Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Inequalities for a class of multivariate operators

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Abstract

This paper introduces and studies a class of generalized multivariate Bernstein operators defined on the simplex. By means of the modulus of continuity and so-called Ditzian-Totik's modulus of function, the direct and inverse inequalities for the operators approximating multivariate continuous functions are simultaneously established. From these inequalities, the characterization of approximation of the operators follows. The obtained results include the corresponding ones of the classical Bernstein operators.

MSC: 41A25; 41A36; 41A60; 41A63

Keywords: generalized Bernstein operators; direct and inverse inequalities; characterization of approximation

1 Introduction

Let $\mathbb{N}$ be the set of natural numbers, and $\{s_n\}_{n=1}^{\infty}$ ($s_n \geq 1$, $s_n \in \mathbb{N}$) be a sequence. In [3], Cao introduced the following generalized Bernstein operators defined on $[0,1]$:

$$(\mathcal{L}_n f)(x) := \frac{1}{s_n}\sum_{k=0}^{n}\left(\sum_{j=0}^{s_n-1} f\left(\frac{k+j}{n+s_n-1}\right)\right)P_{n,k}(x), \tag{1}$$

where $x \in [0,1]$, $f \in C[0,1]$, and

$$P_{n,k}(x) := \binom{n}{k} x^k(1-x)^{n-k}. \tag{2}$$

Clearly, when $s_n = 1$, $\mathcal{L}_n f$ reduce to the classical Bernstein operators, $\mathcal{B}_n f$, given by

$$(\mathcal{B}_n f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)P_{n,k}(x). \tag{3}$$

Furthermore, Cao [3] proved that the necessary and sufficient condition of convergence for the operators is $\lim_{n\to\infty}(s_n/n) = 0$, and he also proved that for $n \in Q = \{n : n \in \mathbb{N}, \text{ and } 0 < (s_n - 1)/n + 1/\sqrt{n} \leq 1\}$ the following estimate of approximation degree holds:

$$\|\mathcal{L}_n f - f\| \leq 4\omega\left(f, \frac{s_n-1}{n} + \frac{1}{\sqrt{n}}\right). \tag{4}$$

Here, $\omega(f,t)$ is the modulus of continuity of first order of the function f. In [4], some approximation properties for the operators were further investigated.

In this paper, we will introduce and study the multivariate version defined on the simplex of the generalized Bernstein operators given by (1). The main aim is to establish the direct and inverse inequalities of approximation, which will imply the characterization of approximation of the operators.

For convenience, we denote by bold letter the vector in $\mathbb{R}^d$. Let

$$\mathbf{e}_i := (0,0,\dots,0,1,0,\dots,0)$$

denote the canonical unit vector in $\mathbb{R}^d$, *i.e.*, its ith component is 1 and the others are 0, and let

$$T := T_d := \left\{\mathbf{x} = (x_1,x_2,\dots,x_d) \in \mathbb{R}^d : x_i \geq 0, i = 1,2,\dots,d, \sum_{i=1}^{d} x_i \leq 1\right\} \tag{5}$$

be the simplex in $\mathbb{R}^d$. For $\mathbf{x} \in T$, $\mathbf{k} := (k_1,k_2,\dots,k_d) \in \mathbb{N}_0^d$, we denote as usual

$$|\mathbf{x}| := \sum_{i=1}^{d} x_i, \qquad \mathbf{x}^{\mathbf{k}} := x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \qquad |\mathbf{k}| := \sum_{i=1}^{d} k_i, \qquad \mathbf{k}! := k_1! k_2! \cdots k_d!. \tag{6}$$

Then the well-known Bernstein basis function on T is given by

$$P_{n,\mathbf{k}}(\mathbf{x}) := \frac{n!}{\mathbf{k}!(n-|\mathbf{k}|)!} \mathbf{x}^{\mathbf{k}} \bigl(1-|\mathbf{x}|\bigr)^{n-|\mathbf{k}|}, \quad \mathbf{x} \in T. \tag{7}$$

By means of the basis function, we define the multivariate generalized Bernstein operators on the simplex T as

$$(\mathcal{L}_{n,d}f)(\mathbf{x}) := \frac{1}{s_n} \sum_{|\mathbf{k}| \leq n} P_{n,\mathbf{k}}(\mathbf{x}) \left(\sum_{|\mathbf{j}| \leq s_n - 1} f\left(\frac{\mathbf{k}+\mathbf{j}}{n+s_n-1}\right)\right). \tag{8}$$

Obviously, when $d = 1$, these operators reduce to the univariate operators defined by (1), and when $s_n = 1$ they are just the well-known multivariate Bernstein operators on the simplex T, $\mathcal{B}_{n,d}$, defined by

$$(\mathcal{B}_{n,d}f)(\mathbf{x}) := \sum_{|\mathbf{k}| \leq n} P_{n,\mathbf{k}}(\mathbf{x}) f\left(\frac{\mathbf{k}}{n}\right). \tag{9}$$

Let $C(T)$ denote the space of continuous functions on T with the norm defined by $\|f\| := \max_{x \in T} |f(x)|, f \in C(T)$. For arbitrary vector $\mathbf{e} \in \mathbb{R}^d$, we write for the rth symmetric difference of a function f in the direction of $\mathbf{e}$

$$\Delta_{h\mathbf{e}}^r f(\mathbf{x}) := \begin{cases} \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(\mathbf{x} + (\frac{r}{2} - i)h\mathbf{e}), & \mathbf{x} \pm \frac{rh\mathbf{e}}{2} \in T, \\ 0, & \text{otherwise}. \end{cases}$$

Then the Ditzian-Totik's modulus of function $f \in C(T)$ is defined by (see [1])

$$\omega_\varphi^r(f,t) := \sup_{0<h\le t} \sum_{1\le i\le j\le d} \left\| \Delta_{h\varphi_{ij}\mathbf{e}_{ij}}^r f \right\|,$$

where the weighted functions

$$\varphi_{ii}(\mathbf{x}) := \sqrt{x_i\bigl(1-|\mathbf{x}|\bigr)}, \quad 1 \le i \le d; \qquad \varphi_{ij}(\mathbf{x}) := \sqrt{x_i x_j}, \quad 1 \le i < j \le d,$$

and

$$\mathbf{e}_{ii} := \mathbf{e}_i, \quad 1 \le i \le d; \qquad \mathbf{e}_{ij} := \mathbf{e}_i - \mathbf{e}_j, \quad 1 \le i < j \le d.$$

Define differential operators:

$$D_i := D_{ii} := \frac{\partial}{\partial x_i}, \quad 1 \le i \le d; \qquad D_{ij} := D_i - D_j, \quad 1 \le i < j \le d;$$
$$D_{ij}^r := D_{ij}\bigl(D_{ij}^{r-1}\bigr), \quad 1 \le i \le j \le d, r \in N,$$

then the weighted Sobolev space can be defined by

$$D_r(T_d) := \bigl\{ g \in C(T) : g \in C^r(\mathring{T}), \text{ and } \varphi_{ij}^r D_{ij}^r g \in C(T), 1 \le i \le j \le d \bigr\},$$

where $\mathring{T}$ is inner of T, and the Peetre K-functional on $C(T)$ is given by

$$\mathcal{K}_\varphi^r\bigl(f,t^r\bigr) := \inf_{g\in D_r(T_d)} \left\{ \|f-g\| + t^r \sum_{1\le i\le j\le d} \left\| \varphi_{ij}^2 D_{ij}^2 f \right\| \right\}, \quad t > 0.$$

Berens and Xu [1] proved that $\mathcal{K}_\varphi^r(f,t^r)$ is equivalent to $\omega_\varphi^r(f,t)$, *i.e.*,

$$C^{-1}\omega_\varphi^r(f,t) \le \mathcal{K}_\varphi^r\bigl(f,t^r\bigr) \le C\omega_\varphi^r(f,t), \tag{10}$$

here and in the following C denotes a positive constant independent of f and n, but its value may be different at a different occurrence.

We also need the usual modulus of continuity of function $f \in C(T)$ defined by (see [8])

$$\omega(f,t) := \sup_{0<|\mathbf{h}|_2\le t} \bigl\| f(\cdot+\mathbf{h}) - f(\cdot) \bigr\|,$$

where $\mathbf{h} = (h_1, h_2, \ldots, h_d) \in \mathbb{R}^d$ and $|\mathbf{h}|_2 := (\sum_{i=1}^d h_i^2)^{1/2}$, and another $\mathcal{K}$-functional given by (see [8])

$$\mathcal{K}(f,t) := \inf_{g\in C^1(T)} \left\{ \|f-g\| + t\sum_{i=1}^d \|D_i g\| \right\}.$$

It is shown in [8] that

$$C^{-1}\omega(f,t) \le \mathcal{K}(f,t) \le C\omega(f,t). \tag{11}$$

Now we state the main results of this paper as follows.

Theorem 1.1 *Let $f \in C(T)$, then for $n \in Q = \{n : n \in \mathbb{N}, \text{ and } 0 < \frac{s_n-1}{n} + \frac{1}{\sqrt{n}} \le 1\}$, there holds*

$$\|\mathcal{L}_{n,d}f - f\| \le 4d\omega\left(f, \frac{s_n-1}{n} + \frac{1}{\sqrt{n}}\right).$$

Theorem 1.2 *If $f \in C(T)$ and $\lim_{n\to\infty}(s_n/n) = 0$, then*

$$\|\mathcal{L}_{n,d}f - f\| \le C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \omega\left(f, \frac{s_n-1}{n+s_n-1}\right) + \frac{1}{n}\|f\|\right).$$

Theorem 1.3 *If $f \in C(T)$ and $\lim_{n\to\infty}(s_n/n) = 0$, then there hold*

$$\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) \le Cn^{-1}\sum_{k=1}^{n}\|\mathcal{L}_{k,d}f - f\|$$

and

$$\omega\left(f, \frac{1}{n}\right) \le Cn^{-1}\left(\sum_{k=1}^{n}\|\mathcal{L}_{k,d}f - f\| + \|f\|\right).$$

From Theorem 1.2 and Theorem 1.3, we easily obtain the following corollaries, which characterize the approximation feature of the multivariate operators $\mathcal{L}_{n,d}$ given by (8).

Corollary 1.1 *Let $f \in C(T)$, $0 < \alpha < 1$. Then, for the Bernstein operators given by (9), the necessary and sufficient condition for which*

$$\|\mathcal{B}_{n,d}f - f\| = \mathcal{O}\left(\frac{1}{n^\alpha}\right), \quad n \to \infty$$

is $\omega_\varphi^2(f,t) = \mathcal{O}(t^{2\alpha})$ ($t \to 0$).

Corollary 1.2 *If $f \in C(T)$, $0 < \alpha < 1$, $s_n > 1$ and $\lim_{n\to\infty}(s_n/n) = 0$, then $\omega_\varphi^2(f,t) = \mathcal{O}(t^{2\alpha})$ and $\omega(f,t) = \mathcal{O}(t^\alpha)(t \to 0)$ imply*

$$\|\mathcal{L}_{n,d}f - f\| = \mathcal{O}\left(\left(\frac{s_n}{n+s_n-1}\right)^\alpha\right), \quad n \to \infty.$$

Corollary 1.3 *If $s_n > 1$ and $s_n = \mathcal{O}(n^{1-\epsilon})$ ($n \to \infty$), $0 < \epsilon \le 1$, then for any $f \in C(T)$ and $0 < \alpha < 1$, the statement*

$$\|\mathcal{L}_{n,d}f - f\| = \mathcal{O}\left(n^{-\epsilon\alpha}\right), \quad n \to \infty$$

implies that $\omega_\varphi^2(f,t) = \mathcal{O}(t^{2\epsilon\alpha})$ and $\omega(f,t) = \mathcal{O}(t^{\epsilon\alpha})$ ($t \to 0$).

From Corollary 1.2 and Corollary 1.3, we have the following.

Corollary 1.4 *Let $0 < \alpha < 1$ and $1 < s_n = \mathcal{O}(1)$ ($n \to \infty$), then, for any $f \in C(T)$ the necessary and sufficient condition for which*

$$\|\mathcal{L}_{n,d}f - f\| = \mathcal{O}\left(\frac{1}{n^\alpha}\right), \quad n \to \infty$$

is $\omega_\varphi^2(f,t) = \mathcal{O}(t^{2\alpha})$ and $\omega(f,t) = \mathcal{O}(t^{\epsilon\alpha})$ ($t \to 0$).

2 Some lemmas

In this section, we prove some lemmas.

Defining the transformation $\mathcal{T}_i$ ($i = 1, 2, \dots, d$) from T to itself, *i.e.*,

$$\mathcal{T}_i(\mathbf{x}) := \mathbf{u}, \quad \mathbf{u} = \bigl(x_1, \dots, x_{i-1}, 1 - |\mathbf{x}|, x_{i+1}, \dots, x_d\bigr), \quad \mathbf{x} \in T,$$

we have the following symmetric property for the operators $\mathcal{L}_{n,d}$, which is similar to the known one of the multivariate Bernstein operators (see [6, 7]).

Lemma 2.1 *For the above transformation $\mathcal{T}_i$, $i = 1, 2, \dots, d$, there holds*

$$(\mathcal{L}_{n,d}f)(\mathbf{x}) = (\mathcal{L}_{n,d}f_i)(\mathbf{u}),$$

where $f_i(\mathbf{x}) = f(\mathcal{T}_i(\mathbf{x}))$, $\mathbf{u} = \mathcal{T}_i(\mathbf{x})$.

Proof It is sufficient to prove the case $i = 1$. Let

$$\begin{aligned}
&\mathbf{l} = (l_1, l_2, \dots, l_d), \qquad \mathbf{l}^* = (l_2, \dots, l_d), \qquad l_1 = n - |\mathbf{k}|, \qquad l_i = k_i, \quad i = 2, 3, \dots, d, \\
&\mathbf{t} = (t_1, t_2, \dots, t_d), \qquad \mathbf{t}^* = (t_2, \dots, t_d), \qquad t_1 = s_n - 1 - |\mathbf{j}|, \qquad t_i = j_i, \quad i = 2, 3, \dots, d,
\end{aligned}$$

and $\mathbf{x} = (x_1, \mathbf{x}^*) \in T$, $\mathbf{x}^* = (x_2, x_3, \dots, x_d)$. Then, from definition (8), it follows that

$$\begin{aligned}
(\mathcal{L}_{n,d}f)(\mathbf{x}) &= \frac{1}{s_n} \sum_{|\mathbf{k}| \le n} P_{n,\mathbf{k}}(\mathbf{x}) \left\{ \sum_{|\mathbf{j}| \le s_n - 1} f\left(\frac{\mathbf{k} + \mathbf{j}}{n + s_n - 1}\right) \right\} \\
&= \frac{1}{s_n} \sum_{|\mathbf{l}| \le n} \frac{n!}{(n - |\mathbf{l}|)! \mathbf{l}^*! l_1!} x_1^{n - |\mathbf{l}|} \bigl(\mathbf{x}^*\bigr)^{\mathbf{l}^*} \bigl(1 - |\mathbf{x}|\bigr)^{l_1} \\
&\quad \times \left\{ \sum_{|\mathbf{t}| \le s_n - 1} f\left(\frac{n - |\mathbf{l}| - |\mathbf{t}|}{n + s_n - 1}, \frac{\mathbf{l}^* + \mathbf{t}^*}{n + s_n - 1}\right) \right\} \\
&= \frac{1}{s_n} \sum_{|\mathbf{l}| \le n} P_{n,\mathbf{l}}\bigl(1 - |\mathbf{x}|, \mathbf{x}^*\bigr) \left\{ \sum_{|t| \le s_n - 1} f\left(1 - \frac{|\mathbf{l}| + |\mathbf{t}|}{n + s_n - 1}, \frac{\mathbf{l}^* + \mathbf{t}^*}{n + s_n - 1}\right) \right\} \\
&= (\mathcal{L}_{n,d}f_1)(\mathbf{u}).
\end{aligned}$$

The proof of Lemma 2.1 is completed. □

To prove Theorem 1.3, we need some the following lemmas. At first, similar to the estimates for the Bernstein operators (see [2, 5, 6]), it is not difficult to derive the following Lemma 2.2.

Lemma 2.2 *The following inequalities hold:*

$$\|D_i(\mathcal{L}_{n,d}f)\| \le \begin{cases} 2n\|f\|, & f \in C(T), \\ \|D_if\|, & f \in C^1(T), \end{cases} \qquad 1 \le i \le d;$$

$$\|D_i^2(\mathcal{L}_{n,d}f)\| \le \begin{cases} 4n^2\|f\|, & f \in C(T), \\ \|D_i^2f\|, & f \in C^2(T), \end{cases} \qquad 1 \le i \le d.$$

Secondly, we need prove two Bernstein type inequalities.

Lemma 2.3 *Let $f \in C(T)$, $1 \le i \le j \le d$. Then*

$$\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\| \le 2n\|f\|.$$

Proof For $d = 1$, by direct computation we have (see [9])

$$(\mathcal{L}_nf)''(x) = \frac{1}{s_n}\varphi^{-4}(x)n^2\sum_{k=1}^{n} r_{n,k}(x)P_{n.k}(x)\left(\sum_{j=0}^{s_n-1} f\left(\frac{k+j}{n+s_n-1}\right)\right),$$

where

$$r_{n,k}(x) = \left(\frac{k}{n}-x\right)^2 - (1-2x)\frac{k}{n^2} - \frac{x^2}{n}.$$

Noting that

$$|r_{n,k}(x)| \le \left(\frac{k}{n}-x\right)^2 + (1-2x)\frac{k}{n^2} + \frac{x^2}{n},$$

we obtain

$$|\varphi^2(x)(\mathcal{L}_nf)''(x)| \le \|f\|\varphi^{-2}(x)n^2\sum_{k=0}^{n}|r_{n,k}(x)|P_{n,k}(x) \le 2n\|f\|.$$

This inequality shows that Lemma 2.3 is valid for $d = 1$. For the proof of the case $d > 1$, we use a decomposition technique and the induction. In fact, let

$$g_{k_1,j_1}(\mathbf{u}) := f\left(\frac{k_1+j_1}{n+s_n-1}, \left(1-\frac{k_1+j_1}{n+s_n-1}\right)\mathbf{u}\right), \qquad \mathbf{u} = (u_1, u_2, \dots, u_{d-1})$$

and

$$\mathbf{z} := (z_1, z_2, \dots, z_{d-1}) := \left(\frac{x_2}{1-x_1}, \frac{x_2}{1-x_1}, \dots, \frac{x_d}{1-x_1}\right),$$

$$\mathbf{k}^* := (k_2, k_3, \dots, k_d), \qquad |\mathbf{k}^*| := \sum_{i=2}^{d} k_i,$$

$$\mathbf{j}^* := (j_2, j_3, \dots, j_d), \qquad |\mathbf{j}^*| := \sum_{i=2}^{d} j_i,$$

then we can decompose the generalized Bernstein operators as

$$\begin{aligned}(\mathcal{L}_{n,d}f)(\mathbf{x}) &= \frac{1}{s_n}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)\sum_{|\mathbf{k}^*|\le n-k_1} P_{n,n-k_1}(\mathbf{z}) \\ &\quad\times\left(\sum_{j_1=0}^{s_n-1}\sum_{|\mathbf{j}^*|\le s_n-1-j_1} f\left(\frac{\mathbf{k}+\mathbf{j}}{n+s_n-1}\right)\right) \\ &= \sum_{k_1=0}^{n} P_{n,k_1}(x_1)\left(\sum_{j_1=0}^{s_n-1}\frac{s_n-j_1}{s_n}(\mathcal{L}_{n-k_1,d-1}g_{k_1,j_1})(\mathbf{z})\right).\end{aligned}$$

Therefore,

$$\varphi_{22}^2(\mathbf{x})D_{22}^2(\mathcal{L}_{n,d}f)(\mathbf{x}) = \sum_{k_1=0}^{n} P_{n,k_1}(x_1)\left(\sum_{j_1=0}^{s_n-1}\frac{s_n-j_1}{s_n}\varphi_{11}^2(\mathbf{z})D_{11}^2(\mathcal{L}_{n-k_1,d-1}g_{k_1,j_1})(\mathbf{z})\right). \tag{12}$$

Now, suppose that Lemma 2.3 is valid for $d-1$, then from (12) it follows that

$$\left|\varphi_{22}^2(\mathbf{x})D_{22}^2(\mathcal{L}_{n,d}f)(\mathbf{x})\right| \le 2\sum_{k_1=0}^{n} P_{n,k_1}(x_1)\left(\sum_{j_1=0}^{s_n-1}\frac{s_n-j_1}{s_n}(n-k_1)\left\|g_{k_1,j_1}(\cdot)\right\|\right) \le 2n\|f\|.$$

So, Lemma 2.3 is true for $i=2$. From the symmetry, the proof of the cases $i=1,3,4,\ldots,d$ is the same. For the cases $1\le i<j\le d$, we use Lemma 2.1 and obtain that

$$\left\|\varphi_{ij}^2D_{ij}^2(\mathcal{L}_{n,d}f)\right\| = \left\|\varphi_{ii}^2D_{ii}^2(\mathcal{L}_{n,d}f_j)\right\| \le 2n\|f_j\| \le 2n\|f\|.$$

So, the proof of Lemma 2.3 is complete. □

Lemma 2.4 *For $f\in C^2(T)$, $1\le i\le j\le d$, one has*

$$\left\|\varphi_{ij}^2D_{ij}^2(\mathcal{L}_{n,d}f)\right\| \le \left\|\varphi_{ij}^2D_{ij}^2f\right\| + \frac{1}{n}\left\|D_{ij}^2f\right\|.$$

Proof We only need to prove the case $s_n>1$ because the case $s_n=1$ has been shown in [1]. Our approach is based on the induction. At first, for $d=1$, let $h=(n+s_n-1)^{-1}$, then by simple calculation we have

$$(\mathcal{L}_nf)''(x) = \frac{n(n-1)}{s_n}\sum_{k=0}^{n-2}\left(\sum_{j=0}^{s_n-1}\Delta_h^2f\left(\frac{k+j+1}{n+s_n-1}\right)\right)P_{n-2,k}(x).$$

Therefore,

$$\begin{aligned}&\left|\varphi^2(x)(\mathcal{L}_nf)''(x)\right| \\ &\quad= \left|\frac{1}{s_n}\sum_{k=0}^{n-2}(k+1)(n-k-1)\left(\sum_{j=0}^{s_n-1}\Delta_h^2f\left(\frac{k+j+1}{n+s_n-1}\right)\right)P_{n,k+1}(x)\right| \\ &\quad= \left|\frac{1}{h^2s_n}\sum_{k=1}^{n-1}\sum_{j=0}^{s_n-1}\frac{k}{n+s_n-1}\frac{n-k}{n+s_n-1}\Delta_h^2f\left(\frac{k+j}{n+s_n-1}\right)P_{n,k}(x)\right|\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{h^2 s_n}\sum_{k=1}^{n-1}\sum_{j=0}^{s_n-1}\varphi^2\left(\frac{k+j}{n+s_n-1}\right)\left|\Delta_h^2 f\left(\frac{k+j}{n+s_n-1}\right)\right|P_{n,k}(x) \\
&= \frac{1}{h^2 s_n}\sum_{k=1}^{n-1}\sum_{j=0}^{s_n-1}\varphi^2\left(\frac{k+j}{n+s_n-1}\right)P_{n,k}(x) \\
&\quad\times\left|\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} f''\left(\frac{k+j}{n+s_n-1}+s+t\right)ds\,dt\right|.
\end{aligned}$$

Let $y = (k+j)/(n+s_n-1)$, then we have for $1 \le k \le n-1$, $0 \le j \le s_n - 1$,

$$h = \frac{1}{n+s_n-1} \le y \le 1 - \frac{1}{n+s_n-1} = 1-h$$

and for $|u| \le h$, there holds $|1-2y-u| \le 1$. Hence,

$$\varphi^2(y) = \varphi^2(y+u) - u(1-2y-u) \le \varphi^2(y+u) + |u| \le \varphi^2(y+u) + h,$$

which implies

$$\begin{aligned}
\varphi^2(y)\left|\Delta_h^2(y)\right| &\le \varphi^2(y)\left|\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} f''(y+s+t)\,ds\,dt\right| \\
&\le \int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\left(\varphi^2(y+s+t)+h\right)\left|f''(y+s+t)\right|ds\,dt \\
&\le h^2\left(\left\|\varphi^2 f''\right\| + h\left\|f''\right\|\right).
\end{aligned}$$

So,

$$\left\|\varphi^2(\mathcal{L}_n f)''\right\| \le \left\|\varphi^2 f''\right\| + \frac{1}{n}\left\|f''\right\|.$$

Now, assume that Lemma 2.4 is valid for $d-1$, then by (12)

$$\begin{aligned}
&\left|\varphi_{22}^2(\mathbf{x})D_{22}^2(\mathcal{L}_{n,d}f)(\mathbf{x})\right| \\
&\quad\le \sum_{k_1=0}^{n} P_{n,k_1}(x_1)\left(\sum_{j_1=0}^{s_n-1}\frac{s_n-j_1}{s_n}\left(\left\|\varphi_{11}^2 D_{11}^2 g_{k_1,j_1}\right\| + \frac{1}{n-k_1+s_n-j_1-1}\left\|D_{11}^2 g_{k_1,j_1}\right\|\right)\right).
\end{aligned}$$

Also, we can check the following inequalities:

$$\begin{aligned}
\left|D_{11}^2 g_{k_1,j_1}(\mathbf{z})\right| &= \left|\left(1-\frac{k_1+j_1}{n+s_n-1}\right)^2 D_{22}^2 f\left(\frac{k_1+j_1}{n+s_n-1},\left(1-\frac{k_1+j_1}{n+s_n-1}\right)\mathbf{z}\right)\right| \\
&\le \left(1-\frac{k_1+j_1}{n+s_n-1}\right)^2\left\|D_{22}^2 f\right\|,
\end{aligned}$$

and

$$\left|\varphi_{11}^2(\mathbf{z})D_{11}^2 g_{k_1,j_1}(\mathbf{z})\right| = \left|\left(\varphi_{22}^2 D_{22}^2 f\right)\left(\frac{k_1+j_1}{n+s_n-1},\left(1-\frac{k_1+j_1}{n+s_n-1}\right)\mathbf{z}\right)\right| \le \left\|\varphi_{22}^2 D_{22}^2 f\right\|.$$

Thus,

$$\left\|\varphi_{22}^2 D_{22}^2(\mathcal{L}_{n,d}f)\right\| \le \left\|\varphi_{22}^2 D_{22}^2 f\right\| + \frac{1}{n}\left\|D_{22}^2 f\right\|.$$

Similarly, the cases $i = 1, 3, 4, \ldots, d$ can be proved. For the case $1 \le i \le j \le d$, we use the transformation $\mathcal{T}_i$ and Lemma 2.1, it is easy to verify

$$\begin{aligned}\left\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\right\| &= \left\|\varphi_{ii}^2 D_{ii}^2(\mathcal{L}_{n,d}f_j)\right\| \\ &\le \left\|\varphi_{ii}^2 D_{ii}^2 f_j\right\| + n^{-1}\left\|D_{ii}^2 f_j\right\| \\ &= \left\|\varphi_{ij}^2 D_{ij}^2 f\right\| + n^{-1}\left\|D_{ij}^2 f\right\|.\end{aligned}$$

Hence, the proof of Lemma 2.4 is complete. □

We also need the following two interesting results related to nonnegative numerical sequence. The proof of the first result can be found in [10], and the proof of the other is similar to Lemma 2.1 of [10] where the proof of case $\nu_1 = 0$ and $C = 1$ was given.

Lemma 2.5 *Let μ_n, ν_n, and ψ_n are all nonnegative numerical sequence, and $\mu_1 = \nu_1 = 0$. If for $0 < r < s$ and $1 \le k \le n$, $n \in \mathbb{N}$, there holds*

$$\mu_n \le \left(\frac{k}{n}\right)^r \mu_k + \nu_k + \psi_k, \qquad \nu_n \le \left(\frac{k}{n}\right)^s \nu_k + \psi_k,$$

then

$$\mu_n \le Cn^{-r}\sum_{k=1}^{n} k^{r-1}\psi_k.$$

Lemma 2.6 *Let ν_n and ψ_n are all nonnegative numerical sequence. If for $s > 0$ and $1 \le k \le n$, $n \in \mathbb{N}$, there holds $\nu_n \le (\frac{k}{n})^s \nu_k + C\psi_k$, then*

$$\nu_n \le Cn^{-s}\left(\sum_{k=1}^{n} k^{s-1}\psi_k + \nu_1\right).$$

3 The proof of main results

First, we prove Theorem 1.1. By straight calculation, we have (see also [3])

$$\begin{aligned}\big(\mathcal{L}_{n,d}(u_i - x_i)\big)(\mathbf{x}) &= \big(\mathcal{L}_n(u_i - x_i)\big)(x_i) = \frac{s_n - 1}{n + s_n - 1}(2x_i - 1), \\ \big(\mathcal{L}_{n,d}\big(|\mathbf{u} - \mathbf{x}|_2^2\big)\big)(\mathbf{x}) &= \sum_{i=1}^{d}\big(\mathcal{L}_n(u_i - x_i)^2\big)(x_i) \\ &= \sum_{i=1}^{d}\frac{x_i(1 - x_i)(n - (s_n - 1)^2)}{(n + s_n - 1)^2} + \frac{(s_n - 1)(2(s_n - 1) + 1)}{6(n + s_n - 1)^2} \\ &\le \sum_{i=1}^{d}\frac{4}{3}\left(\frac{(s_n - 1)^2}{n^2} + \frac{1}{n}\right).\end{aligned}$$

Then we use the same method as Theorem 2 of [3] and obtain easily

$$\|\mathcal{L}_{n,d}f - f\| \le 4d\omega\left(f, \frac{s_n - 1}{n} + \frac{1}{\sqrt{n}}\right).$$

We now prove Theorem 1.2. We use a known estimation on Bernstein operators (see [1]) as an intermediate step to deduce the direct theorem. Since

$$\begin{aligned}\frac{1}{s_n}\sum_{|\mathbf{j}|\le s_n-1}\left|f\left(\frac{\mathbf{k}}{n}\right) - f\left(\frac{\mathbf{k}+\mathbf{j}}{n+s_n-1}\right)\right| &\le \frac{1}{s_n}\sum_{|\mathbf{j}|\le s_n-1}\omega\left(f, \frac{|\mathbf{k}(s_n-1) - n\mathbf{j}|}{n(n+s_n-1)}\right)\\ &\le \omega\left(f, \frac{s_n-1}{n+s_n+1}\right), \quad |\mathbf{k}| \le n\end{aligned}$$

from the fact that (see [1])

$$\|\mathcal{B}_{n,d}f - f\| \le C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\|f\|\right),$$

we get

$$\begin{aligned}\|\mathcal{L}_{n,d}f - f\| &\le \|\mathcal{B}_{n,d}f - f\| + \|\mathcal{B}_{n,d}f - \mathcal{L}_{n,d}f\|\\ &\le C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\|f\|\right)\\ &\quad + \max_{\mathbf{x}\in T}\frac{1}{s_n}\sum_{k=0}^{n}\sum_{j=0}^{s_n-1}\left|f\left(\frac{k}{n}\right) - f\left(\frac{k+j}{n+s_n-1}\right)\right|P_{n,k}(\mathbf{x})\\ &\le C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \omega\left(f, \frac{s_n-1}{n+s_n+1}\right) + \frac{1}{n}\|f\|\right).\end{aligned}$$

This completes the proof of Theorem 1.2.

Finally, we prove Theorem 1.3. Let

$$\begin{aligned}\mu_n &= n^{-1}\left\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\right\|, \quad 1 \le i \le j \le d,\\ \nu_n &= n^{-2}\left\|D_{ij}^2(\mathcal{L}_{n,d}f)\right\|, \quad 1 \le i \le j \le d\end{aligned}$$

and $\psi_n = 4\|(\mathcal{L}_{n,d}f) - f\|$, then $\mu_1 = \nu_1 = 0$ and from Lemma 2.2, Lemma 2.3, and Lemma 2.4, we have for $1 \le k \le n$

$$\begin{aligned}\mu_n &\le n^{-1}\left\|\varphi_{ij}^2 D_{ij}^2\big(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f)\big)\right\| + n^{-1}\left\|\varphi_{ij}^2 D_{ij}^2\big(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f - f)\big)\right\|\\ &\le n^{-1}\left\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{k,d}f)\right\| + \frac{1}{n^2}\left\|D_{ij}^2(\mathcal{L}_{k,d}f)\right\| + 2\|\mathcal{L}_{k,d}f - f\|\\ &\le \left(\frac{k}{n}\right)\mu_k + \nu_k + \psi_k,\\ \nu_n &\le n^{-2}\left\|D_{ij}^2\big(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f)\big)\right\| + n^{-2}\left\|D_{ij}^2\big(\mathcal{L}_{n,d}(\mathcal{L}_{k,d}f - f)\big)\right\|\\ &\le n^{-2}\left\|D_{ij}^2(\mathcal{L}_{k,d}f)\right\| + 4\|\mathcal{L}_{k,d}f - f\|\\ &= \left(\frac{k}{n}\right)^2\nu_k + \psi_k,\end{aligned}$$

which implies from Lemma 2.5 that $\mu_n \le Cn^{-1}\sum_{k=1}^{n}\psi_k$, *i.e.*,

$$\left\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{n,d}f)\right\| \le C\sum_{k=1}^{n}\|\mathcal{L}_{k,d}f-f\|, \quad 1\le i\le j\le d. \tag{13}$$

Let $\nu_n = \|D_i(\mathcal{L}_{n,d}f)\|$ and ψ_n be the same as the above, then we have by Lemma 2.2

$$\nu_n \le \left(\frac{k}{n}\right)\nu_k + \psi_k, \quad 1\le k\le n.$$

So, using Lemma 2.6 gives $\nu_n \le Cn^{-1}(\sum_{k=1}^{n}\psi_k + \nu_1)$, namely,

$$\left\|D_i(\mathcal{L}_{n,d}f)\right\| \le C\left(\sum_{k=0}^{n}\|\mathcal{L}_{k,d}f-f\| + \left\|D_i(\mathcal{L}_{1,d}f)\right\|\right) \le C\left(\sum_{k=0}^{n}\|\mathcal{L}_{k,d}f-f\| + \|f\|\right). \tag{14}$$

For $n\ge 2$, there is an $m\in\mathbb{N}$, such that $n/2\le m\le n$ and $\|\mathcal{L}_{m,d}f-f\| \le \|\mathcal{L}_{k,d}f-f\|$ hold for $1\le k\le n$. Then

$$\|\mathcal{L}_{m,d}f-f\| \le \frac{4}{n}\sum_{k=n/2}^{n}\|\mathcal{L}_{k,d}f-f\|. \tag{15}$$

So, combining (10), (13), and (15) we see

$$\begin{aligned}\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right) &\le C\mathcal{K}_\varphi^2\left(f,\frac{1}{n}\right) \le C\left(\|\mathcal{L}_{m,d}f-f\| + n^{-1}\left\|\varphi_{ij}^2 D_{ij}^2(\mathcal{L}_{m,d}f)\right\|\right)\\ &\le Cn^{-1}\sum_{k=1}^{n}\|\mathcal{L}_{k,d}f-f\|.\end{aligned}$$

Also, collecting (11), (13), and (15) implies

$$\begin{aligned}\omega\left(f,\frac{1}{n}\right) &\le C\mathcal{K}\left(f,\frac{1}{n}\right) \le C\left(\|\mathcal{L}_{m,d}f-f\| + n^{-1}\sum_{1\le i\le d}\left\|D_i(\mathcal{L}_{m,d}f)\right\|\right)\\ &\le Cn^{-1}\sum_{k=1}^{n}\left(\|\mathcal{L}_{k,d}f-f\| + \|f\|\right).\end{aligned}$$

The proof of Theorem 1.3 is complete.

Competing interests
The authors declare that they have no competing interests.

Acknowledgements
This research was supported by the National Nature Science Foundation of China (No. 61101240, 61272023), the Zhejiang Provincial Natural Science Foundation of China (No. Y6110117), and the Science Foundation of the Zhejiang Education Office (No. Y201122002).

Received: 22 January 2012 Accepted: 1 August 2012 Published: 7 August 2012

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doi:10.1186/1029-242X-2012-175
Cite this article as: Zhao: **Inequalities for a class of multivariate operators.** *Journal of Inequalities and Applications* 2012 **2012**:175.