frac{\partial^2 f}{\partial t \partial \lambda}(b, \lambda d + (1-\lambda)c)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a, \lambda d + (1-\lambda)c)\right|\right\} d\lambda \\
&\le \int_0^{\frac{1}{r_2+1}} (1-(r_2+1)\lambda) \left\{\lambda^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right| + (1-\lambda)^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|\right\} d\lambda \\
&+ \int_{\frac{1}{r_2+1}}^1 ((r_2+1)\lambda - 1) \left\{\lambda^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right| + (1-\lambda)^s \left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right|\right\} d\lambda \\
&= \frac{1}{(s+1)(s+2)} \left\{\frac{1}{(r_2+1)^{s+1}} \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|\right. \\
&+ \left[r_2(s+1) + \frac{1}{(r_2+1)^{s+1}} - 1\right] \left|\frac{\partial^2 f}{\partial t \partial f}(a,d)\right| \\
&+ \left[s - r_2 + 1 + r_2\left(\frac{r_2}{r_2+1}\right)^{s+1}\right] \left|\frac{\partial^2 f}{\partial t \partial f}(b,c)\right| \\
&\left. + r_2\left(\frac{r_2}{r_2+1}\right)^{s+1} \left|\frac{\partial^2 f}{\partial t \partial f}(a,c)\right|\right\}.
\end{aligned}
$$

By using these in (3.2), we obtain the inequality (3.1).

Corollary 2

(1) If we choose $r_1 = r_2 = 1$ in (3.1), we have

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2}\left[\frac{1}{d-c}\int_c^d [f(b,y)+f(a,y)]dy\right]\right. \\
&\left. -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,d)+f(x,c)]dx\right] + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\le \frac{(b-a)(d-c)}{4(s+1)^2(s+2)^2}\left(s+\frac{1}{2^s}\right)^2 \\
&\times \left[\frac{1}{2^{s+1}}\left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|\right) + \left(s+\frac{1}{2^{s+1}}\right)\left(\left|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|\right)\right].
\end{aligned}
\tag{3.3}
$$

(2) If we choose $r_1 = r_2 = 0$ in (3.1), we have

$$
\begin{aligned}
&\left|f(a,c) - \frac{1}{d-c}\int_c^d f(a,y)dy - \frac{1}{b-a}\int_a^b f(x,c)dx\right. \\
&\left. + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\le \frac{(b-a)(d-c)}{(s+1)^2(s+2)^2} \\
&\times \left[(s+1)\left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|\right].
\end{aligned}
$$

Theorem 11. *Let $f : \Delta = [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty)$ be an absolutely continuous function on Δ. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^{\frac{p}{p-1}}$ is s-convex function on the co-ordinates on Δ, for* *some fixed* $s \in (0, 1]$ *and* $p > 1$, *then one has the inequality:*

$$|D| \leq \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)} \frac{\left(1+r_1^{p+1}\right)^{\frac{1}{p}}\left(1+r_2^{p+1}\right)^{\frac{1}{p}}}{(r_1+1)^{\frac{1}{p}}(r_2+1)^{\frac{1}{p}}(p+1)^{\frac{2}{p}}} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(b,d)}{(s+1)^2}\right]^{\frac{1}{q}} \tag{3.4}$$

for some fixed $r_1, r_2 \in [0, 1]$, *where* $q = \frac{p}{p-1}$.

Proof. From Lemma 2 and using the Hölder inequality for double integrals, we can write

$$|D| \leq \frac{(b-a)(d-c)}{(r_1+1)(r_2+1)}\left(\int_0^1\int_0^1 \left|((r_1+1)t-1)((r_2+1)\lambda-1)\right|^p dt d\lambda\right)^{\frac{1}{p}} \times \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t \partial \lambda}(tb+(1-t)a, \lambda d+(1-\lambda)c\right|^q dt d\lambda\right)^{\frac{1}{q}}.$$

In above inequality using the *s*-convexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ on Δ and calculating the integrals, then we get the desired result.

Corollary 3

(1) Under the assumptions of Theorem 11, if we choose $r_1 = r_2 = 1$ in (3.4), we have

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\ &-\frac{1}{2}\left[\frac{1}{d-c}\int_c^d [f(b,y)+f(a,y)]dy+\frac{1}{b-a}\int_a^b [f(x,d)+f(x,c)]dx\right] \\ &\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\ &\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \\ &\times\left[\frac{\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q(b,d)}{(s+1)^2}\right]^{\frac{1}{q}}.\end{aligned} \tag{3.5}$$

(2) Under the assumptions of Theorem 11, if we choose $r_1 = r_2 = 0$ in (3.4), we have

$$\begin{aligned}&\left|f(a,c)-\frac{1}{d-c}\int_c^d f(a,y)dy-\frac{1}{b-a}\int_a^b f(x,c)dx\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|\\&=\frac{(b-a)(d-c)}{(p+1)^{\frac{2}{p}}}\\&\times\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,d)}{(s+1)^2}\right]^{\frac{1}{q}}.\end{aligned}$$

Remark 4. *If we choose $s = 1$ in (3.5), we obtain the inequality in (1.3)*

Theorem 12. *Let $f : \Delta = [a, b] \times [c, d] \subset [0, \infty)^2 \to [0, \infty)$ be an absolutely continuous function on Δ. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is s-convex function on the co-ordinates on Δ, for some fixed $s \in (0, 1]$ and $q \geq 1$, then one has the inequality:*

$$\begin{aligned}|D|&\leq\frac{(b-a)(d-c)}{(r_1+1)(r_2+1)}\left(\frac{(1+r_1^2)(1+r_2^2)}{4(r_1+1)(r_2+1)}\right)^{1-\frac{1}{q}}\\&\times\left(\frac{MS\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+ML\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+KR\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+KN\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q}{(s+1)^2(s+2)^2}\right)^{\frac{1}{q}}\end{aligned}$$

for some fixed $r_1, r_2 \in [0, 1]$.

Proof. From Lemma 2 and using the well-known Power-mean inequality, we can write

$$\begin{aligned}|D|&\leq\frac{(b-a)(b-c)}{(r_1+1)(r_2+1)}\left(\int_0^1\int_0^1\left|((r_1+1)t-1)((r_2+1)\lambda-1)\right|dtd\lambda\right)^{1-\frac{1}{q}}\\&\times\left[\int_0^1\int_0^1\left|((r_1+1)t-1)((r_2+1)\lambda-1)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tb+(1-t)a,\lambda d+(1-\lambda)c)\right|^q dtd\lambda\right]^{\frac{1}{q}}.\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is s-convex function on the co-ordinates on Δ, we have

$$\begin{aligned}&\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tb+(1-t)a,\lambda d+(1-\lambda)c\right|^q\\&\leq t^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,\lambda d+(1-\lambda)c)\right|^q+(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,\lambda d+(1-\lambda)c)\right|^q\end{aligned}$$

and

$$\left|\frac{\partial^2 f}{\partial t\partial\lambda}(tb+(1-t)a,\lambda d+(1-\lambda)c\right|^q$$
$$\leq t^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,d)+t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,c)$$
$$+\lambda^s(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,d)+(1-\lambda)^s(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,c)$$

hence, it follows that

$$\begin{aligned}|D|&\leq\frac{(b-a)(d-c)}{(r_1+1)(r_2+1)}\left(\frac{(1+r_1^2)(1+r_2^2)}{4(r_1+1)(r_2+1)}\right)^{1-\frac{1}{q}}\\&\times\left(\int_0^1\int_0^1\left|((r_1+1)t-1)((r_2+1)\lambda-1)\right|\left\{t^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,d)\right.\right.\\&+t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,c)+\lambda^s(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,d)\\&\left.\left.+(1-\lambda)^s(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,c)\right\}dtd\lambda\right)^{\frac{1}{q}}\end{aligned}\tag{3.6}$$

By a simple computation, one can see that

$$\begin{aligned}&\times\left(\int_0^1\int_0^1\left|((r_1+1)t-1)((r_2+1)\lambda-1)\right|\left\{t^s\lambda^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,d)\right.\right.\\&+t^s(1-\lambda)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(b,c)+\lambda^s(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,d)\\&\left.\left.+(1-\lambda)^s(1-t)^s\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q(a,c)\right\}dtd\lambda\right)^{\frac{1}{q}}\\&=\left(\frac{MS\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+ML\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+KR\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+KN\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q}{(s+1)^2(s+2)^2}\right)^{\frac{1}{q}}\end{aligned}$$

where K, L, M, N, R, and S as in Theorem 10. By substituting these in (3.6) and simplifying we obtain the required result.

Corollary 4

(1) Under the assumptions of Theorem 12, if we choose $r_1 = r_2 = 1$, we have

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&-\frac{1}{2}\left[\frac{1}{d-c}\int_c^d\left[f(b,y)+f(a,y)\right]dy+\frac{1}{b-a}\int_a^b\left[f(x,d)+f(x,c)\right]dx\right]\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|\\&\leq\frac{(b-a)(d-c)}{4}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(s+\frac{1}{2^s}\right)^{\frac{1}{q}}\\&\times\left[\frac{\frac{1}{2^{s+1}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q+\left(s+\frac{1}{2^{s+1}}\right)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q}{(s+1)^2(s+2)^2}\right]^{\frac{1}{q}}\end{aligned}$$

(2) Under the assumptions of Theorem 12, if we choose $r_1 = r_2 = 0$, we have

$$\left| f(a,c) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy - \frac{1}{d-c} \int_c^d f(a,y)dy - \frac{1}{b-a} \int_a^b f(x,c)dx \right|$$

$$\leq (b-a)(d-c)\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \times \left[\frac{(s+1)\left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\right|^q}{(s+1)^2(s+2)^2} \right]^{\frac{1}{q}}$$

Remark 5. *Under the assumptions of Theorem 1.2., if we choose* $r_1 = r_2 = 1$ *and* $s = 1$, *we get the inequality in (1.4).*

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Authors' contributions
HK, AOA and MA carried out the design of the study and performed the analysis. MEO (adviser) participated in its design and coordination. All authors read and approved the final manuscript.

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