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Multiple singular integrals and Marcinkiewicz integrals with mixed homogeneity along surfaces

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Abstract

This paper is devoted to studying the singular integrals and Marcinkiewicz integrals with mixed homogeneity along surfaces, which contain many classical surfaces as model examples, on the product domains $\mathbb{R}^m \times \mathbb{R}^n$ ($m, n \geq 2$). Under rather weak size conditions of the kernels, the $L^p(\mathbb{R}^m \times \mathbb{R}^n)$-boundedness for such operators is established. These results essentially extend certain previous results.

MSC: 42B20; 42B25

Keywords: singular integrals; Marcinkiewicz integrals; maximal operators; mixed homogeneity; product domains

1 Introduction

Let $\mathbb{R}^d$ ($d = m$ or n), $d \geq 2$, be the d-dimensional Euclidean space and S^{d-1} be the unit sphere in $\mathbb{R}^d$ equipped with the induced Lebesgue measure $d\sigma_d$. Let $\alpha_{d,1}, \alpha_{d,2}, \ldots, \alpha_{d,d}$ be fixed real numbers, $\alpha_{d,j} \geq 1$ ($j = 1, \ldots, d$). Define the function $F : \mathbb{R}^d \times (0, \infty) \longrightarrow \mathbb{R}$ by $F(x, \rho_d) = \sum_{j=1}^d x_j^2 \rho_d^{-2\alpha_{d,j}}$, $x = (x_1, x_2, \ldots, x_d)$. It is clear that for each fixed $x \in \mathbb{R}^d$, the function $F(x, \rho_d)$ is a decreasing function in $\rho_d > 0$. We let $\rho_d(x)$ denote the unique solution of the equation $F(x, \rho_d) = 1$. Fabes and Riviére [13] showed that $(\mathbb{R}^d, \rho_d)$ is a metric space, which is often called the mixed homogeneity space related to $\{\alpha_{d,j}\}_{j=1}^d$. For $\lambda > 0$, we let $A_{d,\lambda}$ be the diagonal $d \times d$ matrix $A_{d,\lambda} = \mathrm{diag}\{\lambda^{\alpha_{d,1}}, \ldots, \lambda^{\alpha_{d,d}}\}$. Let $\phi : \mathbb{R}^+ \longrightarrow (0, \infty)$, we denote $A_{d,\phi(\rho_d(y))} y'$ by $A_d^{\phi}(y)$ for $y \in \mathbb{R}^d$, where $y' = A_{d,\rho_d(y)^{-1}} y \in S^{d-1}$.

Let $\beta_d = \max_{1 \leq j \leq d} \alpha_{d,j}$, $\gamma_d = \min_{1 \leq j \leq d} \alpha_{d,j}$. It is easy to check that

$$\rho_d(x)^{\gamma_d} < |x| < \rho_d(x)^{\beta_d}, \quad \text{if } \rho_d(x) > 1;$$

$$\rho_d(x)^{\beta_d} < |x| < \rho_d(x)^{\gamma_d}, \quad \text{if } \rho_d(x) < 1;$$

$$\rho_d(x) = |x|, \quad \text{if } \rho_d(x) = 1.$$

The change of variables related to the spaces $(\mathbb{R}^d, \rho_d)$ is given by the transformation

$$x_1 = \rho_d^{\alpha_{d,1}} \cos\theta_1 \cdots \cos\theta_{n-2} \cos\theta_{n-1},$$

$$x_2 = \rho_d^{\alpha_{d,2}} \cos\theta_1 \cdots \cos\theta_{n-2} \sin\theta_{n-1},$$

$$\cdots,$$

$$x_{d-1} = \rho_d^{\alpha_{d,d-1}} \cos\theta_1 \sin\theta_2,$$
$$x_d = \rho_d^{\alpha_{d,d}} \sin\theta_1.$$

Thus $dx = \rho_d^{\alpha_d-1} J_d(x')\, d\rho_d\, d\sigma_d(x')$, where $\rho_d^{\alpha_d-1} J_d(x')$ is the Jacobian of the above transform and $\alpha_d = \sum_{j=1}^d \alpha_{d,j}$, $J_d(x') = \sum_{j=1}^d \alpha_{d,j}(x'_j)^2$. Obviously, $J_d(x') \in C^\infty(S^{d-1})$ and there exists $M_d > 0$ such that

$$1 \le J_d(x') \le M_d, \quad \forall x' \in S^{d-1}.$$

Let $\Omega \in L^1(S^{d-1})$ and satisfy the following conditions:

$$\Omega(A_{d,\lambda}x) = \Omega(x), \quad \forall\lambda > 0 \text{ and } x \neq 0,$$
$$\int_{S^{d-1}} \Omega(y') J_d(y')\, d\sigma_d(y') = 0.$$

Define the parabolic singular integral operator T by

$$Tf(x) := p.v. \int_{\mathbb{R}^d} \frac{\Omega(y')}{\rho_d(y)^{\alpha_d}} f(x-y)\, dy. \tag{1.1}$$

As is well known, a singular integral operator of the type (1.1) originally arose from the study on the existence and regularity results of the heat equation and the more general parabolic differential operator with constant coefficients. In 1966, Fabes and Riviére [13] showed that T is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ if $\Omega \in C^1(S^{d-1})$. Subsequently, Nagel, Riviére and Wainger [18] weakened the regularity condition on Ω to the case $\Omega \in L\log^+ L(S^{d-1})$. Recently, Chen, Ding and Fan [5] extended further the condition to the case $\Omega \in H^1(S^{d-1})$.

In this paper, we will continue the research along this line. We will focus our attention on the multiple singular integrals with mixed homogeneity. Assume that $\Omega \in L^1(S^{m-1} \times S^{n-1})$ and satisfies the following conditions:

$$\Omega(A_{m,s}x, A_{n,t}y) = \Omega(x,y), \quad \forall s,t > 0, (x,y) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{1.2}$$
$$\int_{S^{m-1}} \Omega(u',\cdot) J_m(u')\, d\sigma_m(u') = \int_{S^{n-1}} \Omega(\cdot,v') J_n(v')\, d\sigma_n(v') = 0. \tag{1.3}$$

We consider the multiple singular integral with mixed homogeneity defined by

$$T_\Omega(f)(x,y) := p.v. \iint_{\mathbb{R}^m\times\mathbb{R}^n} \frac{\Omega(u',v')}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}} f(x-u, y-v)\, du\, dv. \tag{1.4}$$

In 2011, Chen and Le [8] showed that if $\Omega \in L(\log^+ L)^2(S^{m-1} \times S^{n-1})$, then T_Ω is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < p < \infty$. On the other hand, in the special case $\alpha_{m,i} = \alpha_{n,j} = 1$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$), T_Ω is the classical multiple singular integral, which is studied extensively by many authors (see [2, 10, 12, 14, 15, 19, 25, 27, 28] for examples). In particular, Ying [28] (also see [27] for a more general case) proved that T_Ω is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $2\beta/(2\beta-1) < p < 2\beta$ and $\beta > 1$ provided that Ω satisfies the following condition:

$$\sup_{(\xi',\eta')\in S^{m-1}\times S^{n-1}} \iint_{S^{m-1}\times S^{n-1}} \left|\Omega(u',v')\right| \left\{G(\xi',\eta';u',v')\right\}^{\beta} d\sigma_m(u')\,d\sigma_n(v') < \infty, \tag{1.5}$$

where

$$G(\xi',\eta';u',v') = \log\frac{1}{|\xi'\cdot u'|} + \log\frac{1}{|\eta'\cdot v'|} + \log\frac{1}{|\xi'\cdot u'|}\cdot\log\frac{1}{|\eta'\cdot v'|}.$$

It should be pointed out that the condition (1.5) for one parameter case was originally defined in Walsh's paper [22] and developed by Grafakos and Stefanov [16]. For the sake of simplicity, we denote that for $\beta > 0$,

$$\mathcal{F}_{\beta}(S^{m-1}\times S^{n-1}) = \left\{\Omega \in L^1(S^{m-1}\times S^{n-1}) : \Omega \text{ satisfies } (1.5)\right\}.$$

Employing the ideas in [16], one easily verifies that for $\beta_1 > \beta_2 > 0$, $\mathcal{F}_{\beta_1}(S^{m-1}\times S^{n-1}) \subsetneq \mathcal{F}_{\beta_2}(S^{m-1}\times S^{n-1})$ and

$$\begin{aligned}\bigcap_{\beta>1}\mathcal{F}_{\beta}(S^{m-1}\times S^{n-1}) &\nsubseteq L(\log^+ L)^2(S^{m-1}\times S^{n-1}) \subsetneq L\log^+ L(S^{m-1}\times S^{n-1}) \\ &\nsubseteq \bigcup_{\beta>1}\mathcal{F}_{\beta}(S^{m-1}\times S^{n-1}).\end{aligned} \tag{1.6}$$

Based on the above, a natural question is as follows.

Question 1.1 For the general case $\alpha_{m,i} \geq 1$ $(i = 1,\dots,m)$ and $\alpha_{n,j} \geq 1$ $(j = 1,\dots,n)$, is T_Ω bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ under the condition (1.5) for some $\beta > 1$?

One of the main purposes of this paper is to give a positive answer to the above question. The method we use allows us to treat a family of operators broader than those given by (1.4). To be precise, for suitable functions $\varphi,\psi : \mathbb{R}^+ \longrightarrow (0,\infty)$ and two real polynomial P_{N_i} on $\mathbb{R}$ with $P_{N_i}(0) = 0$ and $P_{N_i}(t) > 0$ for $t \neq 0$, where N_i is the degree of P_{N_i} $(i = 1,2)$, we define the multiple singular integral operator T_Ω^P along surfaces $S(P_{N_1}(\varphi),P_{N_2}(\psi))$ by

$$\begin{aligned}&T_\Omega^P(f)(x,y)\\ &\quad = p.v.\iint_{\mathbb{R}^m\times\mathbb{R}^n} \frac{\Omega(u',v')}{\rho_m(u)^{\alpha_m}\rho_n(v)^{\alpha_n}} f\left(x - A_m^{P_{N_1}(\varphi)}(u), y - A_n^{P_{N_2}(\psi)}(v)\right) du\,dv,\end{aligned} \tag{1.7}$$

where

$$S(P_{N_1}(\varphi),P_{N_2}(\psi)) := \left\{\left(A_m^{P_{N_1}(\varphi)}(u), A_n^{P_{N_2}(\psi)}(v)\right) : (u,v)\in\mathbb{R}^m\times\mathbb{R}^n\right\}.$$

Obviously, T_Ω is the special case of T_Ω^P for $P_{N_i}(s) = \varphi(s) = \psi(s) = s$ $(i = 1,2)$. Also, in the special case $\alpha_{m,i} = \alpha_{n,j} = 1$ $(i = 1,\dots,m; j = 1,\dots,n)$,

$$S(P_{N_1}(\varphi),P_{N_2}(\psi)) = \left\{\left(P_{N_1}(\varphi(|u|))u', P_{N_2}(\psi(|v|))v'\right) : (u,v)\in\mathbb{R}^m\times\mathbb{R}^n\right\}. \tag{1.8}$$

Moreover, for the special case $\varphi(s) = \psi(s) = s$ and $\alpha_{m,i} = \alpha_{n,j} = 1$ $(i = 1, \dots, m; j = 1, \dots, n)$,

$$S\big(P_{N_1}(\varphi), P_{N_2}(\psi)\big) = \big\{\big(P_{N_1}\big(|u|\big)u', P_{N_2}\big(|v|\big)v'\big) : (u,v) \in \mathbb{R}^m \times \mathbb{R}^n\big\}.$$

Wu and Yang [27] proved that if $\Omega \in \mathcal{F}_\beta(S^{m-1} \times S^{n-1})$ with $\beta > 1$, then T^P_Ω is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$. In this paper, we will extend the result above as follows.

Theorem 1.1 *Let P_{N_1} and P_{N_2} be two real polynomials on $\mathbb{R}$ with $P_{N_i}(0) = 0$ and $P_{N_i}(t) > 0$ for $t \neq 0$, where N_i is the degree of P_{N_i} $(i = 1, 2)$, and let $\varphi, \psi \in \mathfrak{F}$, where $\mathfrak{F}$ is the set of functions ϕ satisfying the following properties:*

- *(i) $\phi : \mathbb{R}^+ \longrightarrow (0, \infty)$ is continuous strictly increasing and $\phi \in \mathcal{C}^1((0, \infty))$ satisfying that ϕ' is monotonous;*
- *(ii) there exist constants $C_\phi, c_\phi > 0$ such that $t\phi'(t) \geq C_\phi\phi(t)$ and $\phi(2t) \leq c_\phi\phi(t)$ for all $t > 0$.*

Suppose that Ω satisfies (1.2)-(1.3) and $\Omega \in \mathcal{F}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > 1$. Then T^P_Ω defined as in (1.7) is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $2\beta/(2\beta - 1) < p < 2\beta$. The bound is independent of the coefficients of P_{N_i} $(i = 1, 2)$, but depends on φ, ψ, N_1, N_2, m, n and β.

Remark 1.1 For any $\phi \in \mathfrak{F}$, there exists a constant $B_\phi > 1$ such that $\phi(2r) \geq B_\phi\phi(r)$ for all $r > 0$. To see this, by the mean-valued theorem, for any $r > 0$, there exists $s \in (r, 2r)$ such that $\phi(2r) - \phi(r) = r\phi'(s)$. The properties (i) and (ii) of ϕ imply that

$$\phi(2r) - \phi(r) = r\phi(s) \geq rC_\phi\frac{\phi(s)}{s} \geq \frac{C_\phi}{2}\phi(r).$$

Taking $B_\phi = 1 + C_\phi/2$, this is the desired constant.

Remark 1.2 We remark that the model examples for functions $\phi \in \mathfrak{F}$ are t^α $(\alpha > 0)$, $t\ln(1 + t)$, $t\ln\ln(e + t)$ and real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ (see [3]). Theorem 1.1 extends the result of [27], which is the multiple-parameter generalization of the result in [11, 16], to the mixed homogeneity setting, even in the special case $\varphi(s) = \psi(s) = s$. Also, by (1.6), Theorem 1.1 is distinct from the result of [8], even in the special case $P_{N_1}(s) = P_{N_2}(s) = \varphi(s) = \psi(s) = s$.

On the other hand, we also consider the multiple Marcinkiewicz integral operator $\mathcal{M}^P_\Omega$ along the surfaces $S(P_{N_1}(\varphi), P_{N_2}(\psi))$ defined by

$$\mathcal{M}^P_\Omega(f)(x,y) = \left(\int_0^\infty \int_0^\infty \big|F^{P_{N_1}(\varphi),P_{N_2}(\psi)}_{s,t}(x,y)\big|^2 \frac{ds\,dt}{s^3t^3}\right)^{1/2}, \tag{1.9}$$

where

$$F^{P_{N_1}(\varphi),P_{N_2}(\psi)}_{s,t}(x,y) = \iint_{\Lambda(s,t)} \frac{\Omega(u,v)}{\rho_m(u)^{\alpha_m-1}\rho_n(v)^{\alpha_n-1}} f\big(x - A^{P_{N_1}(\varphi)}_m(u), y - A^{P_{N_2}(\psi)}_n(v)\big)\,du\,dv,$$

and $\Lambda(s,t) = \{(u,v) \in \mathbb{R}^m \times \mathbb{R}^n : \rho_m(u) \leq s, \rho_n(v) \leq t\}$.

When $P_{N_1}(s) = P_{N_2}(s) = \varphi(s) = \psi(s) = s$, $\alpha_{m,i} = \alpha_{n,j} = 1$ $(i = 1, \dots, m; j = 1, \dots, n)$, we denote $\mathcal{M}^P_\Omega$ by $\mathcal{M}_\Omega$, which is the classical Marcinkiewicz integral on the product domains and is studied extensively by many authors (see [1, 3, 4, 6, 7, 17, 23–26] *et al.*). In particular, Al-Qassem, Al-Salman, Cheng and Pan [1] showed that if $\Omega \in L\log^+ L(S^{m-1} \times S^{n-1})$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < p < \infty$; Hu, Lu and Yan [17] (also see [23, 26]) proved that if $\Omega \in \mathcal{F}_\beta(S^{m-1} \times S^{n-1})$ for $\beta > 1/2$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. For the general operator $\mathcal{M}_\Omega^P$, when $P_{N_i}(t) = t$ $(i = 1, 2)$ and $\varphi, \psi \in \mathfrak{F}$, Al-Salman [3] showed that $\mathcal{M}_\Omega^P$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < p < \infty$ provided that $\Omega \in L\log^+ L(S^{m-1} \times S^{n-1})$.

A natural question which arises from the above is the following:

Question 1.2 Under the condition (1.5) with $\beta > 1/2$, is $\mathcal{M}_\Omega^P$ also bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$?

This question will be addressed by our next theorem.

Theorem 1.2 *Let P_{N_i} $(i = 1, 2)$, φ, ψ be as in Theorem 1.1. Suppose $\Omega \in \mathcal{F}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > 1/2$ and satisfies (1.2)-(1.3). Then $\mathcal{M}_\Omega^P$ defined as (1.9) is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 1 + 2\beta$. The bound is independent of the coefficients of P_{N_i} $(i = 1, 2)$ but depends on φ, ψ, N_1, N_2, m, n and β.*

Remark 1.3 Theorem 1.2 extends the result of [17] to the mixed homogeneity setting, even for the special case $P_{N_1}(s) = P_{N_2}(s) = \varphi(s) = \psi(s) = s$. And by (1.6), Theorem 1.2 is distinct from the result of [3], even in the special case $P_{N_1}(s) = P_{N_2}(s) = s$.

The rest of this paper is organized as follows. After recalling some notation and establishing some preliminary lemmas, we will prove Theorem 1.1 in Section 2. And the proof of Theorem 1.2 will be given in Section 3. We remark that our some ideas in the proofs of our main results are taken from [3, 9, 11, 17], but our methods and technique are more delicate and complex than those used in [3, 9, 11, 17].

Throughout this paper, the letter C or c, sometimes with additional parameters, will stand for positive constants, not necessarily the same at each occurrence but independent of the essential variables.

2 On multiple singular integrals

Let us begin with some notations and lemmas. For given positive polynomials $P_{N_1}(t) = \sum_{i=1}^{N_1} \beta_i t^i$, $P_{N_2}(t) = \sum_{i=1}^{N_2} \gamma_i t^i$ and two smooth functions $\varphi, \psi \in \mathfrak{F}$, we set

$$P_1^l(t) = \big(P_{N_1}(t)\big)^{\alpha_{m,l}} := \sum_{i=1}^{N_1\alpha_{m,l}} a_{i,l} t^i \quad \text{for } l \in \{1, 2, \ldots, m\};$$

$$P_2^k(t) = \big(P_{N_2}(t)\big)^{\alpha_{n,k}} := \sum_{j=1}^{N_1\alpha_{n,k}} b_{j,k} t^j \quad \text{for } k \in \{1, 2, \ldots, n\}.$$

Then for $x, \xi \in \mathbb{R}^m$; $y, \eta \in \mathbb{R}^n$,

$$A_m^{P_{N_1}(\varphi)}(x) \cdot \xi = \sum_{l=1}^{m} P_{N_1}\big(\varphi\big(\rho_m(x)\big)\big)^{\alpha_{m,l}} x_l' \cdot \xi_l = \sum_{\ell=1}^{m} \sum_{i=1}^{N_1\alpha_{m,l}} a_{i,l}\varphi\big(\rho_m(x)\big)^i x_l' \cdot \xi_l,$$

$$A_n^{P_{N_2}(\psi)}(y)\cdot\eta=\sum_{k=1}^{n}P_{N_2}\bigl(\psi\bigl(\rho_n(y)\bigr)\bigr)^{\alpha_{n,k}}y_k'\cdot\eta_k=\sum_{k=1}^{n}\sum_{j=1}^{N_2\alpha_{n,k}}b_{j,k}\psi\bigl(\rho_n(y)\bigr)^j y_k'\cdot\eta_k.$$

We denote $\mathcal{N}_1 := \max\{N_1\alpha_{m,l} : 1 \le l \le m\}$, $\mathcal{N}_2 := \max\{N_2\alpha_{n,k} : 1 \le k \le n\}$ and set $a_{i,l} = 0$ whenever $i > N_1\alpha_{m,l}$; $b_{j,k} = 0$ whenever $j > N_2\alpha_{n,k}$. So we can write

$$A_m^{P_{N_1}(\varphi)}(x)\cdot\xi=\sum_{\ell=1}^{m}\sum_{i=1}^{N_1\alpha_{m,l}}a_{i,l}\varphi\bigl(\rho_m(x)\bigr)^i x_l'\cdot\xi_l=\sum_{i=1}^{\mathcal{N}_1}\bigl(L_i(\xi)\cdot x'\bigr)\varphi\bigl(\rho_m(x)\bigr)^i,$$

where $L_i(\xi) = (a_{i,1}\xi_1, a_{i,2}\xi_2, \dots, a_{i,m}\xi_m)$. Similarly,

$$A_n^{P_{N_2}(\psi)}(y)\cdot\eta=\sum_{j=1}^{\mathcal{N}_2}\bigl(I_j(\eta)\cdot y'\bigr)\psi\bigl(\rho_n(y)\bigr)^j,$$

where $I_j(\eta) = (b_{j,1}\eta_1, b_{j,2}\eta_2, \dots, b_{j,n}\eta_n)$. For $\mu \in \{0,1,\dots,\mathcal{N}_1\}$, $\nu \in \{0,1,\dots,\mathcal{N}_2\}$, we set

$$Q_\mu(x)=\left(\sum_{i=1}^{\mu}a_{i,1}x_1'\varphi\bigl(\rho_m(x)\bigr)^i,\dots,\sum_{i=1}^{\mu}a_{i,m}x_m'\varphi\bigl(\rho_m(x)\bigr)^i\right),$$
$$R_\nu(y)=\left(\sum_{j=1}^{\nu}b_{j,1}y_1'\psi\bigl(\rho_n(y)\bigr)^j,\dots,\sum_{j=1}^{\nu}b_{j,n}y_n'\psi\bigl(\rho_n(y)\bigr)^j\right).$$

Here we use the convention $\sum_{i\in\emptyset}a_i = 0$. Hence,

$$Q_\mu(x)\cdot\xi=\sum_{i=1}^{\mu}\bigl(L_i(\xi)\cdot x'\bigr)\varphi\bigl(\rho_m(x)\bigr)^i,\quad 0\le\mu\le\mathcal{N}_1;$$
$$R_\nu(y)\cdot\eta=\sum_{j=1}^{\nu}\bigl(I_j(\eta)\cdot y'\bigr)\psi\bigl(\rho_n(y)\bigr)^j,\quad 0\le\nu\le\mathcal{N}_2.$$

For any $\kappa,\ell \in \mathbb{Z}$ and $\mu \in \{0,1,\dots,\mathcal{N}_1\}$, $\nu \in \{0,1,\dots,\mathcal{N}_2\}$, we define the measures $\{\sigma_{\kappa,\ell;\mu,\nu}\}$ and $\{|\sigma_{\kappa,\ell;\mu,\nu}|\}$ as follows.

$$\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)=\iint_{\Delta_{\kappa,\ell}}\frac{\Omega(x,y)}{\rho_m(x)^{\alpha_m}\rho_n(y)^{\alpha_n}}\exp\bigl(-i\bigl(Q_\mu(x)\cdot\xi+R_\nu(y)\cdot\eta\bigr)\bigr)\,dx\,dy, \tag{2.1}$$

$$|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}|(\xi,\eta)=\iint_{\Delta_{\kappa,\ell}}\frac{|\Omega(x,y)|}{\rho_m(x)^{\alpha_m}\rho_n(y)^{\alpha_n}}\exp\bigl(-i\bigl(Q_\mu(x)\cdot\xi+R_\nu(y)\cdot\eta\bigr)\bigr)\,dx\,dy, \tag{2.2}$$

where $\Delta_{\kappa,\ell} = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : 2^{\kappa-1} \le \rho_m(x) < 2^\kappa, 2^{\ell-1} \le \rho_n(y) < 2^\ell\}$. By (1.3) and $Q_0(x) = (0,0,\dots,0) \in \mathbb{R}^m$, $R_0(y) = (0,0,\dots,0) \in \mathbb{R}^n$, for $\mu \in \{0,1,\dots,\mathcal{N}_1\}$ and $\nu \in \{0,1,\dots,\mathcal{N}_2\}$ we have

$$\widehat{\sigma_{\kappa,\ell;0,\nu}}(\xi,\eta)=\widehat{\sigma_{\kappa,\ell;\mu,0}}(\xi,\eta)=0. \tag{2.3}$$

Then it is easy to see that

$$T_\Omega^P(f)(x,y)=\sum_{\kappa,\ell\in\mathbb{Z}}\sigma_{\kappa,\ell;\mathcal{N}_1,\mathcal{N}_2}*f(x,y). \tag{2.4}$$

Lemma 2.1 (cf. [21, pp.476-478]) *Let $\mathcal{P}$ be a polynomial mapping $\mathbb{R}^+ \longrightarrow \mathbb{R}^d$, where $\mathcal{P}(t) = (P_1(t), P_2(t), \dots, P_d(t))$ and P_i is a real polynomial defined on $\mathbb{R}^+$ ($i = 1, \dots, d$). Then the maximal function $M_{\mathcal{P}}(f)(x)$ defined by*

$$M_{\mathcal{P}}(f)(x) = \sup_{r>0} \frac{1}{r} \left| \int_{|t|\le r} f\bigl(x - \mathcal{P}(t)\bigr)\, dt \right|$$

is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. The bound is independent of the coefficients of P_i ($i = 1, \dots, d$) and f.

Lemma 2.2 *Let $\mathcal{P}$ be a polynomial mapping $\mathbb{R}^+ \longrightarrow \mathbb{R}^d$, where $\mathcal{P}(t) = (P_1(t), P_2(t), \dots, P_d(t))$ and P_i is a real polynomial defined on $\mathbb{R}^+$ ($i = 1, \dots, d$). Suppose that $\phi \in \mathfrak{F}$. Then the operator M_ϕ defined by*

$$M_{\mathcal{P}(\phi)}(f)(x) = \sup_{r>0} \int_r^{2r} \bigl|f\bigl(x - \mathcal{P}(\phi(t))\bigr)\bigr| \frac{dt}{t}$$

is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. The bound is independent of the coefficients of P_i ($i = 1, \dots, d$) and f, but depends on ϕ.

Proof For any $r > 0$, by the change of variable, it can be easily seen that

$$\begin{aligned}
&\int_r^{2r} \bigl|f\bigl(x - \mathcal{P}(\phi(t))\bigr)\bigr| \frac{dt}{t} \\
&\quad = \int_{\phi(r)}^{\phi(2r)} \bigl|f\bigl(x - \mathcal{P}(s)\bigr)\bigr| \frac{ds}{\phi'(\phi^{-1}(s))\phi^{-1}(s)} \\
&\quad \le \frac{1}{C_\phi} \int_{\phi(r)}^{\phi(2r)} \bigl|f\bigl(x - \mathcal{P}(s)\bigr)\bigr| \frac{ds}{s} \le \frac{1}{C_\phi \phi(r)} \int_{\phi(r)}^{\phi(2r)} \bigl|f\bigl(x - \mathcal{P}(s)\bigr)\bigr|\, ds \\
&\quad \le \frac{\phi(2r)}{C_\phi \phi(r)} \frac{1}{\phi(2r)} \int_0^{\phi(2r)} \bigl|f\bigl(x - \mathcal{P}(s)\bigr)\bigr|\, ds \le \frac{c_\phi}{C_\phi} M_{\mathcal{P}}\bigl(|f|\bigr)(x).
\end{aligned}$$

This implies that $M_{\mathcal{P}(\phi)}(f)(x) \le C(\phi) M_{\mathcal{P}}(|f|)(x)$. Then Lemma 2.2 follows from Lemma 2.1. □

Lemma 2.3 *Let $\varphi, \psi \in \mathfrak{F}$. Suppose that $\Omega \in L^1(S^{m-1} \times S^{n-1})$ and satisfies (1.2)-(1.3). Then, for $\mu \in \{1, 2, \dots, \mathcal{N}_1\}$, $\nu \in \{1, 2, \dots, \mathcal{N}_2\}$, the maximal operator defined by*

$$\sigma^*_{\mu,\nu}(f)(x, y) = \sup_{\kappa,\ell \in \mathbb{Z}} \bigl| |\sigma_{\kappa,\ell;\mu,\nu}| * f(x, y) \bigr|$$

is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < p < \infty$. The bound is independent of the coefficients of P_{N_i} ($i = 1, 2$) and f, but depends on φ, ψ, N_1, N_2, m, n.

Proof By the definition of $|\sigma_{\kappa,\ell;\mu,\nu}|$, we have

$$\begin{aligned}
&\bigl| |\sigma_{\kappa,\ell;\mu,\nu}| * f(\xi, \eta) \bigr| \\
&\quad = \left| \iint_{\Delta_{\kappa,\ell}} \frac{|\Omega(u, v)|}{\rho_m(u)^{\alpha_m} \rho_n(v)^{\alpha_n}} f\bigl(x - Q_\mu(u), y - R_\nu(v)\bigr)\, du\, dv \right|
\end{aligned}$$

$$\begin{aligned}
&\le C \iint_{S^{m-1}\times S^{n-1}} \int_{2^{\kappa-1}}^{2^{\kappa}} \int_{2^{\ell-1}}^{2^{\ell}} \frac{|f(x - Q_\mu(A_{m,\rho_m}u'), y - R_\nu(A_{n,\rho_n}v'))|}{\rho_m\rho_n}\, d\rho_m\, d\rho_n \\
&\quad \times \left|\Omega(u',v')\right| d\sigma_m(u')\, d\sigma_n(v') \\
&\le C \iint_{S^{m-1}\times S^{n-1}} \left|\Omega(u',v')\right| M_{Q_\mu,R_\nu;u',v'}(f)(x,y)\, d\sigma_m(u')\, d\sigma_n(v'),
\end{aligned}$$

where

$$M_{Q_\mu,R_\nu;u',v'}(f)(x,y) := \sup_{s,t>0} \frac{1}{st} \int_s^{2s} \int_t^{2t} \left|f\left(x - Q_\mu\left(A_{m,r}u'\right), y - R_\nu\left(A_{n,h}v'\right)\right)\right| dr\, dh.$$

By Lemma 2.2, using iterated integration, it is easy to see that

$$\left\|M_{Q_\mu,R_\nu;u',v'}(f)\right\|_p \le C\|f\|_p \quad \text{for } 1 < p < \infty,$$

where C is independent of u', v'. Thus

$$\left\|\sigma^*_{\mu,\nu}(f)\right\|_p \le C \iint_{S^{m-1}\times S^{n-1}} \left|\Omega(u',v')\right| \left\|M_{Q_\mu,R_\nu;u',v'}(f)\right\|_p d\sigma_m(u')\, d\sigma_n(v') \le C\|f\|_p,$$

which completes the proof of Lemma 2.3. □

Lemma 2.4 (cf. [20, p.186, Corollary]) *Let* $\Phi(t) = t^{\alpha_1} + \mu_2 t^{\alpha_2} + \cdots + \mu_n t^{\alpha_n}$ *and* $\Psi \in C^1([a,b])$, *where* $\mu_2,\ldots,\mu_n$ *are real parameters, and* $\alpha_1,\ldots,\alpha_n$ *are distinct positive (not necessarily integer) exponents. Then*

$$\left|\int_a^b \exp\left(i\lambda\Phi(t)\right)\Psi(t)\, dt\right| \le C\lambda^{-\epsilon}\left\{\sup_{a\le t\le b}\left|\Psi(t)\right| + \int_a^b \left|\Psi'(t)\right| dt\right\},$$

with $\epsilon = \min\{1/\alpha_1, 1/n\}$ *and* C *does not depend on* $\mu_2,\ldots,\mu_m$ *as long as* $0 \le a < b \le 1$.

Lemma 2.5 *Suppose that* $\varphi, \psi \in \mathfrak{F}$. *Then for any* $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ *and* $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, *there exist* $\epsilon_1 = 1/\mu$ *and* $\epsilon_2 = 1/\nu$ *such that for any* $r > 0$

$$\begin{aligned}
&\int_{r/2}^{r} \exp\left(-iQ_\mu\left(A_{m,\rho_m}x'\right)\cdot\xi\right)\frac{d\rho_m}{\rho_m} \le C(\varphi)\left|\varphi(r)^\mu L_\mu(\xi)\cdot x'\right|^{-\epsilon_1}; \\
&\int_{r/2}^{r} \exp\left(-iR_\nu\left(A_{n,\rho_n}y'\right)\cdot\eta\right)\frac{d\rho_n}{\rho_n} \le C(\psi)\left|\psi(r)^\nu I_\nu(\eta)\cdot y'\right|^{-\epsilon_2}.
\end{aligned}$$

The constant $C(\varphi)$ *is independent of the coefficients of* P_{N_1} *but depends on* φ; *and* $C(\psi)$ *is independent of the coefficients of* P_{N_2} *but depends on* ψ.

Proof We only prove the first inequality, since a similar argument can get the second inequality. By the change of variables, we have

$$\begin{aligned}
&\int_{r/2}^{r} \exp\left(-iQ_\mu\left(A_{m,\rho_m}x'\right)\cdot\xi\right)\frac{d\rho_m}{\rho_m} \\
&\quad = \int_{r/2}^{r} \exp\left(-i\sum_{j=1}^{\mu}\left(L_j(\xi)\cdot x'\right)\varphi(\rho_m)^j\right)\frac{d\rho_m}{\rho_m}
\end{aligned}$$

$$= \int_{\varphi(r/2)}^{\varphi(r)} \exp\left(-i\sum_{j=1}^{\mu}\bigl(L_j(\xi)\cdot x'\bigr)t^j\right)\frac{dt}{\varphi^{-1}(t)\varphi'(\varphi^{-1}(t))}$$
$$= \varphi(r)\int_{\varsigma}^{1} \exp\left(-i\sum_{j=1}^{\mu}\bigl(L_j(\xi)\cdot x'\bigr)\varphi(r)^j t^j\right)\phi(t)g_{r,\varphi}(t)\,dt,$$

where $\varsigma = \varphi(r/2)/\varphi(r)$, $\phi(t) = (\varphi^{-1}(\varphi(r)t))^{-1}$, $g_{r,\varphi}(t) = (\varphi'(\varphi^{-1}(\varphi(r)t)))^{-1}$. Let

$$I(t) = \int_{\varsigma}^{t} \exp\left(-i\sum_{j=1}^{\mu}\bigl(L_j(\xi)\cdot x'\bigr)\varphi(r)^j s^j\right)\phi(s)\,ds, \quad \varsigma \le t \le 1.$$

By Lemma 2.4, there exists $\epsilon_1 = 1/\mu$ such that

$$\begin{aligned}\bigl|I(t)\bigr| &\le C\bigl|\varphi(r)^{\mu}L_{\mu}(\xi)\cdot x'\bigr|^{-\epsilon_1}\left(\sup_{s\in[\varsigma,t]}\bigl|\phi(s)\bigr| + \int_{\varsigma}^{t}\bigl|\phi'(s)\bigr|\,ds\right)\\ &\le C\bigl|\varphi(r)^{\mu}L_{\mu}(\xi)\cdot x'\bigr|^{-\epsilon_1}(2/r+1/r)\\ &\le \frac{C}{r}\bigl|\varphi(r)^{\mu}L_{\mu}(\xi)\cdot x'\bigr|^{-\epsilon_1}.\end{aligned}$$

Thus by integration by parts and the fact that φ' is monotonous, we have

$$\begin{aligned}&\left|\int_{r/2}^{r}\exp\bigl(-iQ_{\mu}\bigl(A_{m,\rho_m}x'\bigr)\cdot\xi\bigr)\frac{d\rho_m}{\rho_m}\right|\\ &\quad= \left|\varphi(r)\int_{\varsigma}^{1} g_{r,\varphi}(t)\,dI(t)\right|\\ &\quad\le \varphi(r)\left(\bigl|I(1)g_{r,\varphi}(1)\bigr| + \int_{\varsigma}^{1}\bigl|I(t)\bigr|\bigl|g'_{r,\varphi}(t)\bigr|\,dt\right)\\ &\quad\le \varphi(r)/r\bigl|\varphi(r)^{\mu}L_{\mu}(\xi)\cdot x'\bigr|^{-\epsilon_1}\bigl\{\bigl(\varphi'(r)\bigr)^{-1} + \bigl(\varphi'(r/2)\bigr)^{-1}\bigr\}.\end{aligned}$$

Using $t\varphi'(t) \ge C_{\varphi}\varphi(t)$, we get

$$\begin{aligned}\int_{r/2}^{r}\exp\bigl(-iQ_{\mu}\bigl(A_{m,\rho_m}x'\bigr)\cdot\xi\bigr)\frac{d\rho_m}{\rho_m} &\le \frac{c}{C_{\varphi}}(1+2c_{\varphi})\bigl|\varphi(r)^{\mu}L_{\mu}(\xi)\cdot x'\bigr|^{-\epsilon_1}\\ &\le C(\varphi)\bigl|\varphi(r)^{\mu}L_{\mu}(\xi)\cdot x'\bigr|^{-\epsilon_1}.\end{aligned}$$

This proves Lemma 2.5. □

Lemma 2.6 *Let $\varphi, \psi \in \mathfrak{F}$. Suppose that $\Omega \in \mathcal{F}_{\beta}(S^{m-1}\times S^{n-1})$ for some $\beta > 1$ and satisfies (1.2)-(1.3). Then for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, there exists a constant $C > 0$ such that*

(i) if $|\psi(2^{\ell})^{\nu}I_{\nu}(\eta)| > 1$, then

$$\begin{aligned}&\bigl|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta)\bigr|\\ &\quad\le C\bigl|\varphi\bigl(2^{\kappa}\bigr)^{\mu}L_{\mu}(\xi)\bigr|\min\bigl\{1,\bigl(\log\bigl|\psi\bigl(2^{\ell}\bigr)^{\nu}I_{\nu}(\eta)\bigr|\bigr)^{-\beta}\bigr\};\end{aligned} \tag{2.5}$$

(ii) *if* $|\varphi(2^\kappa)^\mu L_\mu(\xi)| > 1$, *then*

$$\begin{aligned}&\bigl|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\bigr| \\ &\quad \le C\bigl|\psi\bigl(2^\ell\bigr)^\nu I_\nu(\eta)\bigr| \min\bigl\{1, \bigl(\log\bigl|\varphi\bigl(2^\kappa\bigr)^\mu L_\mu(\xi)\bigr|\bigr)^{-\beta}\bigr\};\end{aligned} \tag{2.6}$$

(iii) *if* $|\varphi(2^\kappa)^\mu L_\mu(\xi)| > 1$ *and* $|\psi(2^\ell)^\nu I_\nu(\eta)| > 1$, *then*

$$\bigl|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\bigr| \le C\min\bigl\{1, \bigl(\log\bigl|\varphi\bigl(2^\kappa\bigr)^\mu L_\mu(\xi)\bigr|\bigr)^{-\beta}, \bigl(\log\bigl|\psi\bigl(2^\ell\bigr)^\nu I_\nu(\eta)\bigr|\bigr)^{-\beta}\bigr\}; \tag{2.7}$$

$$\bigl|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\bigr| \le C\min\bigl\{1, \bigl(\log\bigl|\varphi\bigl(2^\kappa\bigr)^\mu L_\mu(\xi)\bigr|\bigr)^{-\beta}\bigl(\log\bigl|\psi\bigl(2^\ell\bigr)^\nu I_\nu(\eta)\bigr|\bigr)^{-\beta}\bigr\}; \tag{2.8}$$

(iv)

$$\begin{aligned}&\bigl|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta) + \widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\bigr| \\ &\quad \le C\min\bigl\{1, \bigl|\varphi\bigl(2^\kappa\bigr)^\mu L_\mu(\xi)\bigr|, \bigl|\psi\bigl(2^\ell\bigr)^\nu I_\nu(\eta)\bigr|, \\ &\qquad \bigl|\varphi\bigl(2^\kappa\bigr)^\mu L_\mu(\xi)\bigr|\bigl|\psi\bigl(2^\ell\bigr)^\nu I_\nu(\eta)\bigr|\bigr\}.\end{aligned} \tag{2.9}$$

The constant C is independent of the coefficients of P_{N_1} *and* P_{N_2}.

Proof Let

$$\begin{aligned}H_{\kappa,\mu}\bigl(x',\xi\bigr) &= \int_{2^{\kappa-1}}^{2^\kappa} \exp\bigl(-iQ_\mu\bigl(A_{m,\rho_m}x'\bigr)\cdot\xi\bigr)\frac{d\rho_m}{\rho_m}; \\ J_{\ell,\nu}\bigl(y',\eta\bigr) &= \int_{2^{\ell-1}}^{2^\ell} \exp\bigl(-iR_\nu\bigl(A_{n,\rho_n}y'\bigr)\cdot\eta\bigr)\frac{d\rho_n}{\rho_n}.\end{aligned}$$

By Lemma 2.5, there exist $\epsilon_1, \epsilon_2 \in (0,1]$ such that

$$\begin{aligned}\bigl|H_{\kappa,\mu}\bigl(x',\xi\bigr)\bigr| &\le C\min\bigl\{1, \bigl|\varphi\bigl(2^\kappa\bigr)^\mu L_\mu(\xi)\cdot x'\bigr|^{-\epsilon_1}\bigr\}; \\ \bigl|J_{\ell,\nu}\bigl(y',\eta\bigr)\bigr| &\le C\min\bigl\{1, \bigl|\psi\bigl(2^\ell\bigr)^\nu I_\nu(\eta)\cdot y'\bigr|^{-\epsilon_2}\bigr\}.\end{aligned}$$

When $|\varphi(2^\kappa)^\mu L_\mu(\xi)| > 1$, since $t/(\log t)^\beta$ is increasing in (e^β, ∞), we have

$$\bigl|H_{\kappa,\mu}\bigl(x',\xi\bigr)\bigr| \le C\frac{(\log e^\beta|(\varphi(2^\kappa)^\mu L_\mu(\xi))'\cdot x'|^{-\epsilon_1})^\beta}{(\log|\varphi(2^\kappa)^\mu L_\mu(\xi)|)^\beta}.$$

Then

$$\bigl|H_{\kappa,\mu}\bigl(x',\xi\bigr)\bigr| \le C\min\left\{1, \frac{(\log e^\beta|(\varphi(2^\kappa)^\mu L_\mu(\xi))'\cdot x'|^{-\epsilon_1})^\beta}{(\log|\varphi(2^\kappa)^\mu L_\mu(\xi)|)^\beta}\right\}. \tag{2.10}$$

Similarly, when $|\psi(2^\ell)^\nu I_\nu(\eta)| > 1$,

$$\bigl|J_{\ell,\nu}\bigl(y',\eta\bigr)\bigr| \le C\min\left\{1, \frac{(\log e^\beta|(\psi(2^\ell)^\nu I_\nu(\eta))'\cdot y'|^{-\epsilon_2})^\beta}{(\log|\psi(2^\ell)^\nu I_\nu(\eta)|)^\beta}\right\}. \tag{2.11}$$

By the definition of $\sigma_{\kappa,\ell;\mu,\nu}$, we have

$$\begin{aligned}
&\big|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta)\big| \\
&\quad\le C\iint_{S^{m-1}\times S^{n-1}}\int_{1/2}^{1}\big|\exp\big(-iQ_\mu\big(A_{m,2^\kappa\rho_m}x'\big)\cdot\xi\big)-\exp\big(-iQ_{\mu-1}\big(A_{m,2^\kappa\rho_m}x'\big)\cdot\xi\big)\big|\frac{d\rho_m}{\rho_m} \\
&\qquad\times\big|J_{\ell,\nu}\big(y',\eta\big)\big|\big|\Omega\big(x',y'\big)\big|\,d\sigma_m\big(x'\big)\,d\sigma_n\big(y'\big) \\
&\quad\le C\iint_{S^{m-1}\times S^{n-1}}\big|J_{\ell,\nu}\big(y',\eta\big)\big|\big|\varphi\big(2^\kappa\big)^\mu L_\mu(\xi)\big|\big|\Omega\big(x',y'\big)\big|\,d\sigma_m\big(x'\big)\,d\sigma_n\big(y'\big).
\end{aligned}$$

Combining (2.11) with the fact $\Omega\in\mathcal{F}_\beta(S^{m-1}\times S^{n-1})$, we obtain (2.5). Similarly, we can conclude (2.6). To prove (2.7) and (2.8), we write

$$\big|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\big|\le C\iint_{S^{m-1}\times S^{n-1}}\big|\Omega\big(x',y'\big)\big|\big|H_{\kappa,\mu}\big(x',\xi\big)\big|\big|J_{\ell,\nu}\big(y',\eta\big)\big|\,d\sigma_m\big(x'\big)\,d\sigma_n\big(y'\big).$$

Then (2.7) and (2.8) follow from (2.10)-(2.11) with the fact $\Omega\in\mathcal{F}_\beta(S^{m-1}\times S^{n-1})$. Finally, (2.9) follows from the inequality

$$\begin{aligned}
&\big|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta)-\widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)+\widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\big| \\
&\quad\le C\iint_{S^{m-1}\times S^{n-1}}\big|\Omega\big(x',y'\big)\big| \\
&\qquad\times\bigg|\int_{1/2}^{1}\int_{1/2}^{1}\exp\big(-i\big(Q_{\mu-1}\big(A_{m,2^\kappa\rho_m}x'\big)\cdot\xi+R_{\nu-1}\big(A_{n,2^\ell\rho_n}y'\big)\cdot\eta\big)\big) \\
&\qquad\times\big(\exp\big(-i\varphi\big(2^\kappa\rho_m\big)^\mu L_\mu(\xi)\cdot x'\big)-1\big) \\
&\qquad\times\big(\exp\big(-i\psi\big(2^\ell\rho_n\big)^\nu I_\nu(\eta)\cdot y'\big)-1\big)\frac{d\rho_m}{\rho_m}\frac{d\rho_n}{\rho_n}\bigg|\,d\sigma_m\big(x'\big)\,d\sigma_n\big(y'\big).
\end{aligned}$$

This completes the proof of Lemma 2.6. □

Now we take two radial Schwartz functions $\phi_1\in\mathcal{S}(\mathbb{R}^m)$ and $\phi_2\in\mathcal{S}(\mathbb{R}^n)$ such that $\phi_i(t)\equiv 1$ for $|t|\le 1$ and $\phi_i(t)\equiv 0$ for $|t|>\min\{B_\varphi,B_\psi\}$ $(i=1,2)$, where B_φ, B_ψ are as in Remark 1.1. Define the measures $\{\omega_{\kappa,\ell;\mu,\nu}\}$ by

$$\begin{aligned}
\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)&=\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\prod_{i=\mu+1}^{\mathcal{N}_1}\phi_1\big(\varphi\big(2^\kappa\big)^i L_i(\xi)\big)\prod_{j=\nu+1}^{\mathcal{N}_2}\phi_2\big(\psi\big(2^\ell\big)^j I_j(\eta)\big) \\
&\quad-\widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta)\prod_{i=\mu}^{\mathcal{N}_1}\phi_1\big(\varphi\big(2^\kappa\big)^i L_i(\xi)\big)\prod_{j=\nu+1}^{\mathcal{N}_2}\phi_2\big(\psi\big(2^\ell\big)^j I_j(\eta)\big) \\
&\quad-\widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\prod_{i=\mu+1}^{\mathcal{N}_1}\phi_1\big(\varphi\big(2^\kappa\big)^i L_i(\xi)\big)\prod_{j=\nu}^{\mathcal{N}_2}\phi_2\big(\psi\big(2^\ell\big)^j I_j(\eta)\big) \\
&\quad+\widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\prod_{i=\mu}^{\mathcal{N}_1}\phi_1\big(\varphi\big(2^\kappa\big)^i L_i(\xi)\big)\prod_{j=\nu}^{\mathcal{N}_2}\phi_2\big(\psi\big(2^\ell\big)^j I_j(\eta)\big)
\end{aligned}$$

for $\kappa,\ell \in \mathbb{Z}$, $\mu \in \{1,2,\dots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\dots,\mathcal{N}_2\}$, where we use the convention $\prod_{j\in\emptyset} a_j = 1$. By (2.3), it is easy to see that

$$\sigma_{\kappa,\ell;\mathcal{N}_1,\mathcal{N}_2} = \sum_{\mu=1}^{\mathcal{N}_1}\sum_{\nu=1}^{\mathcal{N}_2}\omega_{\kappa,\ell;\mu,\nu}. \tag{2.12}$$

Lemma 2.7 *Let Ω, φ, ψ be as in Lemma 2.6. For $\mu \in \{1,2,\dots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\dots,\mathcal{N}_2\}$, $\kappa,\ell \in \mathbb{Z}$, we have*

(i)

$$\left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right| \le C\left|\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right|\left|\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right|; \tag{2.13}$$

(ii) if $|\varphi(2^\kappa)^\mu L_\mu(\xi)| > B_\varphi$, then

$$\left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right| \le C\left(\log\left|\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right|\right)^{-\beta}\left|\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right|; \tag{2.14}$$

(iii) if $|\psi(2^\ell)^\nu I_\nu(\eta)| > B_\psi$, then

$$\left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right| \le C\left|\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right|\left(\log\left|\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right|\right)^{-\beta}; \tag{2.15}$$

(iv) if $|\varphi(2^\kappa)^\mu L_\mu(\xi)| > B_\varphi$ and $|\psi(2^\ell)^\nu I_\nu(\eta)| > B_\psi$, then

$$\left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right| \le C\left(\log\left|\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right|\right)^{-\beta}\left(\log\left|\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right|\right)^{-\beta}. \tag{2.16}$$

Here and below, B_ϕ ($\phi = \varphi$ or ψ) is as in Remark 1.1, the constant C is independent of the coefficients of P_{N_i} ($i = 1,2$).

Proof We write $\Pi_1(\mu) = \prod_{i=\mu+1}^{\mathcal{N}_1}\phi_1(\varphi(2^\kappa)^i L_i(\xi))$, $\Pi_2(\nu) = \prod_{j=\nu+1}^{\mathcal{N}_2}\phi_2(\psi(2^\ell)^j I_j(\eta))$. Then

$$\begin{aligned}\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta) = {} & \widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu) - \widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu)\\ & - \widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu-1)\\ & + \widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\Pi_1(\mu-1)\Pi_2(\nu-1).\end{aligned} \tag{2.17}$$

Thus, it is easy to see that

$$\begin{aligned}\left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right| = {} & \left|\Pi_1(\mu)\Pi_2(\nu)\right|\Big|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta)\phi_1\left(\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right)\\ & - \widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\phi_2\left(\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right)\\ & + \widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\phi_1\left(\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right)\phi_2\left(\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right)\Big|\\ \le {} & C\left|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta) + \widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\right|\\ & + C\left|\widehat{\sigma_{\kappa,\ell;\mu-1,\nu}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\right|\left|1 - \phi_1\left(\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right)\right|\\ & + C\left|\widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta) - \widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\right|\left|1 - \phi_2\left(\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right)\right|\\ & + C\left|\widehat{\sigma_{\kappa,\ell;\mu-1,\nu-1}}(\xi,\eta)\right|\left|1 - \phi_1\left(\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right)\right|\left|1 - \phi_2\left(\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right)\right|.\end{aligned}$$

Notice that

$$\left|1-\phi_1\left(\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right)\right| \le C\left|\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right|, \tag{2.18}$$

$$\left|1-\phi_2\left(\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right)\right| \le C\left|\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right|. \tag{2.19}$$

Invoking Lemma 2.6, we get (2.13). On the other hand, since

$$\Pi_1(\mu-1)=0, \quad \text{if } \left|\varphi\left(2^\kappa\right)^\mu L_\mu(\xi)\right| > B_\varphi, \tag{2.20}$$

$$\Pi_2(\nu-1)=0, \quad \text{if } \left|\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right| > B_\psi, \tag{2.21}$$

by (2.17) and (2.20), we have

$$\begin{aligned}
\left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right| &= \left|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu)-\widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\Pi_1(\mu)\Pi_2(\nu-1)\right| \\
&\le \left|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\phi_2\left(\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right)\right| \\
&\le \left|\widehat{\sigma_{\kappa,\ell;\mu,\nu}}(\xi,\eta)-\widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\right| + \left|\widehat{\sigma_{\kappa,\ell;\mu,\nu-1}}(\xi,\eta)\right|\left|1-\phi_2\left(\psi\left(2^\ell\right)^\nu I_\nu(\eta)\right)\right|.
\end{aligned}$$

Then (2.14) follows from (2.6)-(2.7) with (2.19). Similarly, we get (2.15). Finally, (2.16) follows from (2.8), (2.17), (2.20) and (2.21). This completes the proof of Lemma 2.7. □

By Lemma 2.3 and the definition of $\{\mu_{\kappa,\ell;\mu,\nu}\}$, it is easy to verify the following lemma.

Lemma 2.8 *Let Ω,φ,ψ be as in Lemma 2.3. Then for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, we have*

$$\left\|\sup_{\kappa,\ell\in\mathbb{Z}}\left||\omega_{\kappa,\ell;\mu,\nu}| * f(\cdot,\cdot)\right|\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}$$

for $1<p<\infty$. The constant C is independent of the coefficients of P_{N_1} and P_{N_2}.

Applying Lemma 2.8 and [9, p.544, Lemma], we can obtain

Lemma 2.9 *Let Ω, φ, ψ be as in Lemma 2.3. Then for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, we have*

$$\left\|\left(\sum_{\kappa,\ell\in\mathbb{Z}}\left|\omega_{\kappa,\ell;\mu,\nu} * g_{\kappa,\ell}(\cdot,\cdot)\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \le C\left\|\left(\sum_{\kappa,\ell\in\mathbb{Z}}\left|g_{\kappa,\ell}(\cdot,\cdot)\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)}$$

for $1<p<\infty$ and any arbitrary functions $\{g_{\kappa,\ell}\}$. The constant C is independent of the coefficients of P_{N_1} and P_{N_2}.

Now we are in the position of proving Theorem 1.1.

Proof of Theorem 1.1 Combining (2.4) with (2.12), we write

$$T_\Omega^P(f) = \sum_{\mu=1}^{\mathcal{N}_1}\sum_{\nu=1}^{\mathcal{N}_2}\sum_{\kappa,\ell\in\mathbb{Z}}\omega_{\kappa,\ell;\mu,\nu} * f := \sum_{\mu=1}^{\mathcal{N}_1}\sum_{\nu=1}^{\mathcal{N}_2} T_{\Omega,\mu,\nu}^P(f). \tag{2.22}$$

It suffices to show that for $\mu \in \{1,2,\dots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\dots,\mathcal{N}_2\}$,

$$\left\| T^{P}_{\Omega,\mu,\nu}(f)\right\|_p \le C\|f\|_p \quad \text{for } 2\beta/(2\beta-1) < p < 2\beta. \tag{2.23}$$

For fixed $\mu \in \{1,2,\dots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\dots,\mathcal{N}_2\}$, choose two collections of C^∞ functions $\{\lambda_i\}_{i\in\mathbb{Z}}$ and $\{\eta_j\}_{j\in\mathbb{Z}}$ on $(0,\infty)$ with the following properties:

(i) $\operatorname{supp}\lambda_i \subset [\varphi(2^{i+1})^{-\mu}, \varphi(2^{i-1})^{-\mu}]$, $\operatorname{supp}\eta_j \subset [\psi(2^{j+1})^{-\nu}, \psi(2^{j-1})^{-\nu}]$;

(ii) $0 \le \lambda_i, \eta_j \le 1$, $\sum_{i\in\mathbb{Z}}\lambda_i(t)^2 = \sum_{j\in\mathbb{Z}}\eta_j(t)^2 = 1$;

(iii) $|\lambda_i'(t)|, |\eta_j'(t)| \le C/t$, where C is a constant.

Define the multiplier operator $S_{i,j}$ on $\mathbb{R}^m \times \mathbb{R}^n$ by

$$\widehat{S_{i,j}f}(x,y) = \lambda_i\big(|L_\mu(x)|\big)\eta_j\big(|I_\nu(y)|\big)\hat{f}(x,y). \tag{2.24}$$

Then

$$\begin{aligned} T^{P}_{\Omega,\mu,\nu}(f)(x,y) &= \sum_{\kappa,\ell\in\mathbb{Z}} \omega_{\kappa,\ell;\mu,\nu} * f(x,y) \\ &= \sum_{\kappa,\ell\in\mathbb{Z}} \omega_{\kappa,\ell;\mu,\nu} * \left(\sum_{i,j\in\mathbb{Z}} S_{i+\kappa,j+\ell}S_{i+\kappa,j+\ell}f\right)(x,y) \\ &= \sum_{i,j\in\mathbb{Z}}\sum_{\kappa,\ell\in\mathbb{Z}} S_{i+\kappa,j+\ell}(\omega_{\kappa,\ell;\mu,\nu} * S_{i+\kappa,j+\ell}f)(x,y) \\ &:= \sum_{i,j\in\mathbb{Z}} T_{i,j}f(x,y). \end{aligned} \tag{2.25}$$

Now we consider the L^p-boundedness of $T_{i,j}$. By the Littlewood-Paley theory and Lemma 2.9, we have

$$\begin{aligned} \|T_{i,j}f\|_p &\le C\left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} \left|S_{i+\kappa,j+\ell}(\omega_{\kappa,\ell;\mu,\nu} * S_{i+\kappa,j+\ell}f)\right|^2\right)^{1/2}\right\|_p \\ &\le C\left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} |\omega_{\kappa,\ell;\mu,\nu} * S_{i+\kappa,j+\ell}f|^2\right)^{1/2}\right\|_p \\ &\le C\left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} |S_{i+\kappa,j+\ell}f|^2\right)^{1/2}\right\|_p \\ &\le C\|f\|_p, \quad 1 < p < \infty, i,j \in \mathbb{Z}. \end{aligned} \tag{2.26}$$

On the other hand, by the Littlewood-Paley theory and Plancherel's theorem, we have

$$\begin{aligned} \|T_{i,j}f\|_2^2 &\le C\left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} |\omega_{\kappa,\ell;\mu,\nu} * S_{i+\kappa,j+\ell}f|^2\right)^{1/2}\right\|_2^2 \\ &= C\sum_{\kappa,\ell}\iint_{\mathbb{R}^m\times\mathbb{R}^n} \left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right|^2 \left|\lambda_{i+\kappa}\big(|L_\mu(\xi)|\big)\eta_{j+\ell}\big(|I_\nu(\eta)|\big)\right|^2 \left|\hat{f}(\xi,\eta)\right|^2 d\xi\, d\eta \\ &\le C\sum_{\kappa,\ell}\iint_{E_{i+\kappa,j+\ell}} \left|\widehat{\omega_{\kappa,\ell;\mu,\nu}}(\xi,\eta)\right|^2 \left|\hat{f}(\xi,\eta)\right|^2 d\xi\, d\eta, \end{aligned}$$

where $E_{i+\kappa,j+\ell} = \{(\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^n : \varphi(2^{i+\kappa+1})^{-\mu} \le |L_\mu(\xi)| \le \varphi(2^{i+\kappa-1})^{-\mu}, \psi(2^{j+\ell+1})^{-\nu} \le |I_\nu(\eta)| \le \psi(2^{j+\ell-1})^{-\nu}\}$. Using Lemma 2.7 and Remark 1.1, we have

$$\|T_{i,j}f\|_2 \le C(\varphi,\psi,\mu,\nu)B_{i,j}\|f\|_2, \tag{2.27}$$

where

$$B_{i,j} = \begin{cases} B_\varphi^{-i\mu}B_\psi^{-j\nu}, & i,j > -2; \\ B_\varphi^{-i\mu}|j|^{-\beta}, & i > -2, j \le -2; \\ |i|^{-\beta}B_\psi^{-j\nu}, & i \le -2, j > -2; \\ |ij|^{-\beta}, & i,j \le -2. \end{cases} \tag{2.28}$$

Interpolating (2.26) and (2.27), for any $p \in (2\beta/(2\beta-1), 2\beta)$, we can obtain $\delta \in (0,1)$ such that $\delta\beta > 1$ and

$$\|T_{i,j}f\|_p \le C(\varphi,\psi,\mu,\nu)^{1-\delta}B_{i,j}^\delta\|f\|_p, \quad 2\beta/(2\beta-1) < p < 2\beta.$$

Then we have

$$\begin{aligned} \sum_{i,j\in\mathbb{Z}} \|T_{i,j}f\|_p &\le C(\varphi,\psi,\mu,\nu)\bigg(\sum_{i,j>-2} B_\varphi^{-i\mu\delta}B_\psi^{-j\nu\delta} + \sum_{i>-2,j\le-2} B_\varphi^{-i\mu\delta}|j|^{-\delta\beta} \\ &\quad + \sum_{i\le-2,j>-2} |i|^{-\delta\beta}B_\psi^{-j\nu\delta} + \sum_{i,j\le-2} |ij|^{-\delta\beta}\bigg)\|f\|_p \\ &\le C(\varphi,\psi,\mu,\nu)\|f\|_p^q, \quad \text{for } 2\beta/(2\beta-1) < p < 2\beta. \end{aligned}$$

This together with (2.22) and (2.25) completes the proof of Theorem 1.1. □

3 On the multiple Marcinkiewicz integrals

This section is devoted to the proof of Theorem 1.2. We first introduce some notations and lemmas. For $\mu \in \{1,2,\dots,\mathcal{N}_1\}$, $\nu \in \{1,2,\dots,\mathcal{N}_2\}$ and $i,j \in \mathbb{Z}$, $s,t \in \mathbb{R}^+$, we define the measures $\{\sigma_{i,j;s,t}^{\mu,\nu}\}$ and $\{|\sigma_{i,j;s,t}^{\mu,\nu}|\}$ by

$$\begin{aligned} \widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta) &= \frac{1}{2^{i+j}st}\iint_{\Delta_{i,j}^{s,t}} \frac{\Omega(x,y)}{\rho_m(x)^{\alpha_m-1}\rho_n(y)^{\alpha_n-1}} \\ &\quad \times \exp\big(-i\big(Q_\mu(x)\cdot\xi + R_\nu(y)\cdot\eta\big)\big)\,dx\,dy, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \big|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}\big|(\xi,\eta) &= \frac{1}{2^{i+j}st}\iint_{\Delta_{i,j}^{s,t}} \frac{|\Omega(x,y)|}{\rho_m(x)^{\alpha_m-1}\rho_n(y)^{\alpha_n-1}} \\ &\quad \times \exp\big(-i\big(Q_\mu(x)\cdot\xi + R_\nu(y)\cdot\eta\big)\big)\,dx\,dy, \end{aligned} \tag{3.2}$$

where $\Delta_{i,j}^{s,t} = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : 2^{i-1}s \le \rho_m(x) \le 2^is, 2^{j-1}t \le \rho_n(y) \le 2^jt\}$ and Q_μ, R_ν were defined as in Section 2. It is obvious that for $\mu \in \{0,1,\dots,\mathcal{N}_1\}$, $\nu \in \{0,1,\dots,\mathcal{N}_2\}$

$$\widehat{\sigma_{i,j;s,t}^{0,\nu}}(\xi,\eta) = \widehat{\sigma_{i,j;s,t}^{\mu,0}}(\xi,\eta) = 0, \quad \forall(\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^n,$$

and

$$F_{s,t}^{P_{N_1}(\varphi),P_{N_2}(\psi)}(x,y) = st \sum_{i,j=-\infty}^{0} 2^{i+j} \sigma_{i,j;s,t}^{\mathcal{N}_1,\mathcal{N}_2} * f(x,y). \tag{3.3}$$

Lemma 3.1 *Let $s,t > 0$, $i,j \in \mathbb{Z}$ and $\varphi, \psi \in \mathfrak{F}$. Suppose that $\Omega \in \mathcal{F}_\beta(S^{m-1} \times S^{n-1})$ for some $\beta > 1/2$ and satisfies (1.2)-(1.3). Then for each pair μ and ν, there exists a constant $C > 0$ such that*

(i) if $|\psi(2^j t)^\nu I_\nu(\eta)| > 1$, then

$$\begin{aligned} &\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta)\right| \\ &\quad \le C\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right| \min\left\{1, \left(\log\left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|\right)^{-\beta}\right\}; \end{aligned} \tag{3.4}$$

(ii) if $|\varphi(2^i s)^\mu L_\mu(\xi)| > 1$, then

$$\begin{aligned} &\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{i,j;s,t}^{\mu,\nu-1}}(\xi,\eta)\right| \\ &\quad \le C\left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right| \min\left\{1, \left(\log\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right|\right)^{-\beta}\right\}; \end{aligned} \tag{3.5}$$

(iii) if $|\varphi(2^i s)^\mu L_\mu(\xi)| > 1$ and $|\psi(2^j t)^\nu I_\nu(\eta)| > 1$, then

$$\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)\right| \le C \min\left\{1, \left(\log\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right|\right)^{-\beta}, \left(\log\left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|\right)^{-\beta}\right\}; \tag{3.6}$$

$$\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)\right| \le C \min\left\{1, \left(\log\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right|\right)^{-\beta} \left(\log\left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|\right)^{-\beta}\right\}; \tag{3.7}$$

(iv)

$$\begin{aligned} &\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta) - \widehat{\sigma_{i,j;s,t}^{\mu,\nu-1}}(\xi,\eta) + \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu-1}}(\xi,\eta)\right| \\ &\quad \le C \min\left\{1, \left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right|, \left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|, \right. \\ &\qquad \left. \left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right| \left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|\right\}. \end{aligned} \tag{3.8}$$

The constant C is independent of the coefficients of P_{N_1} and P_{N_2}.

Proof Set

$$\begin{aligned} U_{i,s}^\mu\left(x',\xi\right) &= \frac{1}{2^i s} \int_{2^{i-1}s}^{2^i s} \exp\left(-iQ_\mu\left(A_{m,\rho_m} x'\right) \cdot \xi\right) d\rho_m; \\ V_{j,t}^\nu\left(y',\eta\right) &= \frac{1}{2^j t} \int_{2^{j-1}t}^{2^j t} \exp\left(-iR_\nu\left(A_{n,\rho_n} y'\right) \cdot \eta\right) d\rho_n. \end{aligned}$$

By Lemma 2.5, there exist $\epsilon_1, \epsilon_2 \in (0,1]$ such that

$$\left|U_{i,s}^\mu\left(x',\xi\right)\right| \le C \min\left\{1, \left|\varphi\left(2^i s\right)^\mu L_\mu(\xi) \cdot x'\right|^{-\epsilon_1}\right\}, \tag{3.9}$$

$$\left|V_{j,t}^\nu\left(y',\eta\right)\right| \le C \min\left\{1, \left|\psi\left(2^j t\right)^\nu I_\nu(\eta) \cdot y'\right|^{-\epsilon_2}\right\}. \tag{3.10}$$

When $|\varphi(2^i s)^\mu L_\mu(\xi)| > 1$, since $t/(\log t)^\beta$ is increasing in (e^β, ∞), we have

$$\left|U_{i,s}^{\mu}(x',\xi)\right| \leq C\min\left\{1, \frac{(\log e^\beta |(\varphi(2^i s)^\mu L_\mu(\xi))' \cdot x'|^{-\epsilon_1})^\beta}{(\log|\varphi(2^i s)^\mu L_\mu(\xi)|)^\beta}\right\}. \tag{3.11}$$

Similarly, when $|\psi(2^j t)^\nu I_\nu(\eta)| > 1$

$$\left|V_{j,t}^{\nu}(y',\eta)\right| \leq C\min\left\{1, \frac{(\log e^\beta |(\psi(2^j t)^\nu I_\nu(\eta))' \cdot y'|^{-\epsilon_2})^\beta}{(\log|\psi(2^j t)^\nu I_\nu(\eta)|)^\beta}\right\}. \tag{3.12}$$

By the definition of $\sigma_{i,j;s,t}^{\mu,\nu}$, we have

$$\begin{aligned}
&\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta)\right| \\
&\quad \leq C \iint_{S^{m-1}\times S^{n-1}} \int_{1/2}^{1} \left|\exp\left(-iQ_\mu\left(A_{m,2^i s\rho_m}x'\right)\cdot\xi\right) - \exp\left(-iQ_{\mu-1}\left(A_{m,2^i s\rho_m}x'\right)\cdot\xi\right)\right| d\rho_m \\
&\qquad \times \left|V_{j,t}^{\nu}(y',\eta)\right|\left|\Omega(x',y')\right| d\sigma_m(x')\, d\sigma_n(y') \\
&\quad \leq C\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right| \iint_{S^{m-1}\times S^{n-1}} \left|V_{j,t}^{\nu}(y',\eta)\right|\left|\Omega(x',y')\right| d\sigma_m(x')\, d\sigma_n(y').
\end{aligned}$$

Combining (3.12) with the fact $\Omega \in \mathcal{F}_\beta(S^{m-1} \times S^{n-1})$, we obtain (3.4). Similarly, we can conclude (3.5). To prove (3.6) and (3.7), we write

$$\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)\right| \leq C \iint_{S^{m-1}\times S^{n-1}} \left|\Omega(x',y')\right|\left|U_{i,s}^{\mu}(x',\xi)\right|\left|V_{j,t}^{\nu}(y',\eta)\right| d\sigma_m(x')\, d\sigma_n(y').$$

Combining (3.11)-(3.12) with the fact that $\Omega \in \mathcal{F}_\beta(S^{m-1} \times S^{n-1})$, we get (3.6) and (3.7). Finally, (3.8) follows from the inequality

$$\begin{aligned}
&\left|\widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta) - \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta) - \widehat{\sigma_{i,j;s,t}^{\mu,\nu-1}}(\xi,\eta) + \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu-1}}(\xi,\eta)\right| \\
&\quad \leq C \iint_{S^{m-1}\times S^{n-1}} \left|\Omega(x',y')\right| \\
&\qquad \times \left| \int_{1/2}^{1}\int_{1/2}^{1} \exp\left(-i\left(Q_{\mu-1}\left(A_{m,2^i s\rho_m}x'\right)\cdot\xi + R_{\nu-1}\left(A_{n,2^j t\rho_n}y'\right)\cdot\eta\right)\right)\right. \\
&\qquad \times \left(\exp\left(-i\varphi\left(2^i s\rho_m\right)^\mu L_\mu(\xi)\cdot x'\right) - 1\right) \\
&\qquad \left. \times \left(\exp\left(-i\psi\left(2^j t\rho_n\right)^\nu I_\nu(\eta)\cdot y'\right) - 1\right) d\rho_m\, d\rho_n \right| d\sigma_m(x')\, d\sigma_n(y').
\end{aligned}$$

This completes the proof of Lemma 3.1. □

We now take two radial Schwartz functions $\phi_1 \in \mathcal{S}(\mathbb{R}^m)$ and $\phi_2 \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi_i(t) \equiv 1$ for $|t| \leq 1$ and $\phi_i(t) \equiv 0$ for $|t| > \min\{B_\varphi, B_\psi\}$ $(i = 1, 2)$, where B_φ, B_ψ are as in Remark 1.1. Define the measures $\{\omega_{i,j;s,t}^{\mu,\nu}\}$ by

$$\begin{aligned}
\widehat{\omega_{i,j;s,t}^{\mu,\nu}}(\xi,\eta) &= \widehat{\sigma_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)\Theta_1(\mu)\Theta_2(\nu) - \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu}}(\xi,\eta)\Theta_1(\mu-1)\Theta_2(\nu) \\
&\quad - \widehat{\sigma_{i,j;s,t}^{\mu,\nu-1}}(\xi,\eta)\Theta_1(\mu)\Theta_2(\nu-1) + \widehat{\sigma_{i,j;s,t}^{\mu-1,\nu-1}}(\xi,\eta)\Theta_1(\mu-1)\Theta_2(\nu-1),
\end{aligned}$$

where $\Theta_1(\mu) = \prod_{\kappa=\mu+1}^{\mathcal{N}_1} \phi_1(\varphi(2^i s)^\kappa L_\kappa(\xi))$, $\Theta_2(\nu) = \prod_{\ell=\nu+1}^{\mathcal{N}_2} \phi_2(\psi(2^j t)^\ell I_\ell(\eta))$, for $i,j \in \mathbb{Z}$, $s,t > 0$, $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$, $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$. Here we use the convention $\Pi_{j\in\emptyset} a_j = 1$. It is easy to see that

$$\sigma_{i,j;s,t}^{\mathcal{N}_1,\mathcal{N}_2} = \sum_{\mu=1}^{\mathcal{N}_1} \sum_{\nu=1}^{\mathcal{N}_2} \omega_{i,j;s,t}^{\mu,\nu}. \tag{3.13}$$

Applying Lemma 3.1 and the same arguments as in proving Lemma 2.7, we have

Lemma 3.2 *Let Ω, φ, ψ be as in Lemma 3.1. Then for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, $i,j \in \mathbb{Z}$; $s,t > 0$, there exists a constant $C > 0$ such that*

(i) $|\widehat{\omega_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)| \le C|\varphi(2^i s)^\mu L_\mu(\xi)||\psi(2^j t)^\nu I_\nu(\eta)|$;

(ii) if $|\varphi(2^i s)^\mu L_\mu(\xi)| > B_\varphi$, *then*

$$\left|\widehat{\omega_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)\right| \le C\left(\log\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right|\right)^{-\beta}\left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|;$$

(iii) if $|\psi(2^j t)^\nu I_\nu(\eta)| > B_\psi$, *then*

$$\left|\widehat{\omega_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)\right| \le C\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right|\left(\log\left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|\right)^{-\beta};$$

(iv) if $|\varphi(2^i s)^\mu L_\mu(\xi)| > B_\varphi$ *and* $|\psi(2^j t)^\nu I_\nu(\eta)| > B_\psi$, *then*

$$\left|\widehat{\omega_{i,j;s,t}^{\mu,\nu}}(\xi,\eta)\right| \le C\left(\log\left|\varphi\left(2^i s\right)^\mu L_\mu(\xi)\right|\right)^{-\beta}\left(\log\left|\psi\left(2^j t\right)^\nu I_\nu(\eta)\right|\right)^{-\beta}.$$

Here the constant C is independent of the coefficients of P_{N_i} ($i = 1,2$).

By Lemma 2.2 and the same arguments as in proving Lemma 2.3, we have the following lemma:

Lemma 3.3 *Let $\varphi,\psi \in \mathfrak{F}$. Suppose that $\Omega \in L^1(S^{m-1} \times S^{n-1})$ and satisfies (1.2)-(1.3). Then for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$, $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, the maximal operator*

$$\tilde{\sigma}_{\mu,\nu}^*(f)(x,y) = \sup_{i,j\in\mathbb{Z}} \sup_{s,t>0} \left|\left|\sigma_{i,j;s,t}^{\mu,\nu}\right| * f(x,y)\right|$$

is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < p < \infty$. The bound is independent of the coefficients of P_{N_i} ($i = 1,2$) but depends on φ, ψ, N_1, N_2, m, n.

Applying Lemma 3.3, we have

Lemma 3.4 *Let Ω, φ, ψ be as in Lemma 3.3. Then for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$,*

$$\left\|\sup_{i,j\in\mathbb{Z}} \sup_{s,t>0} \left|\left|\omega_{i,j;s,t}^{\mu,\nu}\right| * f(\cdot,\cdot)\right|\right\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

The constant C is independent of the coefficients of P_{N_i} ($i = 1,2$).

Furthermore, applying Lemma 3.4 and [9, p.544, Lemma], we can obtain

Lemma 3.5 *Let Ω, φ, ψ be as in Lemma 3.3. For $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$; $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$, $s,t>0$ and any arbitrary functions $g_{i,j}$, then*

$$\begin{aligned}&\Big\|\Big(\sum_{i,j\in\mathbb{Z}}\big|\omega^{\mu,\nu}_{i,j;s,t}*g_{i,j}(\cdot,\cdot)\big|^2\Big)^{1/2}\Big\|_p\\&\quad\le C\Big\|\Big(\sum_{i,j\in\mathbb{Z}}\big|g_{i,j}(\cdot,\cdot)\big|^2\Big)^{1/2}\Big\|_p,\quad 1<p<\infty,\end{aligned}$$

where the constant C is independent of the coefficients of P_{N_i} ($i=1,2$).

Lemma 3.6 *Let Ω, φ, ψ be as in Lemma 3.3. Then for $\mu\in\{1,2,\ldots,\mathcal{N}_1\}$ and $\nu\in\{1,2,\ldots,\mathcal{N}_2\}$, there exists a constant $C_p>0$ such that for $1<p<\infty$*

$$\begin{aligned}&\Big\|\Big(\sum_{\kappa,\ell\in\mathbb{Z}}\int_1^2\int_1^2\big|\omega^{\mu,\nu}_{i,j;s,t}*g_{\kappa,\ell}(\cdot,\cdot)\big|^2\,ds\,dt\Big)^{1/2}\Big\|_p\\&\quad\le C_p\Big\|\Big(\sum_{\kappa,\ell\in\mathbb{Z}}\big|g_{\kappa,\ell}(\cdot,\cdot)\big|^2\Big)^{1/2}\Big\|_p,\end{aligned}$$

where $\{g_{\kappa,\ell}\}_{\kappa,\ell\in\mathbb{Z}}$ are arbitrary functions defined on $\mathbb{R}^m\times\mathbb{R}^n$. The constant C_p is independent of the coefficients of P_{N_i} ($i=1,2$).

Proof We consider the mapping $\mathcal{F}:\{g_{\kappa,\ell}\}_{\kappa,\ell}\longrightarrow\{\omega^{\mu,\nu}_{i,j;s,t}*g_{\kappa,\ell}\}_{\kappa,\ell;s,t}$. By Lemma 3.4, we have for $1<p<\infty$

$$\Big\|\sup_{i,j\in\mathbb{Z}}\sup_{s,t\in[1,2]}\big|\omega^{\mu,\nu}_{i,j;s,t}*g_{\kappa,\ell}(\cdot,\cdot)\big|\Big\|_p\le C_p\Big\|\sup_{\kappa,\ell\in\mathbb{Z}}\big|g_{\kappa,\ell}(\cdot,\cdot)\big|\Big\|_p,$$

which implies $\mathcal{F}:L^p(\mathbb{R}^m\times\mathbb{R}^n)(\ell^\infty)\longrightarrow L^p(\mathbb{R}^m\times\mathbb{R}^n)(\ell^\infty(L^\infty([1,2]\times[1,2])))$.

On the other hand. By the dual argument and Lemma 3.4, we have

$$\Big\|\sum_{\kappa,\ell\in\mathbb{Z}}\int_1^2\int_1^2\big|\omega^{\mu,\nu}_{i,j;s,t}*g_{\kappa,\ell}(\cdot,\cdot)\big|\,ds\,dt\Big\|_p\le C_p\Big\|\sum_{\kappa,\ell\in\mathbb{Z}}\big|g_{\kappa,\ell}(\cdot,\cdot)\big|\Big\|_p,$$

which implies $\mathcal{F}:L^p(\mathbb{R}^m\times\mathbb{R}^n)(\ell^1)\longrightarrow L^p(\mathbb{R}^m\times\mathbb{R}^n)(\ell^1(L^1([1,2]\times[1,2])))$. Then Lemma 3.6 follows from the standard interpolation arguments. □

Lemma 3.7 *Let $S_{i,j}$ be the multiplier operators defined in (2.24) for any $i,j\in\mathbb{Z}$. Then (i) for each fixed $1<p<2$ and for any functions $\{g_{i,j;s,t;\kappa,\ell}\}$*

$$\begin{aligned}&\Big\|\Big(\sum_{\kappa,\ell\in\mathbb{Z}}\int_1^2\int_1^2\Big|\sum_{i,j\in\mathbb{Z}}S_{i+\kappa,j+\ell}g_{i,j;s,t;\kappa,\ell}(\cdot,\cdot)\Big|^2\,ds\,dt\Big)^{1/2}\Big\|_p^q\\&\quad\le C\sum_{i,j\in\mathbb{Z}}\Big\|\Big(\sum_{\kappa,\ell\in\mathbb{Z}}\int_1^2\int_1^2\big|g_{i,j;s,t;\kappa,\ell}(\cdot,\cdot)\big|^2\,ds\,dt\Big)^{1/2}\Big\|_p^q,\quad\forall 1<q<p;\end{aligned}\tag{3.14}$$

(ii) for each fixed $2 < p < \infty$ *and for any functions* $\{g_{i,j;s,t;\kappa,\ell}\}$,

$$\begin{aligned}&\left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} \int_1^2 \int_1^2 \left| \sum_{i,j\in\mathbb{Z}} S_{i+\kappa,j+\ell} g_{i,j;s,t;\kappa,\ell}(\cdot,\cdot) \right|^2 ds\,dt \right)^{1/2} \right\|_p^q \\ &\quad \le C \sum_{i,j\in\mathbb{Z}} \left(\int_1^2 \int_1^2 \left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} |g_{i,j;s,t;\kappa,\ell}(\cdot,\cdot)|^2 \right)^{1/2} \right\|_p^2 ds\,dt \right)^{q/2}, \\ &\quad \forall 1 < q < p' = p/(p-1). \end{aligned} \tag{3.15}$$

By the arguments similar to those used in [17, pp.78-81], we easily establish this lemma. The details are omitted.

Now we turn to prove Theorem 1.2.

Proof of Theorem 1.2 By (3.3) and (3.13), we can write

$$\begin{aligned}\mathcal{M}_\Omega^P(f)(x,y) &= \left(\int_0^\infty \int_0^\infty \left| \sum_{i,j=-\infty}^{0} 2^{i+j} \sigma_{i,j;s,t}^{\mathcal{N}_1,\mathcal{N}_2} * f(x,y) \right|^2 \frac{ds\,dt}{st} \right)^{1/2} \\ &\le \sum_{i,j=-\infty}^{0} 2^{i+j} \left(\int_0^\infty \int_0^\infty \left| \sigma_{i,j;s,t}^{\mathcal{N}_1,\mathcal{N}_2} * f(x,y) \right|^2 \frac{ds\,dt}{st} \right)^{1/2} \\ &= 4 \left(\sum_{\kappa,\ell\in\mathbb{Z}} \int_{2^\kappa}^{2^{\kappa+1}} \int_{2^\ell}^{2^{\ell+1}} \left| \sigma_{0,0;s,t}^{\mathcal{N}_1,\mathcal{N}_2} * f(x,y) \right|^2 \frac{ds\,dt}{st} \right)^{1/2} \\ &= 4 \left(\int_1^2 \int_1^2 \sum_{\kappa,\ell\in\mathbb{Z}} \left| \sigma_{\kappa,\ell;s,t}^{\mathcal{N}_1,\mathcal{N}_2} * f(x,y) \right|^2 \frac{ds\,dt}{st} \right)^{1/2} \\ &\le 4 \sum_{\mu=1}^{\mathcal{N}_1} \sum_{\nu=1}^{\mathcal{N}_2} \left(\int_1^2 \int_1^2 \sum_{\kappa,\ell\in\mathbb{Z}} \left| \omega_{\kappa,\ell;s,t}^{\mu,\nu} * f(x,y) \right|^2 ds\,dt \right)^{1/2} \\ &:= 4 \sum_{\mu=1}^{\mathcal{N}_1} \sum_{\nu=1}^{\mathcal{N}_2} \mathcal{M}_\Omega^{\mu,\nu}(f)(x,y). \end{aligned} \tag{3.16}$$

It suffices to show that $\mathcal{M}_\Omega^{\mu,\nu}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $\mu \in \{1,2,\ldots,\mathcal{N}_1\}$ and $\nu \in \{1,2,\ldots,\mathcal{N}_2\}$. By the definitions of $S_{i,j}$, we can write

$$\begin{aligned}\mathcal{M}_\Omega^{\mu,\nu}(f)(x,y) &= \left(\int_1^2 \int_1^2 \sum_{\kappa,\ell\in\mathbb{Z}} \left| \omega_{\kappa,\ell;s,t}^{\mu,\nu} * \left(\sum_{i,j\in\mathbb{Z}} S_{i+\kappa,j+\ell} S_{i+\kappa,j+\ell} f \right)(x,y) \right|^2 ds\,dt \right)^{1/2} \\ &= \left(\sum_{\kappa,\ell\in\mathbb{Z}} \int_1^2 \int_1^2 \left| \sum_{i,j\in\mathbb{Z}} S_{i+\kappa,j+\ell} \left(\omega_{\kappa,\ell;s,t}^{\mu,\nu} * S_{i+\kappa,j+\ell} f \right)(x,y) \right|^2 ds\,dt \right)^{1/2}. \end{aligned} \tag{3.17}$$

Case 1. $1 + 1/(2\beta) < p < 2$. Combining with (3.17) and Lemma 3.7, we know that for $1 < q < p$

$$\|\mathcal{M}_\Omega^{\mu,\nu}(f)\|_p^q \le C \sum_{i,j\in\mathbb{Z}} \left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} \int_1^2 \int_1^2 \left| \omega_{\kappa,\ell;s,t}^{\mu,\nu} * S_{i+\kappa,j+\ell} f(\cdot,\cdot) \right|^2 ds\,dt \right)^{1/2} \right\|_p^q. \tag{3.18}$$

For fixed $i, j \in \mathbb{Z}$, let

$$M_{i,j}f(x,y) := \left(\sum_{\kappa,\ell \in \mathbb{Z}} \int_1^2 \int_1^2 \left| \omega_{\kappa,\ell;s,t}^{\mu,\nu} * S_{i+\kappa,j+\ell} f(x,y) \right|^2 ds\, dt \right)^{1/2}.$$

By Lemma 3.6 and the Littlewood-Paley theory, we have

$$\|M_{i,j}f\|_p \leq C_p \left\| \left(\sum_{\kappa,\ell \in \mathbb{Z}} \left| S_{i+\kappa,j+\ell} f(\cdot,\cdot) \right|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty. \tag{3.19}$$

On the other hand, by Plancherel's theorem and Lemma 3.2, we know that

$$\begin{aligned} \|M_{i,j}f\|_2^2 &= \sum_{\kappa,\ell \in \mathbb{Z}} \int_1^2 \int_1^2 \iint_{\mathbb{R}^m \times \mathbb{R}^n} \left| \widehat{\omega_{\kappa,\ell;s,t}^{\mu,\nu}}(\xi,\eta) \right|^2 \lambda_{i+\kappa}^2 \left(\left| L_\mu(\xi) \right| \right) \\ &\quad \times \eta_{j+\ell}^2 \left(\left| I_\nu(\eta) \right| \right) \left| \hat{f}(\xi,\eta) \right|^2 d\xi\, d\eta\, ds\, dt \\ &\leq \sum_{\kappa,\ell \in \mathbb{Z}} \int \int_{E_{i+\kappa,j+\ell}} \left| \hat{f}(\xi,\eta) \right|^2 \int_1^2 \int_1^2 \left| \widehat{\omega_{\kappa,\ell;s,t}^{\mu,\nu}}(\xi,\eta) \right|^2 ds\, dt\, d\xi\, d\eta, \end{aligned}$$

where $E_{i+\kappa,j+\ell}$ is as in the proof of Theorem 1.1. Then

$$\|M_{i,j}f\|_2 \leq C(\varphi,\psi,\mu,\nu) B_{i,j} \|f\|_2, \tag{3.20}$$

where $B_{i,j}$ is as in (2.28). Interpolating between (3.19) and (3.20), there exists $\delta \in (2/(2\beta+1), 1)$ such that

$$\|M_{i,j}f\|_p \leq C(\varphi,\psi,\mu,\nu)^{1-\delta} B_{i,j}^{\delta} \|f\|_p, \quad 1 + 1/(2\beta) < p < 2.$$

For fixed $1 + 1/(2\beta) < p < 2$, we can choose $1 < q < p$ such that $q\delta\beta > 1$. Then

$$\begin{aligned} \sum_{i,j \in \mathbb{Z}} \|M_{i,j}f\|_p^q &\leq C(\varphi,\psi,\mu,\nu) \left(\sum_{i,j > -2} B_\varphi^{-i\mu\delta q} B_\psi^{-j\nu\delta q} + \sum_{i > -2, j \leq -2} B_\varphi^{-iq\mu\delta} |j|^{-q\delta\beta} \right. \\ &\quad \left. + \sum_{i \leq -2, j > -2} |i|^{-q\delta\beta} B_\psi^{-jq\nu\delta} + \sum_{i,j \leq -2} |ij|^{-q\delta\beta} \right) \|f\|_p^q \\ &\leq C(\varphi,\psi,\mu,\nu) \|f\|_p^q, \quad 1 + 2\beta < p < 2, \end{aligned}$$

which implies

$$\left\| \mathcal{M}_\Omega^{\mu,\nu}(f) \right\|_p \leq C(\varphi,\psi,\mu,\nu) \|f\|_p, \quad 1 + 1/(2\beta) < p < 2. \tag{3.21}$$

Case 2. $2 < p < 1 + 2\beta$. By (3.17) and Lemma 3.7, we have, for $2 < p < \infty$ and any $1 < q < p' = p/(p-1)$,

$$\left\| \mathcal{M}_\Omega^{\mu,\nu}(f) \right\|_p^q \leq C \sum_{i,j \in \mathbb{Z}} \left(\int_1^2 \int_1^2 \left\| \left(\sum_{\kappa,\ell \in \mathbb{Z}} \left| \omega_{\kappa,\ell;s,t}^{\mu,\nu} * S_{i+\kappa,j+\ell} f(\cdot,\cdot) \right|^2 \right)^{1/2} \right\|_p^2 ds\, dt \right)^{q/2}. \tag{3.22}$$

Let

$$J_{i,j;s,t}f(x,y) := \left(\sum_{\kappa,\ell\in\mathbb{Z}} \left| \omega^{\mu,\nu}_{\kappa,\ell;s,t} * S_{i+\kappa,j+\ell}f(x,y) \right|^2 \right)^{1/2}.$$

By Lemma 3.5 and the Littlewood-Paley theory, we have that for $i,j \in \mathbb{Z}$ and $s,t \in [1,2]$,

$$\begin{aligned} \|J_{i,j;s,t}f\|_p &\le C_p \left\| \left(\sum_{\kappa,\ell\in\mathbb{Z}} \left| S_{i+\kappa,j+\ell}f(\cdot,\cdot) \right|^2 \right)^{1/2} \right\|_p \\ &\le C_p\|f\|_p, \quad 1 < p < \infty. \end{aligned} \tag{3.23}$$

On the other hand, by the same argument as in getting (3.20), we have

$$\|J_{i,j;s,t}f\|_2 \le C(\varphi,\psi,\mu,\nu)B_{i,j}\|f\|_2, \tag{3.24}$$

where $B_{i,j}$ is as in (2.28). By interpolating between (3.23) and (3.24), for fixed $p \in (2, 1+2\beta)$, we can choose $q \in (1,p')$ and $\gamma \in (2/(2\beta+1), 1)$ such that $q\gamma\beta > 1$ and

$$\|J_{i,j;s,t}f\|_p \le C(\varphi,\psi,\mu,\nu)^{1-\gamma}B^{\gamma}_{i,j}\|f\|_p, \quad 2 < p < 1+2\beta.$$

Combining this with (3.22), we have

$$\begin{aligned} \left\| \mathcal{M}^{\mu,\nu}_{\Omega}(f) \right\|^q_p &\le C(\varphi,\psi,\mu,\nu) \Bigg(\sum_{i,j>-2} B_\varphi^{-i\mu\gamma q} B_\psi^{-j\nu\gamma q} + \sum_{i>-2, j\le -2} B_\varphi^{-iq\mu\gamma}|j|^{-q\gamma\beta} \\ &\quad + \sum_{i\le -2, j>-2} |i|^{-q\gamma\beta} B_\psi^{-jq\nu\gamma} + \sum_{i,j\le -2} |ij|^{-q\gamma\beta} \Bigg) \|f\|^q_p \\ &\le C(\varphi,\psi,\mu,\nu)\|f\|^q_p, \quad 2 < p < 1+2\beta. \end{aligned}$$

This together with (3.21) completes the proof of Theorem 1.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors worked jointly in drafting and approving the final manuscript.

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