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Weighted Trudinger inequality associated with rough multilinear fractional type operators

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Abstract

Let $I_{\Omega,\alpha}^{\Theta}$ be the multilinear fractional type operator defined by $I_{\Omega,\alpha}^{\Theta}(\vec{f})(x) = \int_{\mathbb{R}^n} \Omega(y) \prod_{j=1}^{m} f_j(x-\theta_j y)|y|^{(\alpha-n)}\,dy$. In this paper, we study the weighted estimates for the Trudinger inequality associated to $I_{\Omega,\alpha}^{\Theta}$ with rough homogeneous kernels, which improve some known results significantly. A similar Trudinger inequality holds for another type of fractional integral defined by $\overline{I}_{\Omega,\alpha}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{j=1}^{m}|f_j(y_j)||\Omega_j(x-y_j)|}{|(x-y_1,x-y_2,\ldots,x-y_m)|^{mn-\alpha}}\,d\vec{y}$, where $d\vec{y} = dy_1 \cdots dy_m$.

Keywords: Riesz potential; multilinear fractional integral; A_p weights; $A_{p,q}$ weights; Trudinger inequality

1 Introduction

The Trudinger inequality (also sometimes called the Moser-Trudinger inequality) is named after N. Trudinger who first put forward this inequality in [22]. Later, J. Moser [14] gave a sharp form of this Trudinger inequality. It provides an inequality between a certain Sobolev space norm and an Orlicz space norm of a function. In [14], J. Moser gave the largest positive number β_0, such that if $u \in C^1(\mathbb{R}^n)$, normalized and supported in a domain D with finite measure in $\mathbb{R}^n$, such that $\int_D |\nabla u(x)|^n\,dx \le 1$, then there is a constant c_0 depending only on n such that for all $\beta \le \beta_0 = nw_{n-1}^{1/(n-1)}$, where w_{n-1} is the area of the surface of the unit n-ball. The following inequality holds:

$$\int_D \exp\bigl(\beta\bigl|u(x)\bigr|^{n/(n-1)}\bigr)\,dx \le c_0|D|. \tag{1.1}$$

In 1971, D. Adams [1] considered the similar inequality of J. Moser for higher order derivatives. The key, for him, was to write the function u as a potential I_α (see the definition below) and prove the analogue of (1.1) as follows:

$$\int_D \exp\biggl(\frac{n}{w_{n-1}}\biggl|\frac{I_\alpha f(x)}{\|f\|_p}\biggr|^{n/(n-\alpha)}\biggr)\,dx \le c_0|D|, \quad \text{for } \alpha = n/p, f \in L^p\ (1 < p < \infty). \tag{1.2}$$

Variant forms of the Trudinger inequality as a generalization of the classical results, especially in the literature associated with multilinear Riesz potential or multilinear fractional integral, have been studied in recently years (see, for example, [2, 3, 6, 7, 10, 14, 16–18, 20, 21]). This kind of inequality plays an important role in Harmonic analysis and other fields, such as PDE.

We begin by introducing a class of multilinear maximal function and multilinear fractional integral operators. Suppose that $n \geq 2$, $0 < \alpha < n$, Ω is homogeneous of degree zero, and $\Omega \in L^s(S^{n-1})$ $(s > 1)$, where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$. The multilinear maximal function and multilinear fractional integral is defined by

$$I^{\Theta}_{\Omega,\alpha}(\vec{f})(x) = \int_{\mathbb{R}^n} \Omega(y) \prod_{j=1}^{m} f_j(x - \theta_j y)|y|^{(\alpha-n)}\, dy \tag{1.3}$$

and the fractional maximal operator $M_{\Omega,\alpha}$ defined by

$$M^{\Theta}_{\Omega,\alpha}(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} \left|\Omega(y)\right| \prod_{j=1}^{m} \left|f_j(x - \theta_j y)\right| dy. \tag{1.4}$$

Multilinear fractional integral $I^{\Theta}_{\Omega,\alpha}$ can be looked at as a natural generalization of the classical fractional integral, which has a very profound background of partial differential equations and is a very important operator in Harmonic analysis. In fact, if we take $K = 1$, $\theta_j = 1$, and $\Omega = 1$, then $I^{\Theta}_{\Omega,\alpha}$ is just the well-known classical fractional integral operator studied by Muckenhoupt and Wheeden in [15]. We denote it by I_α. If $\Omega \equiv 1$, we simply denote $I^{\Theta}_{\Omega,\alpha} = I^{\Theta}_{\alpha}$. In recent years, the study of the Trudinger inequality associated to multilinear type operators has received increasing attention. Among them, it is well known that Grafakos considered the boundedness of a family of related fractional integrals in [7]. After that, in [6], Y. Ding and S. Lu gave the following Trudinger inequality with rough kernels.

Theorem A ([6]) *Let $0 < \alpha < n$, $s = \frac{n}{\alpha}$, $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, $p_j > 1$, $j = 1, 2, \ldots, m$, $m \geq 2$. Denote B as a ball with a radius R in $\mathbb{R}^n$. If $f_j \in L^{p_j}(B)$, $\operatorname{supp}(f_j) \subset B$, and $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, then for any $\gamma < 1$, there is a constant C, independent of n, α, θ_j, γ, such that*

$$\int_B \exp\left(n\gamma \left(\frac{L I^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}}}\right)^{n/(n-\alpha)}\right) dx \leq CR^n,$$

where $L = \prod_{j=1}^{m} |\theta_j|^{n/p_j}$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$, $\vec{f} = (f_1, f_2, \ldots, f_m)$ and

$$\|\Omega\|_{L^{n/(n-\alpha)}} = \left(\int_{S^{n-1}} \left|\Omega(x)\right|^{n/(n-\alpha)} d\sigma(x)\right)^{(n-\alpha)/n}.$$

The definition of multiple weights $A_{\vec{p},q}$ was given in [5] and [13] independently, including some weighted estimates for a class of multilinear fractional type operators. These results together with [12] answered an open problem in [8], namely the existence of the multiple weights.

In 2010, W. Li, Q. Xue, and K. Yabuta [16] obtained the weighted estimates for the Trudinger inequality associated to I^{Θ}_{α} as follows.

Theorem B ([16]) *Let $0 < \alpha < n$, $s = \frac{n}{\alpha}$, $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, $p_j > 1$, $\omega_j(x) \in A_{p_j}$, and $\omega_j \geq 1$, $j = 1, 2, \ldots, m$, $m \geq 2$, $\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{s/p_j}$. Denote B as a ball with the radius R in $\mathbb{R}^n$, if $f_j \in L^{p_j}_{\omega_j}(B)$,*

$\operatorname{supp}(f_j) \subset B$, *$j = 1, 2, \ldots, m$, then for any $\gamma < 1$, there is a constant C, independent of n, α, θ_j, γ, such that*

$$\int_B \exp\left(\frac{n}{\omega_{n-1}}\gamma\left(\frac{LI_\alpha^\Theta(\vec{f})(x)}{\prod_{j=1}^m \|f_j\|_{L_{\omega_j}^{p_j}}}\right)^{n/(n-\alpha)}\right)\nu_{\vec{\omega}}\,dx \le C\prod_{j=1}^m \omega_j(B),$$

where $L = \prod_{j=1}^m |\theta_j|^{n/p_j}$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$, $\vec{f} = (f_1, f_2, \ldots, f_m)$.

On the other hand, in 1999, Kenig and Stein [11] considered another more general type of multilinear fractional integral which was defined by

$$I_{\alpha,A}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|(y_1, \ldots, y_m)|^{mn-\alpha}} \prod_{i=1}^m f_i\bigl(\ell_i(y_1, \ldots, y_m, x)\bigr)\,dy_i,$$

where ℓ_i is a linear combination of y_js and x depending on the matrix A. They showed that $I_{\alpha,A}$ was of strong type $(L^{p_1} \times \cdots \times L^{p_m}, L^q)$ and weak type $(L^{p_1} \times \cdots \times L^{p_m}, L^{q,\infty})$. When $\ell_i(y_1, \ldots, y_m, x) = x - y_i$, we denote this multilinear fractional type operator by $\bar{I}_\alpha$. In 2008, L. Tang [20] obtained the estimation of the exponential integrability of the above operator $\bar{I}_\alpha$, which is quite similar to Theorem B.

Thus, it is natural to ask whether Theorem B is true or not for $I_{\Omega,\alpha}^\Theta$ with rough kernels. Moreover, one may ask if Theorem B still holds or not for the operator with rough kernels defined by

$$\bar{I}_{\Omega,\alpha}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{j=1}^m |f_j(y_j)||\Omega_j(x - y_j)|}{|(x - y_1, x - y_2, \ldots, x - y_m)|^{mn-\alpha}}\,d\vec{y}.$$

Inspired by the works above, in this paper, we study the Trudinger inequality associated to multilinear fractional integral operators $I_{\Omega,\alpha}^\Theta$ and $\bar{I}_{\Omega,\alpha}$ with rough homogeneous kernels. Precisely, we obtain the following theorems, which give a positive answer to the above questions.

Theorem 1.1 *Let $0 < \alpha < n$, $s = \frac{n}{\alpha}$, $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, $p_j > 1$, $j = 1, 2, \ldots, m$, $m \ge 2$. Denote B as a ball with radius R in $\mathbb{R}^n$; if $f_j \in L_{\omega_j}^{p_j}(B)$,* $\operatorname{supp}(f_j) \subset B$ *$(j = 1, 2, \ldots, m)$, $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, and $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{s}{p_j}}$, where $\omega_j \in A_s$, $\omega_j \ge 1$. Then for any $\gamma < 1$, there is a constant C, independent of n, α, θ_j, γ, such that*

$$\int_B \exp\left(n\gamma\left(\frac{LI_{\Omega,\alpha}^\Theta(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^m \|f_j\|_{L_{\omega_j}^{p_j}}}\right)^{n/(n-\alpha)}\right)\nu_{\vec{\omega}}\,dx \le C\prod_{j=1}^m \omega_j(B),$$

where $L = \prod_{j=1}^m |\theta_j|^{n/p_j}$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$, $\vec{f} = (f_1, f_2, \ldots, f_K)$.

Remark 1.1 If we take $\Omega = 1$, then Theorem 1.1 coincides with Theorem B. If $w_j \equiv 1$ for $j = 1, \ldots, K$, then Theorem 1.1 is just Theorem A that appeared in [6]. We give an example of $\nu_{\vec{\omega}}$ as follows: Let $\omega_j(x) = (1 + |x|)^{\alpha_j}$ ($\alpha_j \ge 0$ for each j), then $\nu_\omega(x)$ satisfy the conditions of the above Theorem 1.1.

Remark 1.2 Assume $m = 1$, $\omega_j = 1$. If $\alpha = 1$, Trudinger [20] proved exponential integrability of $I_\alpha(\vec{f})$, and Strichartz [19] for other α. In 1972, Hedberg [9] gave a simpler proof for all α. In 1970, Hempel-Morris-Trudinger [10] showed that if $\gamma > 1$, for $\alpha = 1$ the inequality in Theorem 1.1 cannot hold, and later Adams [1] obtained the same conclusion for all α; meanwhile, in the endpoint case $\gamma = 1$, it is true. In 1985, Chang and Marshall [4] proved a similar sharp exponential inequality concerning the Dirichlet integral. Assume $m \geq 2$, $w_j = 1$, then the result was obtained by Grafakos [7] as we have already mentioned above.

Corollary 1.2 *Let B, f_j, p_j, s, and $\nu_{\vec{\omega}}$ be the same as in Theorem 1.1, then $I^{\Theta}_{\Omega,\alpha}(\vec{f})$ is in $L^q(\nu_{\vec{\omega}}(B))$ for every $q > 0$, that is,*

$$\left\| I^{\Theta}_{\Omega,\alpha}(\vec{f}) \right\|_{L^q(\nu_{\vec{\omega}}(B))} \leq C \|\Omega\|_{L^{n/(n-\alpha)}(S^{n-1})} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{\omega_j}}$$

for some constant C depending only on q on n on α and on the θ_j's.

Theorem 1.3 *Let $m \geq 2$, $0 < \alpha < mn$, $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m = \alpha/n$ with $1 < p_i < \infty$ for $i = 1, 2, \ldots, m$. Let B be a ball with radius R in $\mathbb{R}^n$ and let $f_j \in L^{p_j}(B)$ be supported in B, and if Ω_j is homogeneous of degree zero, and $\Omega_j \in L^{p_j'}(S^{n-1})$, where S^{n-1} denotes the sphere of $\mathbb{R}^n$, and $\nu_{\vec{\omega}}(\vec{y}) = \prod_{j=1}^m \omega_j^{1/p_j}(y_j)$, where $\vec{y} = (y_1, y_2, \ldots, y_m)$ and $\omega_j \in A_s$, $\omega_j \geq 1$. Then there exist constants k_1, k_2 depending only on n, m, α, p, and the p_j such that*

$$\int_B \exp\left(k_1 \left(\frac{|\bar{I}_{\Omega,\alpha}(\vec{f})(x)|}{\prod_{j=1}^m \|\Omega_j\|_{L^{p_j'}(S^{n-1})} \|f_j\|_{L^{p_j}_{\omega_j}}} \right)^{n/(mn-\alpha)} \right) \nu_{\vec{\omega}}(x)\, dx \leq k_2 \prod_{j=1}^{m} \omega_j(B).$$

Remark 1.3 If we take $\Omega = 1$, $w_j \equiv 1$ for $j = 1, \ldots, m$, then Theorem 1.3 is just as Theorem 1.3 appeared in [20]. But there is something that needs to be changed in the proof of Theorem 1.3 in [20]. In the case $r_1 = r_2 = \cdots = r_{m-1} = 0$, one cannot obtain the conclusion that $F_2 \leq C_2[\log \frac{2\sqrt{m}R}{\delta}]^{(mn-\alpha)/n}$. Thus, our proof gives an alternative correction of Theorem 1.3 in [20].

Corollary 1.4 *Let B, f_j, p_j, s, and $\nu_{\vec{\omega}}$ be the same as in Theorem 1.3. Then $\bar{I}_{\Omega,\alpha}(\vec{f})$ is in $L^q(\nu_{\vec{\omega}}(B))$ for every $q > 0$, that is,*

$$\left\| \bar{I}_{\Omega,\alpha}(\vec{f}) \right\|_{L^q(\nu_{\vec{\omega}}(B))} \leq C \prod_{j=1}^{m} \|\Omega_j\|_{L^{p_j'}(S^{n-1})} \|f_j\|_{L^{p_j}_{\omega_j}}$$

for some constant C depending only on q on n on α.

Corollary 1.2 and Corollary 1.4 follow since exponential integrability of $\bar{I}_{\Omega,\alpha}(\vec{f})$ implies integrability to any power q.

On the other hand, we shall study the boundedness of the multilinear fractional maximal operator with a weighted norm. It follows the following theorem.

Theorem 1.5 *If* $1 < p_j < \infty$, $\frac{1}{s} = \sum_{j=1}^{m} \frac{1}{p_j}$, $\frac{1}{r} = \frac{1}{s} - \frac{\alpha}{n}$, $\omega_j^{\frac{p_j}{s}} \in A(s, \frac{sr_j}{p_j})$, $1/r_j = 1/p_j(1 - \alpha s/n)$, $j = 1, 2, \dots, m$, $\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j$, *then there is a constant C, independent* f_j, *such that*

$$\left(\int_{\mathbb{R}^n} \left(M_{1,\alpha}^{\Theta}(\vec{f})(x)\nu_{\vec{\omega}}(x)\right)^r dx\right)^{\frac{1}{r}} \le C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^n} \left|f_j(x)\omega_j(x)\right|^{p_j} dx\right)^{\frac{1}{p_j}},$$

where $\vec{f} = (f_1, f_2, \dots, f_m)$, $f_j \in L_{\omega_j}^{p_j}(\mathbb{R}^n)$.

2 The proof of Theorem 1.1

In this section, we will prove Theorem 1.1.

Proof For any $\delta > 0$,

$$\left|I_{\Omega,\alpha}^{\Theta}(\vec{f})(x)\right| \le C\delta^{\alpha} M_{\Omega}(\vec{f})(x) + \int_{|y|\ge\delta} \frac{|\Omega(y)|}{|y|^{n-\alpha}} \prod_{j=1}^{m} f_j(x - \theta_j y)\, dy.$$

Set $P = 2\min\{\frac{1}{\theta_j} : j = 1, 2, \dots, K\}$. For any $R > 0$, denote $B(R)$ as a ball with radius R in $\mathbb{R}^n$, then for any $x \in B(R)$, when $|x - \theta_j y| < R$, $|\theta_j y| < 2R$ for $j = 1, \dots, m$. Therefore, $|y| < RP$. So,

$$\int_{|y|\ge\delta} \prod_{j=1}^{m} f_j(x - \theta_j y)|y|^{\alpha-n}\, dy = \int_{\delta\le|y|<PR} \prod_{j=1}^{m} f_j(x - \theta_j y)|y|^{\alpha-n}\, dy.$$

According to the relationship between s and p_j: $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} + \frac{1}{n/(n-\alpha)} = 1$, from the Hölder's inequality and $\nu_{\vec{\omega}} \ge 1$, it follows that

$$\begin{aligned}
&\int_{\delta\le|y|<PR} \Omega(y) \prod_{j=1}^{m} f_j(x - \theta_j y)|y|^{\alpha-n}\, dy \\
&\quad\le \left(\int_{\delta\le|y|\le PR} \left(\prod_{j=1}^{m} f_j(x - \theta_j y)\right)^s dy\right)^{1/s} \left(\int_{\delta\le|y|\le PR} \left(\frac{|\Omega(y)|}{|y|^{n-\alpha}}\right)^{s'} dy\right)^{1/s'} \\
&\quad\le \left(\int_{\delta\le|y|\le PR} \prod_{j=1}^{m} f_j(x - \theta_j y)^s \nu_{\vec{\omega}}(x - \theta_j y)\, dy\right)^{1/s} \|\Omega\|_{L^{s'}} \left(\ln\frac{PR}{\delta}\right)^{\frac{n-\alpha}{n}} \\
&\quad\le \prod_{j=1}^{m} \left(\int_{\delta\le|y|\le PR} \left|f_j(x - \theta_j y)\right|^{p_j} \omega_j(x - \theta_j y)\, dy\right)^{\frac{1}{p_j}} \|\Omega\|_{L^{s'}} \left(\frac{1}{n}\ln\left(\frac{PR}{\delta}\right)^n\right)^{\frac{n-\alpha}{n}} \\
&\quad\le L^{-1} \prod_{j=1}^{m} \|f_j\|_{L_{\omega_j}^{p_j}} \|\Omega\|_{L^{s'}} \left(\frac{1}{n}\ln\left(\frac{PR}{\delta}\right)^n\right)^{\frac{n-\alpha}{n}}.
\end{aligned}$$

Hence, we obtain that

$$\left|I_{\Omega,\alpha}^{\Theta}(\vec{f})(x)\right| \le C\delta^{\alpha} M_{\Omega}\vec{f}(x) + L^{-1} \prod_{j=1}^{m} \|f_j\|_{L_{\omega_j}^{p_j}} \|\Omega\|_{L^{s'}} \left(\frac{1}{n}\ln\left(\frac{PR}{\delta}\right)^n\right)^{\frac{n-\alpha}{n}}.$$

Set $\delta = \varepsilon(|I^{\Theta}_{\Omega,\alpha}(\vec{f})(x)|/CM_{\Omega}(\vec{f})(x))^{1/\alpha}$, then

$$\exp\left\{n\gamma\left(\frac{LI^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{s'}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{\frac{n}{n-\alpha}}\right\} \le \ln CR^n\left(\frac{M_{\Omega}(\vec{f})(x)}{I^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}\right)^{n/\alpha}.$$

Now we put $B_1 = \{x \in B : \frac{I^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{\omega_j}}} \ge 1\}$, $B_2 = B - B_1$, thus

$$\begin{aligned}
&\int_{B_1}\exp\left(n\gamma\left(\frac{LI^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/(n-\alpha)}\right)\nu_{\vec{\omega}}(x)\,dx \\
&\quad\le CR^n\int_{B_1}\left(\frac{M_{\Omega}(\vec{f})(x)}{I^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}\right)^{n/\alpha}\nu_{\vec{\omega}}(x)\,dx \\
&\quad\le CR^n\int_{B_1}\left(\frac{M_{\Omega}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/\alpha}\nu_{\vec{\omega}}(x)\,dx.
\end{aligned}$$

By the fact that

$$\begin{aligned}
M_{\Omega}(\vec{f})(x) &= \sup_{r>0}\int_{|y|<r}\left|\Omega(y)\right|^{\sum_{j=1}^{m}\frac{s}{p_j}}\prod_{j=1}^{m}f_j(x-\theta_j y)\,dy \\
&\le \sup_{r>0}\prod_{j=1}^{m}\left(\frac{1}{r^n}\int_{|y|<r}\left|\Omega(y)\right|f_j^{\frac{p_j}{s}}(x-\theta_j y)\,dy\right)^{\frac{s}{p_j}} \\
&\le \prod_{j=1}^{m}\left(M_{\Omega}\left(f^{\frac{p_j}{s}}\right)(x)\right)^{\frac{s}{p_j}}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\int_{B_1}\exp\left(n\gamma\left(\frac{LI^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/(n-\alpha)}\right)\nu_{\vec{\omega}}(x)\,dx \\
&\quad\le \frac{CR^n}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^{m}\|f_j\|^{s}_{L^{p_j}_{\omega_j}}}\int_{B_1}\prod_{j=1}^{m}\left(M_{\Omega}\left(f_j^{\frac{p_j}{s}}(x)\right)\right)^{\frac{s^2}{p_j}}\nu_{\vec{\omega}}(x)\,dx \\
&\quad\le \frac{CR^n}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^{m}\|f_j\|^{s}_{L^{p-j}_{\omega_j}}}\prod_{j=1}^{m}\left(\int_{B_1}\left(M_{\Omega}\left(f_j^{\frac{p_j}{s}}(x)\right)\right)^{s}\omega_j(x)\,dx\right)^{\frac{1}{s}\frac{s^2}{p_j}} \\
&\quad\le \frac{CR^n}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^{m}\|f_j\|^{s}_{L^{p_j}_{\omega_j}}}\prod_{j=1}^{m}\|f_j^{\frac{p_j}{s}}\|^{\frac{s^2}{p_j}}_{L^{s}_{\omega_j}} \\
&\quad\le CR^n.
\end{aligned}$$

Here, in the above third inequality, we have used the well-known weighted result of Hardy-Littlewood maximal function.

From $\omega_j \geq 1$ ($j = 1, 2, \ldots, m$), we get

$$R^n = c\int_B dx \leq c\int_B \omega_j(x)\,dx = c\omega_j(B).$$

Hence,

$$\int_{B_1} \exp\left(n\gamma\left(\frac{LI^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^m \|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/(n-\alpha)}\right)\nu_{\vec{\omega}}(x)\,dx \leq C'\prod_{j=1}^m \omega_j(B).$$

On the other hand,

$$\begin{aligned}
&\int_{B_2} \exp\left(n\gamma\left(\frac{LI^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^m \|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/(n-\alpha)}\right)\nu_{\vec{\omega}}(x)\,dx \\
&\quad\leq \exp(n\gamma)\left(\frac{L}{\|\Omega\|_{L^{s'}}}\right)^{\frac{n}{n-\alpha}}\int_{B_2}\nu_{\vec{\omega}}(x)\,dx \\
&\quad\leq C\prod_{j=1}^m \omega_j(B).
\end{aligned}$$

From the above all, we obtain that

$$\int_{B} \exp\left(n\gamma\left(\frac{LI^{\Theta}_{\Omega,\alpha}(\vec{f})(x)}{\|\Omega\|_{L^{n/(n-\alpha)}}\prod_{j=1}^m \|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/(n-\alpha)}\right)\nu_{\vec{\omega}}(x)\,dx \leq C\prod_{j=1}^m \omega_j(B).$$

□

3 The proof of Theorem 1.5

In this section, we will prove Theorem 1.5.

Proof By the well-known Hölder's inequality, we get

$$\begin{aligned}
M_{1,\alpha}(\vec{f})(x) &= \sup_{r>0}\frac{1}{|r|^{n-\alpha}}\int_{|y|<r}\prod_{j=1}^m f_j(x-y)\,dy \\
&\leq \sup_{r>0}\frac{1}{|r|^{n-\alpha}}\prod_{j=1}^m\left(\int_{|y|<r} f_j^{\frac{p_j}{s}}(x-y)\,dy\right)^{\frac{s}{p_j}} \\
&\leq \prod_{j=1}^m\left(\sup_{r>0}\frac{1}{|r|^{n-\alpha}}\int_{|y|<r} f_j^{\frac{p_j}{s}}(x-y)\,dy\right)^{\frac{s}{p_j}} \\
&= \prod_{j=1}^m\left(M_{1,\alpha}\left(f^{p_j/s}\right)(x)\right)^{\frac{s}{p_j}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}\left(M_{1,\alpha}(\vec{f})(x)\nu_{\vec{\omega}}(x)\right)^r dx\right)^{1/r} &\leq \left[\int_{\mathbb{R}^n}\left(\prod_{j=1}^m\left[M_{1,\alpha}\left(f^{p_j/s}\right)(x)\omega_j^{p_j/s}(s)\right]^{\frac{s}{p_j}}\right)^r dx\right]^{1/r} \\
&\leq \prod_{j=1}^m\left[\int_{\mathbb{R}^n}\left(M_{1,\alpha}\left(f_j^{p_j/s}\right)(x)\omega^{p_j/s}(x)\right)^{sr_j/p_j}dx\right]^{\frac{p_j}{sr_j}\frac{s}{p_j}}.
\end{aligned}$$

In addition, from the condition $\omega_j^{p_j/s}(x) \in A(s, \frac{sr_j}{p_j})$, it follows that

$$\left[\int_{\mathbb{R}^n} \left(M_{1,\alpha}\left(f_j^{p_j/s}\right)(x)\omega^{p_j/s}(x)\right)^{sr_j/p_j} dx\right]^{\frac{p_j}{sr_j}\frac{s}{p_j}} \le C_j\left[\int_{\mathbb{R}^n} \left(f_j^{p_j/s}(x)\omega_j^{p_j/s}(x)\right)^s dx\right]^{1/p_j}.$$

According to the above, we obtain that

$$\left(\int_{R^n} \left(M_{1,\alpha}(\vec{f})(x)\nu_{\vec{\omega}}(x)\right)^r dx\right)^{1/r} = C\prod_{j=1}^m \left(\int_{\mathbb{R}^n} \left(f_j(x)\omega_j(x)\right)^{p_j} dx\right)^{1/p_j}.$$

It is easy to see that

$$M_{1,\alpha}^{\Theta}(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} \prod_{j=1}^m \left|f_j(x-\theta_j y)\right| dy,$$

where $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$, $\theta_j \in \mathbb{R}$ holds, also. □

4 The proof of Theorem 1.3

In this section, we will prove Theorem 1.3.

Proof For any $\delta > 0$ and $x \in B$,

$$\begin{aligned}
&\left|\bar{I}_{\Omega,\alpha}(f_1, f_2, \ldots, f_m)(x)\right| \\
&\quad \le \int_{|(x-y_1,x-y_2,\ldots,x-y_m)|<\delta} \frac{\prod_{j=1}^m |\Omega_j(y_j)f_j(y_j)|}{|(x-y_1, x-y_2, \ldots, x-y_m)|^{mn-\alpha}} \, d\vec{y} \\
&\qquad + \int_{|(x-y_1,x-y_2,\ldots,x-y_m)|\ge\delta} \frac{\prod_{j=1}^m |\Omega_j(y_j)f_j(y_j)|}{|(x-y_1, x-y_2, \ldots, x-y_m)|^{mn-\alpha}} \, d\vec{y} \\
&\quad := F_1 + F_2.
\end{aligned}$$

For F_1, let $\alpha = \sum_{j=1}^m \alpha_j$ with $\alpha_j = n/p_j$ for $j = 1, 2, \ldots, m$. Then

$$\begin{aligned}
F_1 &\le \int_{|(x-y_1,x-y_2,\ldots,x-y_m)|<\delta} \frac{|\Omega_j(y_j)f_j(y_j)|}{\prod_{j=1}^m |x-y_j|^{n-\alpha_j}} \, d\vec{y} \\
&\le \prod_{j=1}^m \int_{|x-y_j|<\delta} \frac{|\Omega_j(y_j)f_j(y_j)|}{|x-y_j|^{n-\alpha_j}} \, dy_j \\
&\le C\prod_{j=1}^m \delta^{\alpha_j} M_{\Omega_j}(f_j)(x) \\
&:= C_1\delta^{\alpha} \prod_{j=1}^m M_{\Omega_j}(f_j)(x),
\end{aligned}$$

where M_Ω denotes as $M_\Omega(f)(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(y)f(y)| \, dy$.

For F_2, if $(y_1, y_2, \ldots, y_m)$ satisfies $|(x-y_1, x-y_2, \ldots, x-y_m)| \ge \delta$, then for some $j \in 1, 2, \ldots, m$, $|x-y_j| \le \frac{\delta}{\sqrt{m}}$. Without losing the generalization, we set $j = m$.

Thus,

$$F_2 \le \int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\int_{(\mathbb{R}^n)^{m-1}} \frac{\prod_{j=1}^m |\Omega_j(y_j)f_j(y_j)|}{|(x-y_1,x-y_2,\dots,x-y_m)|^{mn-\alpha}}\,d\vec{y}.$$

Define that $f_j^0 = f_j\chi_{B(x,\delta/\sqrt{m})}$ and $f_j^\infty = f - f_j^0$ for $j = 1,2,\dots,m$. By the condition of $\nu_{\vec{\omega}}$, we have

$$\begin{aligned} F_2 &\le \sum_{\vec{r}\in\{0,\infty\}^m}\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\int_{(\mathbb{R}^n)^{m-1}} \frac{\prod_{j=1}^{m-1}|\Omega_j(y_j)f_j^{r_j}(y_j)||\Omega_m(y_m)f_m(y_m)|}{|(x-y_1,x-y_2,\dots,x-y_m)|^{mn-\alpha}}\,d\vec{y} \\ &\le \sum_{\vec{r}\in\{0,\infty\}^m}\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\int_{(\mathbb{R}^n)^{m-1}} \frac{\prod_{j=1}^{m-1}|\Omega_j(y_j)f_j^{r_j}(y_j)||\Omega_m(y_m)f_m(y_m)|}{|(x-y_1,x-y_2,\dots,x-y_m)|^{mn-\alpha}}\nu_{\vec{\omega}}(\vec{y})\,d\vec{y}, \end{aligned}$$

where $\vec{r} = (r_1,r_2,\dots,r_m)$. In the case that $r_1 = r_2 = \cdots = r_{m-1} = 0$, by the fact that

$$\begin{aligned} \big|(x-y_1,x-y_2,\dots,x-y_m)\big|^{mn-\alpha} &\ge |x-y_m|^{mn-\alpha} \\ &= |x-y_m|^{n-\alpha_m}|x-y_m|^{\sum_{j=1}^{m-1} n/p_j'} \\ &\ge |x-y_m|^{n-\alpha_m}\left(\frac{\delta}{\sqrt{m}}\right)^{\sum_{j=1}^{m-1} n/p_j'}, \end{aligned}$$

we have

$$\begin{aligned} &\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\int_{(\mathbb{R}^n)^{m-1}} \frac{\prod_{j=1}^{m-1}|\Omega_j(y_j)f_j^0(y_j)||\Omega(y_m)f_m(y_m)|}{|(x-y_1,x-y_2,\dots,x-y_m)|^{mn-\alpha}}\nu_{\vec{\omega}}(\vec{y})\,d\vec{y} \\ &\quad\le \prod_{j=1}^{m-1}\delta^{-\frac{n}{p_j'}}\int_{\frac{\delta}{\sqrt{m}}\le|x-y_m|\le 2R}\frac{|\Omega_m(y_m)f_m(y_m)|}{|x-y_m|^{n-\alpha_m}}\omega_m^{1/p_m}(y_m)\,dy_m \\ &\qquad\times\prod_{j=1}^{m-1}\int_{|x-y_j|<\delta/\sqrt{m}}\big|\Omega_j(y_j)f_j(y_j)\big|\omega_j^{1/p_j}(y_j)\,dy_j \\ &\quad\le C\prod_{j=1}^m\|\Omega_j\|_{L^{p_j'}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}}\left(\log\frac{2R\sqrt{m}}{\delta}\right)^{1/p_m'} \\ &\quad\le C\prod_{j=1}^m\|\Omega_j\|_{L^{p_j'}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}}\left(\log\frac{2R\sqrt{m}}{\delta}\right)^{(mn-\alpha)/n}. \end{aligned}$$

Consider the case where exactly l of the r_j are ∞ for some $1\le l\le m$. Without losing the generalization, we only give the argument for $r_j = \infty$, $j = 1,2,\dots,l$, then

$$\begin{aligned} &\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\int_{(\mathbb{R}^n)^{m-1}} \frac{\prod_{j=1}^m\Omega_j(y_j)\prod_{j=1}^l|f_j^\infty(y_j)\prod_{k=l+1}^{m-1}f_k^0(y_k)f_m(y_m)|}{|(x-y_1,x-y_2,\dots,x-y_m)|^{mn-\alpha}}\nu_{\vec{\omega}}\,d\vec{y} \\ &\quad\le\prod_{k=l+1}^{m-1}\int_{|x-y_k|<\delta/\sqrt{m}}\big|\Omega_k(y_k)f_k(y_k)\big|\omega_k^{1/p_m}(y_k)\,dy_k \\ &\qquad\times\prod_{j=1}^l\int_{\delta/\sqrt{m}\le|x-y_j|\le 2R}\frac{|\Omega_j(y_j)f_j(y_j)|}{|x-y_j|^{n-\alpha_j}}\omega_j^{1/p_j}(y_j)\,dy_j \end{aligned}$$

$$\begin{aligned}&\times \int_{\delta/\sqrt{m}\le|x-y_m|\le 2R} \frac{|\Omega_m(y_m)f_m(y_m)|}{|x-y_m|^{(m-l)n-\sum_{k=l+1}^m \alpha_k}} \omega_m^{1/p_m}(y_m)\,dy_m \\ &\le C\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{\sum_{k=1}^l \frac{1}{p'_m}} \prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}} \\ &\le C\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}}\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{(mn-\alpha)/n}.\end{aligned}$$

Combining the above cases, we obtain

$$F_2 \le C_2\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}}\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{(mn-\alpha)/n}.$$

Thus, by the estimates for F_1, F_2, we have

$$\begin{aligned}\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x) &\le C_1\delta^\alpha \prod_{j=1}^m M_{\Omega_j}(f_j)(x) \\ &\quad + C_2\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}}\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{(mn-\alpha)/n}.\end{aligned}$$

In particular, we chose $\delta = 2\sqrt{m}R$ for all $x \in B$, then

$$\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x) \le C_1\delta^\alpha \prod_{j=1}^m M_{\Omega_j}(f_j)(x).$$

Now, we set

$$\delta = \delta(x) = \varepsilon\left[\left|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)\right| \Big/ C_1\prod_{j=1}^m M_{\Omega_j}(f_j)(x)\right]^{1/\alpha},$$

where $\varepsilon < 1$.

Then

$$\begin{aligned}&\left|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)\right| \\ &\quad\le \varepsilon^\alpha\left|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)\right| \\ &\qquad + C_2\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}}\left[\frac{1}{n}\log\left(\frac{(2\sqrt{m}R)^n[C_1\prod_{j=1}^m M_{\Omega_j}(f_j)(x)]^{n/\alpha}}{\varepsilon^n|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)|^{n/\alpha}}\right)\right]^{(mn-\alpha)/n}.\end{aligned}$$

Hence,

$$\exp\left(k_1\left(\frac{|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)|}{\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})}\|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/(mn-\alpha)}\right) \le \frac{C[\prod_{j=1}^m M_{\Omega_j}(f_j)(x)]^{n/\alpha}}{|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)|^{n/\alpha}}.$$

Let $B_1 = \{x \in B : \frac{|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)|}{\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})} \|f_j\|_{L^{p_j}}} \geq 1\}$ and $B_2 = B - B_1$, then

$$\begin{aligned}
&\int_{B_1} \exp\left(k_1\left(\frac{|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)|}{\prod_{j=1}^m \|\Omega_j\|_{L^{p_j'}(S^{n-1})} \|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/(mn-\alpha)}\right) \nu_{\vec{\omega}}\,dx \\
&\quad \leq CR^n \int_{B_1} \left(\frac{\prod_{j=1}^m M_{\Omega_j}(f_j)(x)}{\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})} \|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/\alpha} \nu_{\vec{\omega}}\,dx \\
&\quad \leq CR^n \left(\prod_{j=1}^m \frac{\|M_{\Omega_j}(f_j)\|_{L^{p_j}_{\omega_j}}}{\|\Omega_j\|_{L^{p'_j}(S^{n-1})} \|f_j\|_{L^{p_j}_{\omega_j}}}\right)^{n/\alpha} \\
&\quad \leq CR^n \\
&\quad \leq C \prod_j^m \omega_j(B).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{B_2} \exp\left(k_1\left(\frac{|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)|}{\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})} \|f_j\|_{L^{p_j}}}\right)^{n/(mn-\alpha)}\right) \nu_{\vec{\omega}}(x)\,dx \\
&\quad \leq \exp(k_1) \prod_{j=1}^m \int_{B_2} \omega_j(x)\,dx \\
&\quad \leq C \prod_{j=1}^m \omega_j(B).
\end{aligned}$$

Combining the above results, we obtain

$$\int_{B} \exp\left(k_1\left(\frac{|\bar{I}_{\Omega,\alpha}(f_1,f_2,\ldots,f_m)(x)|}{\prod_{j=1}^m \|\Omega_j\|_{L^{p'_j}(S^{n-1})} \|f_j\|_{L^{p_j}}}\right)^{n/(mn-\alpha)}\right) \nu_{\vec{\omega}}(x)\,dx \leq k_2 \prod_{j=1}^m \omega_j(B),$$

where k_1, k_2 are constants depending only on n, m, α, p, and the p_j. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References

1. Adams, D: A sharp inequality of J. Moser for higher order derivatives. Ann. Math. **128**, 385-398 (1988)
2. Cerny, R, Gurka, P, Hencl, S: Concentration compactness principle for generalized Trudinger inequalities. Z. Anal. Anwend. **30**(3), 355-375 (2011)
3. Carleson, L, Chang, S: On the existence of an extremal function for an inequality of J. Moser. Bull. Sci. Math. (2) **110**(2), 113-127 (1986)
4. Chang, S, Marshall, D: On a sharp inequality concerning the Dirichlet integral. Am. J. Math. **107**, 1015-1033 (1985)
5. Chen, X, Xue, Q: Weighted estimates for a class of multilinear fractional type operators. J. Math. Anal. Appl. **362**(2), 355-373 (2010)
6. Ding, Y, Lu, S: The $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}$ boundedness for some rough operators. J. Math. Anal. Appl. **203**, 166-186 (1996)
7. Grafakos, L: On multilinear fractional integrals. Stud. Math. **102**(1), 49-56 (1992)
8. Grafakos, L, Torres, RH: On multilinear singular integrals of Calderón-Zygmund type. In: Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorual). Publ. Mat. **Vol. Extra**, 57-91 (2002)
9. Hedberg, LI: On certain convolution inequalities. Proc. Am. Math. Soc. **36**, 505-510 (1972)
10. Hempel, J, Morris, G, Trudinger, N: On the sharpness of a limiting case of the Sobolev embedding theorem. Bull. Aust. Math. Soc. **3**, 369-373 (1970)
11. Kenig, C, Stein, E: Multilinear estimates and fractional integration. Math. Res. Lett. **6**, 1-15 (1999)
12. Lerner, AK, Ombrosi, S, Pérez, C, Torres, RH, Trujillo-González, R: New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. Adv. Math. **220**(4), 1222-1264 (2009)
13. Moen, K: Weighted inequalities for multilinear fractional integral operators. Collect. Math. **60**, 213-238 (2009)
14. Moser, J: A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. **20**, 1077-1092 (1971)
15. Muckenhoupt, B, Wheeden, R: Weighted norm inequalities for fractional integrals. Trans. Am. Math. Soc. **192**, 261-274 (1974)
16. Li, W, Xue, Q, Yabuta, K: Multilinear Calderón-Zygmund operators on weighted Hardy spaces. Stud. Math. **199**(1), 1-16 (2010)
17. Lu, G, Yang, Y: Sharp constant and extremal function for the improved Moser-Trudinger inequality involving L^p norm in two dimension. Discrete Contin. Dyn. Syst., Ser. A **25**, 963-979 (2009)
18. Ruf, B: A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$. J. Funct. Anal. **219**(2), 340-367 (2005)
19. Strichartz, RS: A note on Trudinger's extension of Sobolev's inequalities. Indiana Univ. Math. J. **21**, 841-842 (1972)
20. Tang, L: Endpoint estimates for multilinear fractional integrals. J. Aust. Math. Soc. **84**, 419-429 (2008)
21. Tian, G, Zhu, X: A nonlinear inequality of Moser-Trudinger type. Calc. Var. Partial Differ. Equ. **10**(4), 349-354 (2000)
22. Trudinger, N: On imbedding into Orlicz spaces and some applications. J. Math. Mech. **17**, 473-483 (1967)

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