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Stability results in non-Archimedean $\mathcal{L}$-fuzzy normed spaces for a cubic functional equation

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Abstract

We establish some stability results concerning the functional equation

$$nf(x+ny)+f(nx-y)=\frac{n(n^2+1)}{2}\big[f(x+y)+f(x-y)\big]+(n^4-1)f(y),$$

where $n \geq 2$ is a fixed integer in the setting of non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

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1 Introduction

The theory of fuzzy sets was introduced by Zadeh [34] in 1965. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development has been made in the field of fuzzy topology [1, 8, 9, 12–14, 18, 22, 30]. One of the problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces. In 2004, Park [23] introduced and studied the notion of intuitionistic fuzzy metric spaces. In 2006, Saadati and Park [28] introduced and studied the notion of intuitionistic fuzzy normed spaces.

On the other hand, the study of stability problems for a functional equation is related to the question of Ulam [33] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [19]. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. We refer the interested readers for more information on such problems to the papers [4, 6, 11, 20, 21, 25, 29, 32, 33].

Let X and Y be real linear spaces and $f : X \to Y$ a mapping. If $X = Y = \mathbb{R}$, the cubic function $f(x) = cx^3$, where c is a real constant, clearly satisfies the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x).$$

Hence, the above equation is called the *cubic functional equation*. Recently, Cho, Saadati and Wang [5] introduced the functional equation

$$3f(x+3y)+f(3x-y)=15\bigl[f(x+y)+f(x-y)\bigr]+80f(y),$$

which has $f(x)=cx^3$ $(x\in\mathbb{R})$ as a solution for $X=Y=\mathbb{R}$.

In this paper, we investigate the Hyers-Ulam stability of the functional equation as follows:

$$nf(x+ny)+f(nx-y)=\frac{n(n^2+1)}{2}\bigl[f(x+y)+f(x-y)\bigr]+\bigl(n^4-1\bigr)f(y), \tag{1.1}$$

where $n\geq 2$ is a fixed integer.

2 Preliminaries

In this section, we recall some definitions and results for our main result in this paper.

A triangular norm (shorter t-norm) is a binary operation on the unit interval $[0,1]$, *i.e.*, a function $T:[0,1]\times[0,1]\to[0,1]$ satisfying the following four axioms: for all $a,b,c\in[0,1]$,

(i) $T(a,b)=T(b,a)$ (commutativity);
(ii) $T(a,T(b,c))=T(T(a,b),c)$ (associativity);
(iii) $T(a,1)=a$ (boundary condition);
(iv) $T(a,b)\leq T(a,c)$ whenever $b\leq c$ (monotonicity).

Basic examples are the Łukasiewicz t-norm T_L and the t-norms T_P, T_M and T_D, where $T_L(a,b):=\max\{a+b-1,0\}$, $T_P(a,b):=ab$, $T_M(a,b):=\min\{a,b\}$ and

$$T_D(a,b):=\begin{cases}\min\{a,b\} & \text{if } \max\{a,b\}=1,\\ 0 & \text{otherwise}\end{cases}$$

for all $a,b\in[0,1]$.

For all $x\in[0,1]$ and all t-norms T, let $x_T^{(0)}:=1$. For all $x\in[0,1]$ and all t-norms T, define $x_T^{(r)}$ by the recursion equation $x_T^{(r)}=T(x_T^{(r-1)},x)$ for all $r\in\mathbb{N}$. A t-norm T is said to be *Hadžić* type (we denote it by $T\in\mathcal{H}$) if the family $(x_T^{(r)})_{r\in\mathbb{N}}$ is equicontinuous at $x=1$ (see [15]).

Other important triangular norms are as follows (see [16]):

- The *Sugeno-Weber family* $\{T_\lambda^{SW}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{SW}:=T_D$, $T_\infty^{SW}:=T_P$ and

  $$T_\lambda^{SW}(x,y):=\max\left\{0,\frac{x+y-1+\lambda xy}{1+\lambda}\right\}$$

  if $\lambda\in(-1,\infty)$.
- The *Domby family* $\{T_\lambda^{D}\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda=0$, T_M, if $\lambda=\infty$ and

  $$T_\lambda^{D}(x,y):=\frac{1}{1+[(\frac{1-x}{x})^\lambda+(\frac{1-y}{y})^\lambda]^{1/\lambda}}$$

  if $\lambda\in(0,\infty)$.

- The *Aczel-Alsina family* $\{T_\lambda^{AA}\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{AA}(x,y) := e^{-(|\log x|^\lambda+|\log y|^\lambda)^{1/\lambda}}$$

  if $\lambda \in (0,\infty)$.

A t-norm T can be extended (by associativity) in a unique way to an r-array operation by taking, for any $(x_1,\dots,x_r) \in [0,1]^r$, the value $T(x_1,\dots,x_r)$ defined by

$$T_{j=1}^0 x_j := 1, \qquad T_{j=1}^r x_j := T\bigl(T_{j=1}^{r-1} x_j, x_r\bigr) = T(x_1,\dots,x_r).$$

A t-norm T can also be extended to a countable operation by taking, for any sequence $(x_r)_{r\in\mathbb{N}}$ in $[0,1]$, the value

$$T_{j=1}^\infty x_j := \lim_{r\to\infty} T_{j=1}^r x_j. \tag{2.1}$$

The limit on the right side of (2.1) exists since the sequence $\{T_{j=1}^r x_j\}_{r\in\mathbb{N}}$ is non-increasing and bounded below.

Proposition 2.1 [16]

(1) For $T \geq T_L$, the following equivalence holds:

$$\lim_{r\to\infty} T_{j=1}^\infty x_{r+j} = 1 \quad\Longleftrightarrow\quad \sum_{r=1}^\infty (1-x_r) < \infty.$$

(2) If T is of Hadžić type, then

$$\lim_{r\to\infty} T_{j=1}^\infty x_{r+j} = 1$$

for all sequence $\{x_r\}_{r\in\mathbb{N}}$ in $[0,1]$ such that $\lim_{r\to\infty} x_r = 1$.

(3) If $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^D\}_{\lambda\in(0,\infty)}$, then

$$\lim_{r\to\infty} T_{j=1}^\infty x_{r+j} = 1 \quad\Longleftrightarrow\quad \sum_{r=1}^\infty (1-x_r)^\alpha < \infty.$$

(4) If $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, then

$$\lim_{r\to\infty} T_{j=1}^\infty x_{r+j} = 1 \quad\Longleftrightarrow\quad \sum_{r=1}^\infty (1-x_r) < \infty.$$

3 $\mathcal{L}$-fuzzy normed spaces

In this section, we give some definitions and related lemmas for our main result.

Definition 3.1 [10] Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and U a non-empty set called a universe. An *$\mathcal{L}$-fuzzy set* $\mathcal{A}$ on U is defined by a mapping $\mathcal{A} : U \to L$. For any $u \in U$, $\mathcal{A}(u)$ represents the *degree* (in L) to which u satisfies $\mathcal{A}$.

Lemma 3.2 [7] *Consider the set L^* and the operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, \qquad x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 3.3 [3] An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ on a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ for all $u \in U$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore, they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

In Section 2, we presented the classical definition of t-norms, which can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} := \inf L$ and $1_{\mathcal{L}} := \sup L$.

Definition 3.4 A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $\mathcal{T}(x, 1_{\mathcal{L}}) = x$ for all $x \in L$ (boundary condition);
(ii) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ for all $(x, y) \in L^2$ (commutativity);
(iii) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ for all $(x, y, z) \in L^3$ (associativity);
(iv) $x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y')$ for all $(x, x', y, y') \in L^4$ (monotonicity).

A t-norm can also be defined recursively as an $(r+1)$-array operation for each $r \in \mathbb{N}$ by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^r(x_1, \ldots, x_{r+1}) = \mathcal{T}\left(\mathcal{T}^{r-1}(x_1, \ldots, x_r), x_{r+1}\right)$$

for all $r \geq 2$ and $x_j \in L$.

The t-norm $\mathcal{M}$ defined by

$$\mathcal{M}(x, y) = \begin{cases} x & \text{if } x \leq_L y, \\ y & \text{if } y \leq_L x \end{cases}$$

is a continuous t-norm.

Definition 3.5 A t-norm $\mathcal{T}$ on L^* is said to be *t-representable* if there exist a t-norm T and a t-conorm S on $[0,1]$ such that

$$\mathcal{T}(x, y) = \left(T(x_1, y_1), S(x_2, y_2)\right)$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$.

Definition 3.6

(1) A *negator* on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.
(2) If a negator $\mathcal{N}$ on $\mathcal{L}$ satisfies $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involution negator*.

(3) The negator $\mathcal{N}_s$ on $([0,1],\leq)$ defined as $\mathcal{N}_s(x) = 1 - x$ for all $x \in [0,1]$ is called the *standard negator* on $([0,1],\leq)$.

Definition 3.7 The 3-tuple $(V,\mathcal{P},\mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy normed space* if V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0,\infty)$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in (0,\infty)$,

(a) $0_{\mathcal{L}} <_L \mathcal{P}(x,t)$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}} \Leftrightarrow x = 0$;
(c) $\mathcal{P}(\alpha x,t) = \mathcal{P}(x,\frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{T}(\mathcal{P}(x,t),\mathcal{P}(y,s)) \leq_L \mathcal{P}(x+y,t+s)$;
(e) $\mathcal{P}(x,\cdot) : (0,\infty) \to L$ is continuous;
(f) $\lim_{t\to 0}\mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called an *$\mathcal{L}$-fuzzy norm*. If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set and the t-norm $\mathcal{T}$ is t-representable, then the 3-tuple $(V,\mathcal{P}_{\mu,\nu},\mathcal{T})$ is said to be an *intuitionistic fuzzy normed space*.

Definition 3.8 (see [27]) Let $(V,\mathcal{P},\mathcal{T})$ be an *$\mathcal{L}$-fuzzy normed space*.

(1) A sequence $\{x_r\}_{r\in\mathbb{N}}$ in $(V,\mathcal{P},\mathcal{T})$ is called a *Cauchy sequence* if, for any $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and for any $t > 0$, there exists a positive integer r_0 such that

$$\mathcal{N}(\varepsilon) <_L \mathcal{P}(x_{r+p} - x_r, t)$$

for all $r \geq r_0$ and $p > 0$, where $\mathcal{N}$ is a negator on $\mathcal{L}$.

(2) A sequence $\{x_r\}_{r\in\mathbb{N}}$ in $(V,\mathcal{P},\mathcal{T})$ is said to be *convergent* to a point $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V,\mathcal{P},\mathcal{T})$ (denoted by $x_r \xrightarrow{\mathcal{P}} x$) if $\mathcal{P}(x_r - x, t) \to 1_{\mathcal{L}}$ wherever $r \to \infty$ for all $t > 0$.

(3) If every Cauchy sequence in $(V,\mathcal{P},\mathcal{T})$ is convergent in V, then the $\mathcal{L}$-fuzzy normed space $(V,\mathcal{P},\mathcal{T})$ is said to be *complete* and the $\mathcal{L}$-fuzzy normed space is called an *$\mathcal{L}$-fuzzy Banach space*.

Lemma 3.9 [26] *Let $\mathcal{P}$ be an $\mathcal{L}$-fuzzy norm on V. Then we have the following:*

(1) $\mathcal{P}(x,t)$ is non-decreasing with respect to $t \in (0,\infty)$ for all x in V.
(2) $\mathcal{P}(x-y,t) = \mathcal{P}(y-x,t)$ for all x, y in V and all $t \in (0,\infty)$.

Definition 3.10 Let $(V,\mathcal{P},\mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space. For any $t \in (0,\infty)$, we define the *open ball* $B(x,r,t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}},1_{\mathcal{L}}\}$ as

$$B(x,r,t) = \{y \in V : \mathcal{N}(r) <_L \mathcal{P}(x-y,t)\}.$$

A subset $A \subseteq V$ is called *open* if, for all $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}},1_{\mathcal{L}}\}$ such that $B(x,r,t) \subseteq A$.

Let $\tau_{\mathcal{P}}$ denote the family of all open subsets of V. Then $\tau_{\mathcal{P}}$ is called the *topology induced by the $\mathcal{L}$-fuzzy norm $\mathcal{P}$*.

4 $\mathcal{L}$-fuzzy Hyers-Ulam stability for cubic functional equations in non-Archimedean $\mathcal{L}$-fuzzy normed spaces

In 1897, Hensel [17] introduced the field with valuation in which does not have the Archimedean property.

Definition 4.1 Let $\mathcal{K}$ be a field. A *non-Archimedean absolute value* on $\mathcal{K}$ is a function $|\cdot| : \mathcal{K} \to [0, \infty)$ such that, for any $a, b \in \mathcal{K}$,

(1) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(2) $|ab| = |a||b|$,
(3) $|a + b| \leq \max\{|a|, |b|\}$ (the strict triangle inequality).

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is non-trivial, *i.e.*, there is $a_0 \in \mathcal{K}$ such that $|a_0| \notin \{0, 1\}$.

Definition 4.2 A *non-Archimedean $\mathcal{L}$-fuzzy normed space* is a triple $(V, \mathcal{P}, \mathcal{T})$, where V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, \infty)$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in (0, \infty)$,

(a) $0_{\mathcal{L}} <_L \mathcal{P}(x, t)$;
(b) $\mathcal{P}(x, t) = 1_{\mathcal{L}} \Leftrightarrow x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, \max\{t, s\})$;
(e) $\mathcal{P}(x, \cdot) : (0, \infty) \to L$ is continuous;
(f) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.

From now on, let $\mathcal{K}$ be a non-Archimedean field, X a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$. We investigate the Hyers-Ulam stability of the cubic functional equation (1.1).

Next, we define an $\mathcal{L}$-fuzzy approximately cubic mapping. Let Ψ be an $\mathcal{L}$-fuzzy set on $X \times X \times (0, \infty)$ such that $\Psi(x, y, \cdot)$ is non-decreasing,

$$\Psi(cx, cx, t) \geq_L \Psi\left(x, x, \frac{t}{|c|}\right)$$

and

$$\lim_{t\to\infty} \Psi(x, y, t) = 1_{\mathcal{L}}$$

for all $x, y \in X$, all $t > 0$ and all $c \in \mathcal{K} \setminus \{0\}$.

Definition 4.3 A mapping $f : X \to Y$ is said to be Ψ*-approximately cubic* if

$$\begin{aligned} &\mathcal{P}\left(nf(x + ny) + f(nx - y) - \frac{n(n^2 + 1)}{2}\big[f(x + y) + f(x - y)\big] - \big(n^4 - 1\big)f(y), t\right) \\ &\quad \geq_L \Psi(x, y, t) \end{aligned} \tag{4.1}$$

for all $x, y \in X$ and all $t > 0$.

The following is the main result in this paper.

Theorem 4.4 *Let $f : X \to Y$ be a Ψ-approximately cubic mapping. If $n \in \mathcal{K} \setminus \{0\}$ and there are $\alpha \in (0, \infty)$ and an integer k ($k \geq 2$) with $|n|^k < \alpha$ such that*

$$\Psi\left(n^{-k}x, n^{-k}y, t\right) \geq_L \Psi(x, y, \alpha t) \tag{4.2}$$

and

$$\lim_{r\to\infty} \mathcal{T}_{j=r}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|n|^{kj}}\right) = 1_{\mathcal{L}}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}\left(f(x) - C(x), t\right) \geq_L \mathcal{T}_{j=1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{j+1}}{|n|^{kj}} t\right) \tag{4.3}$$

for all $x \in X$ and all $t > 0$, where

$$\mathcal{M}(x, t) := \mathcal{T}\left(\Psi(x, 0, t), \Psi(nx, 0, t), \ldots, \Psi\left(n^{k-1}x, 0, t\right)\right)$$

for all $x \in X$ and all $t > 0$.

Proof Let $\mathcal{T}^0 : L \to L$ be the identity mapping $I_L : L \to L$. First, we show, by induction on j, that

$$\mathcal{P}\left(f\left(n^j x\right) - n^{3j} f(x), t\right) \geq_L \mathcal{M}_j(x, t) := \mathcal{T}^{j-1}\left(\Psi(x, 0, t), \ldots, \Psi\left(n^{j-1}x, 0, t\right)\right) \tag{4.4}$$

for all $x \in X$, all $t > 0$ and all $j \geq 1$. Putting $y = 0$ in (4.1), we obtain

$$\mathcal{P}\left(f(nx) - n^3 f(x), t\right) \geq_L \Psi(x, 0, t)$$

for all $x \in X$ and all $t > 0$. This proves (4.4) for $j = 1$. Let (4.4) hold for some $j \geq 1$. Replacing y by 0 and x by $3^j x$ in (4.1), we get

$$\mathcal{P}\left(f\left(n^{j+1}x\right) - n^3 f\left(n^j x\right), t\right) \geq_L \Psi\left(n^j x, 0, t\right)$$

for all $x \in X$ and all $t > 0$. Since $|n|^3 \leq 1$, it follows that

$$\begin{aligned}
&\mathcal{P}\left(f\left(n^{j+1}x\right) - n^{3(j+1)} f(x), t\right) \\
&\quad \geq_L \mathcal{T}\left(\mathcal{P}\left(f\left(n^{j+1}x\right) - n^3 f\left(n^j x\right), t\right), \mathcal{P}\left(n^3 f\left(n^j x\right) - n^{3(j+1)} f(x), t\right)\right) \\
&\quad = \mathcal{T}\left(\mathcal{P}\left(f\left(n^{j+1}x\right) - n^3 f\left(n^j x\right), t\right), \mathcal{P}\left(f\left(n^j x\right) - n^{3j} f(x), \frac{t}{|n|^3}\right)\right) \\
&\quad \geq_L \mathcal{T}\left(\mathcal{P}\left(f\left(n^{j+1}x\right) - n^3 f\left(n^j x\right), t\right), \mathcal{P}\left(f\left(n^j x\right) - n^{3j} f(x), t\right)\right) \\
&\quad \geq_L \mathcal{T}\left(\Psi\left(n^j x, 0, t\right), \mathcal{M}_j(x, t)\right) = \mathcal{M}_{j+1}(x, t)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus (4.4) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}\left(f\left(n^k x\right) - n^{3k} f(x), t\right) \geq_L \mathcal{M}(x, t)$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{n^{k(r+1)}}$ in the above inequality and using the inequality (4.2), we obtain

$$\mathcal{P}\left(f\left(\frac{x}{n^{kr}}\right) - n^{3k}f\left(\frac{x}{n^{k(r+1)}}\right), t\right) \geq_L \mathcal{M}\left(\frac{x}{n^{k(r+1)}}, t\right) \geq_L \mathcal{M}\left(x, \alpha^{r+1}t\right)$$

for all $x \in X$, all $t > 0$ and all $r \geq 0$, and so

$$\begin{aligned}&\mathcal{P}\left(n^{3kr}f\left(\frac{x}{n^{kr}}\right) - n^{3k(r+1)}f\left(\frac{x}{n^{k(r+1)}}\right), t\right)\\ &\quad\geq_L \mathcal{M}\left(x, \frac{\alpha^{r+1}}{|n|^{3kr}}t\right) \geq_L \mathcal{M}\left(x, \frac{\alpha^{r+1}}{|n|^{kr}}t\right)\end{aligned}$$

for all $x \in X$, all $t > 0$ and all $r \geq 0$. Hence it follows that

$$\begin{aligned}&\mathcal{P}\left(n^{3kr}f\left(\frac{x}{n^{kr}}\right) - n^{3k(r+p)}f\left(\frac{x}{n^{k(r+p)}}\right), t\right)\\ &\quad\geq_L \mathcal{T}_{j=r}^{r+p-1}\mathcal{P}\left(n^{3kj}f\left(\frac{x}{n^{kj}}\right) - n^{3k(j+1)}f\left(\frac{x}{n^{k(j+1)}}\right), t\right)\\ &\quad\geq_L \mathcal{T}_{j=r}^{r+p-1}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|n|^{kj}}t\right)\end{aligned}$$

for all $x \in X$, all $t > 0$ and all $r \geq 0$. Since $\lim_{r\to\infty} \mathcal{T}_{j=r}^{\infty}\mathcal{M}(x, \frac{\alpha^{j+1}}{|n|^{kj}}t) = 1_{\mathcal{L}}$ for all $x \in X$ and all $t > 0$, $\{n^{3kr}f(\frac{x}{n^{kr}})\}_{r\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$. Hence we can define a mapping $C : X \to Y$ such that

$$\lim_{r\to\infty} \mathcal{P}\left(n^{3kr}f\left(\frac{x}{n^{kr}}\right) - C(x), t\right) = 1_{\mathcal{L}} \tag{4.5}$$

for all $x \in X$ and all $t > 0$. Next, for all $r \geq 1$, all $x \in X$ and all $t > 0$, we have

$$\begin{aligned}\mathcal{P}\left(f(x) - n^{3kr}f\left(\frac{x}{n^{kr}}\right), t\right) &= \mathcal{P}\left(\sum_{j=0}^{r-1}\left[n^{3kj}f\left(\frac{x}{n^{kj}}\right) - n^{3k(j+1)}f\left(\frac{x}{n^{k(j+1)}}\right)\right], t\right)\\ &\geq_L \mathcal{T}_{j=0}^{r-1}\mathcal{P}\left(n^{3kj}f\left(\frac{x}{n^{kj}}\right) - n^{3k(j+1)}f\left(\frac{x}{n^{k(j+1)}}\right), t\right)\\ &\geq_L \mathcal{T}_{j=0}^{r-1}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|n|^{kj}}t\right)\end{aligned}$$

for all $x \in X$ and all $t > 0$, and so

$$\begin{aligned}\mathcal{P}\big(f(x) - C(x), t\big) &\geq_L \mathcal{T}\left(\mathcal{P}\left(f(x) - n^{3kr}f\left(\frac{x}{n^{kr}}\right), t\right),\right.\\ &\qquad\left.\mathcal{P}\left(n^{3kr}f\left(\frac{x}{n^{kr}}\right) - C(x), t\right)\right)\\ &\geq_L \mathcal{T}\left(\mathcal{T}_{j=0}^{r-1}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|n|^{kj}}t\right),\right.\\ &\qquad\left.\mathcal{P}\left(n^{3kr}f\left(\frac{x}{n^{kr}}\right) - C(x), t\right)\right)\end{aligned}$$

for all $x \in X$ and all $t > 0$. Taking the limit as $r \to \infty$ in the above inequality, we obtain

$$\mathcal{P}\big(f(x) - C(x), t\big) \geq_L \mathcal{T}_{j=0}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{j+1}}{|n|^{kj}} t\right)$$

for all $x \in X$ and all $t > 0$, which proves (4.3).

Since $\mathcal{T}$ is continuous, from the well-known result in an $\mathcal{L}$-fuzzy (probabilistic) normed space (see [31], Chapter 12), it follows that

$$\begin{aligned}
&\lim_{r\to\infty} \mathcal{P}\left(n \cdot n^{3kr} f\big(n^{-kr}(x+ny)\big) + n^{3kr} f\big(n^{-kr}(nx-y)\big) - \frac{n(n^2+1)}{2} n^{3kr} f\big(n^{-kr}(x+y)\big)\right. \\
&\qquad \left. - \frac{n(n^2+1)}{2} n^{3kr} f\big(n^{-kr}(x-y)\big) - \big(n^4-1\big) n^{3kr} f\big(n^{-kr} y\big), t\right) \\
&\quad = \mathcal{P}\left(n \cdot C(x+ny) + C(nx-y) - \frac{n(n^2+1)}{2} C(x+y) - \frac{n(n^2+1)}{2} C(x-y)\right. \\
&\qquad \left. - \big(n^4-1\big) C(y), t\right)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. On the other hand, replacing x, y by $n^{-kr}x$, $n^{-kr}y$ in (4.1) and (4.2), we get

$$\begin{aligned}
&\mathcal{P}\left(n \cdot n^{3kr} f\big(n^{-kr}(x+ny)\big) + n^{3kr} f\big(n^{-kr}(nx-y)\big) - \frac{n(n^2+1)}{2} n^{3kr} f\big(n^{-kr}(x+y)\big)\right. \\
&\qquad \left. - \frac{n(n^2+1)}{2} n^{3kr} f\big(n^{-kr}(x-y)\big) - \big(n^4-1\big) n^{3kr} f\big(n^{-kr} y\big), t\right) \\
&\quad \geq_L \Psi\left(n^{-kr}x, n^{-kr}y, \frac{t}{|n|^{3kr}}\right) \geq_L \Psi\left(x, y, \frac{\alpha^r t}{|n|^{3kr}}\right)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Since $\lim_{r\to\infty} \Psi(x, y, \frac{\alpha^r t}{|n|^{3kr}}) = 1_{\mathcal{L}}$, we infer that C is a cubic mapping.

For the uniqueness of C, let $C' : X \to Y$ be another cubic mapping such that

$$\mathcal{P}\big(C'(x) - f(x), t\big) \geq_L \mathcal{M}(x, t)$$

for all $x \in X$ and all $t > 0$. Then we have, for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned}
&\mathcal{P}\big(C(x) - C'(x), t\big) \\
&\quad \geq_L \mathcal{T}\left(\mathcal{P}\left(C(x) - n^{3kr} f\left(\frac{x}{n^{kr}}\right), t\right), \mathcal{P}\left(n^{3kr} f\left(\frac{x}{n^{kr}}\right) - C'(x), t\right)\right).
\end{aligned}$$

Therefore, from (4.5), we conclude that $C = C'$. This completes the proof. □

Corollary 4.5 *Let $\mathcal{T} \in \mathcal{H}$ and let $f : X \to Y$ be a Ψ-approximately cubic mapping. If $n \in \mathcal{K} \setminus \{0\}$ and there are $\alpha \in (0, \infty)$ and an integer k ($k \geq 2$) with $|n|^k < \alpha$ such that*

$$\Psi\big(n^{-k}x, n^{-k}y, t\big) \geq_L \Psi(x, y, \alpha t)$$

and

$$\lim_{r\to\infty} \mathcal{M}\left(x, \frac{\alpha^r t}{|n|^{kr}}\right) = 1_{\mathcal{L}}$$

for all $x, y \in X$ *and all* $t > 0$, *then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$\mathcal{P}\big(f(x) - C(x), t\big) \geq_L \mathcal{T}_{j=1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{j+1}}{|n|^{kj}} t\right)$$

for all $x \in X$ *and all* $t > 0$, *where*

$$\mathcal{M}(x, t) := \mathcal{T}\big(\Psi(x, 0, t), \Psi(nx, 0, t), \ldots, \Psi\big(n^{k-1}x, 0, t\big)\big)$$

for all $x \in X$ *and all* $t > 0$.

Proof Since

$$\lim_{r\to\infty} \mathcal{M}\left(x, \frac{\alpha^r}{|n|^{kr}} t\right) = 1_{\mathcal{L}}$$

for all $x \in X$ and all $t > 0$ and $\mathcal{T}$ is of *Hadžić* type, it follows from Proposition 2.1 that

$$\lim_{r\to\infty} \mathcal{T}_{j=r+1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j}{|n|^{kj}} t\right) = 1_{\mathcal{L}}$$

for all $x \in X$ and all $t > 0$. Therefore, if we apply Theorem 4.4, then we get the conclusion. This completes the proof. □

Example 4.6 Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space, $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space) in which

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right)$$

for all $x \in X$ and all $t > 0$ and $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ a complete non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space). Define

$$\Psi(x, y, t) = \left(\frac{t}{1 + t}, \frac{1}{1 + t}\right)$$

for all $x, y \in X$ and all $t > 0$. It is easy to see that (4.2) holds for $\alpha = 1$. Also, since

$$\mathcal{M}(x, t) = \left(\frac{t}{1 + t}, \frac{1}{1 + t}\right)$$

for all $x \in X$ and all $t > 0$, we have

$$\begin{aligned}
\lim_{r\to\infty} (\mathcal{T}_M)_{j=r}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j}{|n|^{kj}} t\right) &= \lim_{r\to\infty}\left[\lim_{s\to\infty} (\mathcal{T}_M)_{j=r}^{s} \mathcal{M}\left(x, \frac{\alpha^j}{|n|^{kj}} t\right)\right] \\
&= \lim_{r\to\infty}\lim_{s\to\infty}\left(\frac{t}{t + |n|^{kj}}, \frac{|n|^{kr}}{t + |n|^{kr}}\right) = (1, 0) = 1_{L^*}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Let $f : X \to Y$ be a Ψ-approximately cubic mapping. A straightforward computation shows that

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}\left(nf(x+ny)+f(nx-y)-\frac{n(n^2+1)}{2}\big[f(x+y)+f(x-y)\big]-\big(n^4-1\big)f(y),t\right)\\&\quad\geq_{L^*}\mathcal{P}_{\mu,\nu}(x+y,t)\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Therefore, all the conditions of Theorem 4.4 hold, and so there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}\big(f(x)-C(x),t\big)\geq_{L^*}\left(\frac{t}{t+|n|^k},\frac{|n|^k}{t+|n|^k}\right)$$

for all $x \in X$ and all $t > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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