Journal of Inequalities and Applications
a SpringerOpen Journal

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Estimates for the composition of the carathéodory and homotopy operators

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Abstract
In the present paper, we deal with the composition of carathéodory and homotopy operators for differential forms satisfying the A-harmonic equation in the bounded and convex domain. We obtain estimates for the composition and the form of inequalities with weights. Moreover, we also obtain the composition for the gradient, carathéodory, and homotopy operators. Then we obtain the $W^{1,p}$ norm estimates for the composition operators.

Keywords: differential forms; composition operators; norm estimates

1 Introduction

The purpose of this paper is to establish the inequalities for the composition of the homotopy operator T and the carathéodory operator G applied to differential forms in $\mathbf{R}^n$, $n \geq 2$. The homotopy operator T is widely used in the decomposition and the L^p-theory of differential forms. And in [3], we have extended the homotopy operator to the domain that is deformed to every point. In the meanwhile, the carathéodory operator G form classic examples to discuss boundedness and continuity of nonlinear operators and play an important part in advanced functional analysis, and in [4] we have extended it to differential forms. In many situations, we need to estimate the various norms of the operators and their compositions.

Throughout this paper, we always assume that Ω is a bounded and convex domain and B is a ball in $\mathbf{R}^n$, $n \geq 2$. Let σB be the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$, $\sigma > 0$. We do not distinguish the balls from cubes in this paper. For any subset $E \subset \mathbf{R}^n$, we use $|E|$ to denote the Lebesgue measure of E. In [2], we have the estimate for $\|T(u)\|$:

$$\|T(u)\|_{p,F} \leq 2^n \sigma_{n-1} \mu(\Omega)(\operatorname{diam}\Omega)\|u\|_{p,F} \tag{1.1}$$

for all $u \in L^p_{loc}(\Omega, \wedge^l)$, where $F \subset \Omega$ is bounded and convex. And for carathéodory operator, we obtain

$$|f(s,\omega)| \leq a(s) + b|\omega|^{p_1/p_2}, \quad s \in \Omega, \omega \in L^{p_1}(\Omega, \wedge^l).$$

With these estimates, we can obtain the estimates for the composition of them. Finally, we obtain the $W^{1,p}$ norm estimates for the composition operator.

The main theorems are proved by reference to Chap. 7 of [1].

2 Some preliminaries about differential forms

The majority of notations and preliminaries used throughout this paper can be found in [1]. For the sake of convenience, we list briefly them in this section.

Let $e_1, e_2, \ldots, e_n$ denote the standard orthogonal basis of $\mathbf{R}^n$. Suppose that $\Lambda^l = \Lambda^l(\mathbf{R}^n)$ is the linear space of l-covectors, generated by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = 0, 1, \ldots, n$. The Grassmann algebra $\Lambda = \bigoplus_{l=0}^n \Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \Sigma \alpha^I e_I \in \Lambda$ and $\beta = \Sigma \beta^I e_I \in \Lambda$, the inner product in Λ is given by $\langle \alpha, \beta \rangle = \Sigma \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integrals $l = 0, 1, \ldots, n$. We define the Hodge star operator $\star : \Lambda \to \Lambda$ by

$$\star\omega = \operatorname{sign}(\pi)\alpha_{i_1,i_2,\ldots,i_k}(x_1, x_2, \ldots, x_n)\, dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}},$$

where $\omega = \alpha_{i_1,i_2,\ldots,i_k}(x_1, x_2, \ldots, x_n)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ is a k-form, $\pi = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is a permutation of $(1, 2, \ldots, n)$ and $\operatorname{sign}(\pi)$ is the signature of the permutation. The norm of $\alpha \in \Lambda$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \Lambda^0 = \mathbf{R}$.

A differential l-form ω is a Schwartz distribution on Ω with values in $\Lambda^l(\mathbf{R}^n)$. We use $D'(\Omega, \Lambda^l)$ to denote the space of all differential l-forms, and $L^p(\Omega, \Lambda)$ to denote the l-forms

$$\omega(x) = \sum_I \omega(x)\, dx_I = \sum \omega_{i_1,i_2,\ldots,i_l}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$$

with all coefficients $\omega_I \in L^p(\Omega, \mathbf{R})$. Thus, $L^p(\Omega, \Lambda^l)$, $p \ge 1$, is a Banach space with norm

$$\|\omega\|_p = \big\|\omega(x)\big\|_{p,\Omega} = \left(\int_\Omega \big|\omega(x)\big|^p\right)^{1/p} = \left(\int_\Omega \left(\sum_I \big|\omega_I(x)\big|^2\right)^{p/2} dx\right)^{1/p}.$$

The space $L_1^p(\Omega, \Lambda^l)$ is the subspace of $D'(\Omega, \Lambda^l)$ with the condition

$$\|\alpha\|_{L_1^p(\Omega)} = \left(\int_\Omega \left(\sum_{i=1}^n \left|\frac{\partial \alpha}{\partial x_i}\right|^2\right)^{p/2} dx\right)^{1/p} < \infty.$$

The Sobolev space $W^{1,p}(\Omega, \Lambda^l)$ of l-forms is $W^{1,p}(\Omega, \Lambda^l) = L^p(\Omega, \Lambda) \cap L_1^p(\Omega, \Lambda^l)$. The norms are given by

$$\|\omega\|_{W^{1,p}(\Omega,\wedge^l)} = (\operatorname{diam}\Omega)^{-1}\|\omega\|_{p,\Omega} + \|\nabla\omega\|_{p,\Omega}. \tag{2.1}$$

We denote the exterior derivative by $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n-1$, which means

$$d\omega(x) = \sum_{k=1}^n \sum_{1 \le i_1 < \cdots < i_l \le n} \frac{\partial \omega_{i_1,i_2,\ldots,i_l}(x)}{\partial x_k}\, dx_k \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}.$$

Its formal adjoint operator is defined by

$$d^* = (-1)^{nl+1} \star d \star : D'\big(\Omega, \Lambda^{l+1}\big) \to D'\big(\Omega, \Lambda^l\big), \quad l = 0, 1, \ldots, n-1,$$

which is called the Hodge codifferential.

In [3], we define an operation K_y for any $y \in \Omega$ and we construct a homotopy operator $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^l)$ by averaging K_y over all points $y \in \Omega$:

$$T\omega = \int_\Omega \psi(y) K_y\big(\phi^* \omega\big)\, dy, \tag{2.2}$$

where ψ in $C_0^\infty(U)$ is normalized so that $\int_\Omega \psi(y)\, dy = 1$. We obtain the following decomposition for the operator T:

$$\omega = d(T\omega) + T(d\omega). \tag{2.3}$$

From [2], we know that for any differential form $u \in L^p_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \dots, n$, $1 < p < \infty$, we have

$$\big\|\nabla(Tu)\big\|_{p,\Omega} \le C\mu(\Omega)\|u\|_{p,\Omega}, \tag{2.4}$$

$$\|Tu\|_{p,\Omega} \le C\mu(\Omega)\operatorname{diam}(B)\|u\|_{p,\Omega}. \tag{2.5}$$

where $\mu(B)$ is flatness of Ω (see [2]). See [5–13] for more details of differential forms and its applications.

Then we define the carathéodory conditions and carathéodory operator for differential forms (see [4]).

Definition 2.1 For a mapping $f : \Omega \times \wedge^l \to \wedge^l$, where Ω is an open set in $\mathbf{R}^n$, we say that f satisfies carathéodory conditions if

1. For all most $s \in \Omega$, $f(s, \omega)$ is continuous with respect to ω, which means that f can be expanded as $f(s, \omega) = \sum_J f_J(s, \omega)\, dx_J$, where $f_J : \Omega \times \wedge^l \to \mathbf{R}$ and $f_J(s, \omega)$ is continuous about ω for all most $s \in \Omega$; and
2. For any fixed $\omega = \sum_I \omega\, dx_I \in \wedge^l$, $f(s, \omega)$ is measurable about s, which means that each coefficient function $f_J(s, \omega)$ is measurable about s for any fixed $\omega \in \wedge^l$.

Throughout this paper, we assume that $f(s, \omega)$ satisfies the carathéodory condition (C-condition). Similarly, we can define the continuity of $f(s, \omega)$ about $(s, \omega) \in \Omega \times \wedge^l$.

Definition 2.2 Suppose that $\Omega \subset \mathbf{R}^n$ is a measurable set ($0 < \operatorname{mes}\Omega \le +\infty$), and $f : \Omega \times \wedge^l \to \wedge^l$. We define the carathéodory operator $G : \wedge^l \to \wedge^l$ for differential forms by

$$G\omega(s) = f\big(s, \omega(s)\big).$$

For the carathéodory operator, we have the similar result for differential forms as for the functions (see [4]).

Theorem 2.1 *The carathéodory operator G maps continuously and boundedly $L^{p_1}(\Omega, \wedge^l)$ into $L^{p_2}(\Omega, \wedge^l)$, if and only if, there exists $b > 0$, $a(x) \ge 0$, $a(x) \in L^{p_2}(\Omega)$ satisfying the following inequality:*

$$\big|f(x, \omega)\big| \le a(x) + b|\omega|^{\frac{p_1}{p_2}} \quad \big(x \in \Omega, \omega \in \wedge^l\big). \tag{2.6}$$

Here, we suppose $p_1 = p_2$.

We define Muckenhoupt weights (see [1]).

Definition 2.3 A weight ω satisfies the $A_r(\Omega)$-condition in a subset $\Omega \subset \mathbf{R}^n$, where $r > 1$, and write $\omega \in A_r(\Omega)$ when

$$\sup_B \left(\frac{1}{|B|} \int_B \omega \, dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-r)} \, dx \right)^{r-1} < \infty, \tag{2.7}$$

where the supremum is over all balls $B \subset \Omega$.

The following class of two-weight or $A_{r,\lambda}(\Omega)$-weights appeared in [1] and [13].

Definition 2.4 A pair of weights (ω_1, ω_2) satisfy the $A_{r,\lambda}(\Omega)$-condition in a set $B \subset R^N$, write $(\omega_1, \omega_2) \in A_{r,\lambda}(B)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$, if

$$\sup_{B \subset \Omega} \left(\frac{1}{|B|} \int_B \omega_1^{\lambda} \, dx \right)^{1/\lambda r} \left(\frac{1}{|B|} \int_B (1/\omega_2)^{1/(r-1)} \, dx \right)^{\lambda(r-1)} < \infty$$

for all balls $B \subset \Omega$.

In the present paper, we deal with the A-harmonic equations formulated by $d^* A(x, du) = B(x, du)$.

We also need the following weak reverse Hölder inequality (Lemma 3.1.1 of [1]).

Theorem 2.2 *Let u be a solution of the nonhomogeneous A-harmonic equation in a domain Ω and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st} \|u\|_{t,\rho B} \tag{2.8}$$

for all balls B with $\rho B \subset \Omega$ for some $\rho > 1$.

For A_r-weights ω, we have the following reverse Hölder inequality (Lemma 1.4.7 of [1]).

Theorem 2.3 *If $\omega \in A_r$, $r > 1$, then there exist constants $\beta > 1$ and C, independent of ω, such that*

$$\|\omega\|_{\beta,Q} \leq C|Q|^{(1-\beta)/\beta} \|\omega\|_{1,Q} \tag{2.9}$$

for all balls $Q \subset \mathbf{R}^N$.

3 Main results and proofs

Theorem 3.1 *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the A-harmonic equation in domain Ω is bounded and convex and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator. Assume that $\rho > 1$ and $\omega \in A_r(\Omega)$ for some $1 < r < \infty$. Then $T(G(u)) \in L^s_{loc}(\Omega, \wedge^l)$. Moreover, there exists a constant C, independent of u, such that*

$$\left\| T\big(G(u)\big) \right\|_{s,B,\omega^\alpha} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,\omega^\alpha} \tag{3.1}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha \leq 1$.

Proof We only need to prove the inequality holds. With (2.6) and (2.7), we have

$$\begin{aligned}\|T(G(u))\|_{s,B} &\le C\mu(\Omega)\operatorname{diam} B\|G(u)\|_{s,B}\\ &\le C\mu(\Omega)\operatorname{diam} B\|a(x)+b|u|\|_{s,B}\\ &\le C\mu(\Omega)\operatorname{diam} B(\|a(x)\|_{s,B}+b\|u\|_{s,B})\\ &\le C_1\mu(\Omega)\operatorname{diam} B\|u\|_{s,B}.\end{aligned} \tag{3.2}$$

Then just like the process of the proof for Theorem 7.3.14 in [1], we obtain the inequality.

We discuss the inequality with $0<\alpha<1$ and $\alpha=1$ separately. For $0<\alpha<1$, first we set $t=s/(1-\alpha)$. With Hölder inequality, we obtain

$$\begin{aligned}\|T(G(u))\|_{s,B,\omega^\alpha} &= \left(\int_B \left(\left(|T(G(u))|\right)\omega^{\alpha/s}\right)^s dx\right)^{1/s}\\ &\le \|T(G(u))\|_{t,B}\left(\int_B \omega^{t\alpha/(t-s)}\,dx\right)^{(t-s)/st}\\ &= \|T(G(u))\|_{t,B}\left(\int_B \omega\,dx\right)^{\alpha/s}.\end{aligned} \tag{3.3}$$

By (3.2), we obtain

$$\|T(G(u))\|_{t,B} \le C_3\mu(\Omega)\operatorname{diam} B\|u\|_{t,B}. \tag{3.4}$$

Let $m=s/(1+\alpha(r-1))$, then $m<s$. With (3.3) and (3.4) and using Theorem 2.2, we have

$$\begin{aligned}\|T(G(u))\|_{s,B,\omega^\alpha} &\le C_3\mu(\Omega)\operatorname{diam} B\|u\|_{t,B}\left(\int_B \omega\,dx\right)^{\alpha/s}\\ &\le C_4\mu(\Omega)\operatorname{diam} B|B|^{(m-t)/mt}\|u\|_{m,\rho B}\left(\int_B \omega\,dx\right)^{\alpha/s}.\end{aligned} \tag{3.5}$$

And using Hölder's inequality again, we obtain

$$\begin{aligned}\|u\|_{m,\rho B} &= \left(\int_{\rho B} |u|^m\,dx\right)^{1/m}\\ &= \left(\int_{\rho B} \left(|u|\omega^{\alpha/s}\omega^{-\alpha/s}\right)^m dx\right)^{1/m}\\ &\le \left(\int_{\rho B} |u|^s\omega^\alpha\,dx\right)^{1/s}\left(\int_{\rho B} (1/\omega)^{1/(r-i)}\,dx\right)^{\alpha(r-1)/s}\end{aligned} \tag{3.6}$$

for all balls B with $\rho B\subset\Omega$. With (3.5) and (3.6), we find that

$$\begin{aligned}\|T(G(u))\|_{s,B,\omega^\alpha} &\le C_4\mu(\Omega)\operatorname{diam} B|B|^{(m-t)/mt}\|u\|_{s,\rho B,\omega^\alpha}\\ &\quad\times\left(\int_{\rho B} (1/\omega)^{1/(r-1)}\,dx\right)^{\alpha(r-1)/s}\left(\int_B \omega\,dx\right)^{\alpha/s}.\end{aligned} \tag{3.7}$$

As $\omega \in A_r(\Omega)$, we have

$$\begin{aligned}
&\left(\int_{\rho B} (1/\omega)^{1/(r-1)}\,dx\right)^{\alpha(r-1)/s}\left(\int_B \omega\,dx\right)^{\alpha/s} \\
&\quad\le \left(\left(\int_{\rho B} \omega\,dx\right)\left(\int_{\rho B} (1/\omega)^{1/(r-1)}\,dx\right)^{r-1}\right)^{\alpha/s} \\
&\quad= \left(|\rho B|^r\left(\frac{1}{|\rho B|}\int_{\rho B} \omega\,dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B} (1/\omega)^{1/(r-1)}\,dx\right)^{r-1}\right)^{\alpha/s} \\
&\quad\le C_5|B|^{\alpha r/s}. \qquad (3.8)
\end{aligned}$$

With (3.7) and (3.8), we have

$$\left\|T\big(G(u)\big)\right\|_{s,B,\omega^\alpha} \le C_6\mu(B)\operatorname{diam}B\|u\|_{s,\rho B,\omega^\alpha} \qquad (3.9)$$

for all balls B with $\rho B \subset \Omega$. This is just (3.1) with $0 < \alpha < 1$. Then we prove the case of $\alpha = 1$. First, with Theorem 2.3, we know

$$\|\omega\|_{\beta,B} \le C_7|B|^{(1-\beta)/\beta}\|\omega\|_{1,B}, \qquad (3.10)$$

here $\beta > 1$ and $C_7 > 0$ are all constants. Let $t = s\beta/(\beta-1)$, then we know $1 < s < t$ and $\beta = t/(t-s)$. With Hölder's inequality (3.2), and (3.10), we obtain

$$\begin{aligned}
\left\|T\big(G(u)\big)\right\|_{s,B,\omega} &= \left(\int_B \left(\left|T\big(G(u)\big)\right|\omega^{1/s}\right)^s dx\right)^{1/s} \\
&\le \left(\left(\int_B \left|T\big(G(u)\big)\right|^t dx\right)^{1/t}\left(\int_B \left(\omega^{1/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st}\right) \\
&= C_8\left\|T\big(G(u)\big)\right\|_{t,B}\|\omega\|_{\beta,B}^{1/s} \\
&\le C_8\mu(B)\operatorname{diam}B\|u\|_{t,B}\|\omega\|_{\beta,B}^{1/s} \\
&\le C_9\mu(B)\operatorname{diam}B|B|^{(1-\beta)/\beta s}\|\omega\|_{\beta,B}^{1/s}\|u\|_{t,B} \\
&\le C_9\mu(B)\operatorname{diam}B|B|^{-1/t}\|\omega\|_{\beta,B}^{1/s}\|u\|_{t,B}. \qquad (3.11)
\end{aligned}$$

Set $m = s/r$. With Theorem 2.2, we have

$$\|u\|_{t,B} \le C_{10}|B|^{(m-t)/mt}\|u\|_{m,\rho B}. \qquad (3.12)$$

And we use Hölder's inequality again

$$\begin{aligned}
\|u\|_{m,\rho B} &= \left(\int_{\rho B} \left(|u|\omega^{1/s}\omega^{-1/s}\right)^m dx\right)^{1/m} \\
&\le \left(\int_{\rho B} |u|^s\omega\,dx\right)^{1/s}\left(\int_{\rho B} (1/\omega)^{1/(r-1)}\,dx\right)^{(r-1)/s}. \qquad (3.13)
\end{aligned}$$

With $\omega \in A_r(\Omega)$, we have

$$\|\omega\|_{1,B}^{1/s}\|1/\omega\|_{1/(r-1),\rho B}^{1/s} \le \left(\left(\int_{\rho B} \omega\,dx\right)\left(\int_{\rho B} (1/\omega)^{1/(r-1)}\,dx\right)^{r-1}\right)^{1/s}$$

$$= \left(|\rho B|^r \left(\frac{1}{|\rho B|}\int_{\rho B}\omega\,dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B}(1/\omega)^{1/(r-1)}\,dx\right)^{r-1}\right)^{1/s}$$
$$\le C_{11}|B|^{r/s}. \tag{3.14}$$

With (3.11)-(3.14), we have

$$\begin{aligned}\left\|T\big(G(u)\big)\right\|_{s,B,\omega} &\le C_{12}\mu(B)\operatorname{diam}B|B|^{-1/t}\|\omega\|_{1,B}^{1/s}|B|^{(m-t)/mt}\|u\|_{m,\rho B}\\ &\le C_{12}\mu(B)\operatorname{diam}B|B|^{-1/m}\|\omega\|_{1,B}^{1/s}\|1/\omega\|_{1/(r-1),\rho B}^{1/s}\|u\|_{s,\rho B,\omega}\\ &\le C_{13}\mu(B)\operatorname{diam}B\|u\|_{s,\rho B,\omega}\end{aligned} \tag{3.15}$$

for all balls B with $\rho B\subset\Omega$. Thus, we complete the proof. □

Actually by the method developed in [1], for the two weight $(\omega_1,\omega_2)\in A_{r,\lambda}(\Omega)$, we have the following inequality.

Theorem 3.2 *Let $u\in L^s_{loc}(\Omega,\wedge^l)$, $l=1,2,\dots,n$, $1<s<\infty$, be a solution of the A-harmonic equation in domain Ω is bounded and convex and $T: C^\infty(\Omega,\wedge^l)\to C^\infty(\Omega,\wedge^{l-1})$ be the homotopy operator. Assume that $\rho>1$ and $(\omega_1,\omega_2)\in A_{r,\lambda}(\Omega)$ for some $1<r<\infty$, $\lambda\ge 1$. Then, $T(G(u))\in L^S_{loc}(\Omega,\wedge^l)$. Moreover, there exists a constant C, independent of u, such that*

$$\left\|T\big(G(u)\big)\right\|_{s,B,\omega_1^\alpha}\le C|B|\operatorname{diam}(B)\|u\|_{s,\rho B,\omega_2^\alpha} \tag{3.16}$$

for all balls B with $\rho B\subset\Omega$ and any real number α with $0<\alpha\le 1$.

Proof Let $t=\lambda s/(\lambda-\alpha)$. As $\frac{1}{s}=\frac{1}{t}+\frac{(t-s)}{st}$, with Hölder inequality, we have

$$\begin{aligned}\left(\int_B\left|T(Gu)\right|\omega_1^\alpha\,dx\right)^{1/s} &= \left(\int_B\left(\left|T(Gu)\right|\omega_1^{\alpha/s}\right)^s dx\right)^{1/s}\\ &\le \left(\int_B\left|T(Gu)\right|^t dx\right)^{1/t}\left(\int_B\left(\omega_1^{\alpha/s}\right)^{st/(t-s)}dx\right)^{(t-s)/st}\\ &\le \left\|T(Gu)\right\|_{t,B}\left(\int_B\omega_1^\lambda\,dx\right)^{\alpha/\lambda s}\end{aligned} \tag{3.17}$$

for all balls $B\subset\Omega$. Then, from (3.2), we obtain

$$\left\|T(Gu)\right\|_{t,B}\le C_1\mu(\Omega)\operatorname{diam}B\|u\|_{t,B}. \tag{3.18}$$

Let $m=\frac{\lambda s}{(\lambda+\alpha(r-1))}$, then we know $m<s<t$. With (3.17) and (3.18) and Theorem 2.2, we have

$$\begin{aligned}&\left(\int_B\left|T(Gu)\right|^s\omega_1^\alpha\,dx\right)^{1/s}\\ &\quad\le C_1|B|\mu(\Omega)\operatorname{diam}B\|u\|_{t,B}\left(\int_B\omega_1^\lambda\,dx\right)^{\alpha/\lambda s}\\ &\quad\le C_2|B||B|^{\frac{m-t}{mt}}\|u\|_{m,\rho B}\left(\int_B\omega_1^\lambda\,dx\right)^{\alpha/\lambda s}.\end{aligned} \tag{3.19}$$

Then by the generalized Hölder's inequality, we have

$$\begin{aligned}
\|u\|_{m,\rho B} &= \left(\int_{\rho B} |u|^m \, dx\right)^{1/m} \\
&= \left(\int_{\rho B} \left(|u|\omega_2^{\alpha/s}\omega_2^{-\alpha/s}\right)^m \, dx\right)^{1/m} \\
&\le \left(\int_{\rho B} |u|^s \omega_2^\alpha \, dx\right)^{1/s} \left(\int_{\rho B} (1/\omega_2)^{\lambda/(r-1)} \, dx\right)^{\alpha(r-1)/\lambda s}
\end{aligned} \tag{3.20}$$

for all balls B with $\rho B \subset \Omega$, where we use $\frac{1}{m} = \frac{1}{s} + \frac{s-m}{sm}$. Then with (3.19) and (3.20), we obtain

$$\begin{aligned}
&\left(\int_B \left|T(Gu)\right|^s \omega_1^\alpha \, dx\right)^{1/s} \\
&\quad \le C_3|B||B|^{\frac{m-t}{mt}} \mu(\Omega) \operatorname{diam} B \left(\int_{\rho B} |u|^s \omega_2^\alpha \, dx\right)^{1/s} \\
&\qquad \times \left(\int_{\rho B} (1/\omega_2)^{\lambda/(r-1)} \, dx\right)^{\frac{\alpha(r-1)}{\lambda s}} \left(\int_B \omega_1^\lambda \, dx\right)^{\alpha/\lambda s}.
\end{aligned} \tag{3.21}$$

Then, with $(\omega_1, \omega_2) \in A_{r,\lambda}(\Omega)$, we have

$$\begin{aligned}
&\left(\left(\int_B \omega_1^\lambda \, dx\right)^{\alpha/\lambda s} \left(\int_{\rho B} (1/\omega_2)^{\lambda/(r-1)} \, dx\right)^{r-1}\right)^{\alpha/\lambda s} \\
&\quad \le \left(\left(\int_{\rho B} \omega_1^\lambda \, dx\right)^{\alpha/\lambda s} \left(\int_{\rho B} (1/\omega_2)^{\lambda/(r-1)} \, dx\right)^{r-1}\right)^{\alpha/\lambda s} \\
&\quad = |\rho B|^r \left(\frac{1}{|\rho B|} \int_{\rho B} \omega_1^\lambda \, dx\right) \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{\omega_2^\lambda}\right)^{1/(r-1)}\right)^{\alpha/\lambda s} \\
&\quad \le C_4 |B|^{\alpha r/\lambda s}.
\end{aligned} \tag{3.22}$$

With (3.21) and (3.22), we have

$$\left(\int_B \left|T(Gu)\right|^s \omega_1^\alpha \, dx\right)^{1/s} \le C\mu(\Omega) \operatorname{diam} B \left(\int_{\rho B} |u|^s \omega_2^\alpha \, dx\right)^{1/s} \tag{3.23}$$

for all balls B with $\rho B \subset \Omega$. □

For the compositions of the gradient operator ∇, the homotopy operator T, the carathéodory operator G, $\nabla \circ T \circ G$, we obtain the local *Sobolev-Poincaré* embedding theorem.

Theorem 3.3 *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the A-harmonic equation in bounded and convex domain Ω, $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator, ∇ be the gradient operator and G be the carathéodory operator. Then $\nabla(T(G(u))) \in L^s_{loc}(\Omega, \wedge^l)$ and $T(G(u)) \in W^{1,s}(B)$. Moreover, there exists a constant C, inde-*

pendent of u, such that

$$\|\nabla(T(G(u)))\|_{s,B} \le C\mu(\Omega)\|u\|_{s,B} \tag{3.24}$$

and

$$\|T(G(u))\|_{W^{1,s}(B)} \le C\mu(\Omega)\|u\|_{s,B}. \tag{3.25}$$

Proof Actually, we only need to prove (3.16) and (3.17). From these two inequalities, the remaining part of the theorem follows. From (2.4), we obtain

$$\|\nabla(T(\omega))\|_{p,\Omega} \le C\mu(\Omega)\|\omega\|_{p,\Omega} \tag{3.26}$$

for any $\omega \in L^p_{loc}(\wedge^l B)$. Let $G(u) = \omega$, we have

$$\begin{aligned}\|\nabla(T(G(u)))\|_{s,B} &\le C\mu(\Omega)\|G(u)\|_{s,B}\\ &\le C\mu(\Omega)\big(\|a(x)\|_{s,B} + b\|u\|_{s,B}\big)\\ &\le C_1\mu(\Omega)\|u\|_{s,B}.\end{aligned} \tag{3.27}$$

With the definition of $W^{1,p}$ norm, (3.2), and (3.27), we have

$$\begin{aligned}\|T(G(u))\|_{W^{1,p}(B)} &= \operatorname{diam}(B)^{-1}\|T(G(u))\|_{s,B} + \|\nabla(T(G(u)))\|_{s,B}\\ &\le \operatorname{diam}(B)^{-1}C_1\mu(\Omega)\operatorname{diam}B\|u\|_{s,B} + C\mu(\Omega)\|u\|_{s,B}\\ &\le C_2\mu(\Omega)\|u\|_{s,B}.\end{aligned} \tag{3.28}$$

Thus, we obtain the inequality. □

Using the same method as in the proof of Theorem 3.1, we obtain the weighted inequality for $\|\nabla(T(G(u)))\|_{s,B,\omega^\alpha}$.

Corollary 3.4 *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \dots, n$, $1 < s < \infty$, be a solution of the A-harmonic equation in bounded and convex domain Ω, $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator, ∇ be the gradient operator and G be the carathéodory operator. Assume that $\rho > 1$ and $\omega \in A_r(\Omega)$ for some $1 < r < \infty$. Then $\nabla(T(G(u))) \in L^s_{loc}(\Omega, \wedge^l)$. Moreover, there exists a constant C, independent of u, such that*

$$\|\nabla(T(G(u)))\|_{s,B,\omega^\alpha} \le C\mu(\Omega)\|u\|_{s,B,\omega^\alpha}. \tag{3.29}$$

For $G(T(u))$, we also have the similar result.

Corollary 3.5 *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \dots, n$, $1 < s < \infty$, be a solution of the A-harmonic equation in a bounded, convex domain Ω and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator. Assume that $\rho > 1$ and $\omega \in A_r(\Omega)$ for some $1 < r < \infty$. Then $G(T(u)) \in L^s_{loc}(\Omega, \wedge^l)$. Moreover, there exists a constant C, independent of u, such that*

$$\|G(T(u))\|_{s,B,\omega^\alpha} \le C|B|\operatorname{diam}(B)\|u\|_{s,\rho B,\omega^\alpha} \tag{3.30}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha \le 1$.

Proof If (3.22) holds, then $G(T(u)) \in L^s_{loc}(\Omega, \wedge^l)$ follows. Hence, we only need to prove (3.22). From (2.6) and (2.7), we have

$$\begin{aligned} \|G(T(u))\|_{s,B} &\leq \|a(x)\|_{s,B} + b\|T(u)\|_{s,B} \\ &\leq C\mu(\Omega)\,\mathrm{diam}(B)\|u\|_{s,B}. \end{aligned} \tag{3.31}$$

Using the method in the proof of Theorem 3.1, we obtain the inequality. □

Actually for two weight $(\omega_1, \omega_2) \in A_{r,\lambda}(\Omega)$, for some $\lambda \leq 1$ and $1 < r < \infty$, we have the similar inequalities, with the method developed in the proof of Theorem 3.2.

Corollary 3.6 *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying A-harmonic equation in a bounded, convex domain $\Omega \subset R^N$ and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2.2). Assume $\rho > 1$ and $(\omega_1, \omega_2) \in A_{r,\lambda}(\Omega)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B \left|\nabla\left(T(u)\right)\right|^s \omega_1^\alpha \, dx\right)^{1/s} \leq C|B|\left(\int_{\rho B} |u|^s \omega_2^\alpha \, dx\right)^{1/s} \tag{3.32}$$

for all balls B with $\rho B \subset \Omega$ and all real number α with $0 < \alpha < \lambda$.

The above inequality is an extension of the usual inequality of A_r-weights. If we choose $\omega_1(x) = \omega_2(x) = \omega(x)$ and $\lambda = 1$ in the two weighted inequalities, we obtain the $A_r(\Omega)$ weight case.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Professor JZ gave the ideas. ZT gave the proofs and completed the manuscript.

Acknowledgement
The research of the first author was supported by the NSF of Hunan Province (No. 11JJ3004) and NSF of NUDT (No. JC10-02-02).

Received: 18 October 2011 Accepted: 6 June 2012 Published: 31 August 2012

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doi:10.1186/1029-242X-2012-188
Cite this article as: Tang and Zhu: Estimates for the composition of the carathéodory and homotopy operators. *Journal of Inequalities and Applications* 2012 **2012**:188.