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Inclusion results for certain subclasses of p-valent meromorphic functions associated with a new operator

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Abstract
In this paper, we introduce new subclasses of p-valent starlike, p-valent convex, p-valent close-to-convex, and p-valent quasi-convex meromorphic functions and investigate some inclusion properties of these subclasses and investigate various inclusion properties and integral-preserving properties for the p-valent meromorphic function classes.

MSC: 30C45

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1 Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in $U = U^* \cup \{0\}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$ $(z \in U)$. Furthermore, if $g(z)$ is univalent in U, then the following equivalence relationship holds true (see [3] and [8]):

$$f(z) \prec g(z) \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For functions $f(z) \in \Sigma_p$, given by (1.1) and $g(z) \in \Sigma_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.3}$$

Aqlan *et al.* [1] defined the operator $Q^{\alpha}_{\beta,p}:\Sigma_p\to\Sigma_p$ by:

$$Q^{\alpha}_{\beta,p}f(z)=\begin{cases} z^{-p}+\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)}a_{k-p}z^{k-p} & (\alpha>0;\beta>-1;p\in\mathbb{N};f\in\Sigma_p),\\ f(z) & (\alpha=0;\beta>-1;p\in\mathbb{N};f\in\Sigma_p).\end{cases} \tag{1.4}$$

Now, we define the operator $H^{\alpha}_{p,\beta,\mu}:\Sigma_p\to\Sigma_p$ as follows:
First, put

$$G^{\alpha}_{\beta,p}(z)=z^{-p}+\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)}z^{k-p}\quad (p\in\mathbb{N}) \tag{1.5}$$

and let $G^{\alpha*}_{\beta,p,\mu}$ be defined by

$$G^{\alpha}_{\beta,p}(z)*G^{\alpha*}_{\beta,p,\mu}(z)=\frac{1}{z^p(1-z)^{\mu}}\quad (\mu>0;p\in\mathbb{N}). \tag{1.6}$$

Then

$$H^{\alpha}_{p,\beta,\mu}f(z)=G^{\alpha*}_{\beta,p}(z)*f(z)\quad (f\in\Sigma_p). \tag{1.7}$$

Using (1.5)-(1.7), we have

$$H^{\alpha}_{p,\beta,\mu}f(z)=z^{-p}+\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta+\alpha)(\mu)_k}{\Gamma(k+\beta)(1)_k}a_{k-p}z^{k-p}, \tag{1.8}$$

where $f\in\Sigma_p$ is in the form (1.1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n=\frac{\Gamma(\nu+n)}{\Gamma(\nu)}=\begin{cases}1 & (n=0),\\ \nu(\nu+1)\cdots(\nu+n-1) & (n\in\mathbb{N}).\end{cases}$$

It is readily verified from (1.8) that

$$z\big(H^{\alpha}_{p,\beta,\mu}f(z)\big)'=(\alpha+\beta)H^{\alpha+1}_{p,\beta,\mu}f(z)-(\alpha+\beta+p)H^{\alpha}_{p,\beta,\mu}f(z) \tag{1.9}$$

and

$$z\big(H^{\alpha}_{p,\beta,\mu}f(z)\big)'=\mu H^{\alpha}_{p,\beta,\mu+1}f(z)-(\mu+p)H^{\alpha}_{p,\beta,\mu}f(z). \tag{1.10}$$

It is noticed that, putting $\mu=1$ in (1.8), we obtain the operator

$$H^{\alpha}_{p,\beta,1}f(z)=z^{-p}+\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}a_{k-p}z^{k-p}. \tag{1.11}$$

Let M be the class of analytic functions $h(z)$ with $h(0)=1$, which are convex and univalent in U and satisfy $\mathrm{Re}\{h(z)\}>0$ $(z\in U)$.

For $0\le\eta,\ \gamma<p$, we denote by $\Sigma_pS(\eta)$, $\Sigma_pK(\eta)$, $\Sigma_pC(\eta,\gamma)$, and $\Sigma_pC^*(\eta,\gamma)$, the subclasses of Σ_p consisting of all p-valent meromorphic functions which are, respectively,

starlike of order η, convex of order η, close-to-convex functions of order γ and type η, and quasi-convex functions of order γ and type η in U.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $\Sigma_p S(\eta;\phi)$, $\Sigma_p C(\eta;\phi)$, $\Sigma_p K(\eta,\gamma;\phi,\psi)$, and $\Sigma_p K^*(\eta,\gamma;\phi,\psi)$ $(0 \le \eta,\ \gamma < p$ and $\phi,\psi \in M)$ of the class Σ_p which are defined by:

$$\Sigma_p S(\eta;\phi) = \left\{ f \in \Sigma_p : \frac{1}{p-\eta}\left(-\frac{zf'(z)}{f(z)} - \eta\right) \prec \phi(z), \text{ in } U \right\},$$
$$\Sigma_p K(\eta;\phi) = \left\{ f \in \Sigma_p : \frac{1}{p-\eta}\left(-\left[1 + \frac{zf''(z)}{f'(z)}\right] - \eta\right) \prec \phi(z), \text{ in } U \right\},$$
$$\Sigma_p C(\eta,\gamma;\phi,\psi) = \left\{ f \in \Sigma_p : \exists g \in \Sigma_p S(\eta;\phi) \text{ s.t. } \frac{1}{p-\gamma}\left(-\frac{zf'(z)}{g(z)} - \gamma\right) \prec \psi(z), \text{ in } U \right\},$$

and

$$\Sigma_p C^*(\eta,\gamma;\phi,\psi) = \left\{ f \in \Sigma_p : \exists g \in \Sigma_p K(\eta;\phi) \text{ s.t. } \frac{1}{p-\gamma}\left(-\frac{(zf'(z))'}{g'(z)} - \gamma\right) \prec \psi(z), \text{ in } U \right\}.$$

From these definitions, we can obtain some well-known subclasses of Σ_p by special choices of the functions ϕ and ψ as well as special choices of η, γ, and p (see [2, 5], and [10]).

Now, by using the linear operator $H^{\alpha}_{p,\beta,\mu}$ $(\alpha \ge 0,\ \mu > 0,\ \beta > -1;\ p \in \mathbb{N})$ and for $\phi,\psi \in M$, $0 \le \eta,\ \gamma < p$, we define new subclasses of meromorphic functions of Σ_p by:

$$\Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi) = \left\{ f \in \Sigma_p : H^{\alpha}_{p,\beta,\mu} f \in \Sigma_p S(\eta;\phi) \right\},$$
$$\Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi) = \left\{ f \in \Sigma_p : H^{\alpha}_{p,\beta,\mu} f \in \Sigma_p K(\eta;\phi) \right\},$$
$$\Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi) = \left\{ f \in \Sigma_p : H^{\alpha}_{p,\beta,\mu} f \in \Sigma_p C(\eta,\gamma;\phi,\psi) \right\},$$

and

$$\Sigma_p C^{\alpha*}_{\beta,\mu}(\eta,\gamma;\phi,\psi) = \left\{ f \in \Sigma_p : H^{\alpha}_{p,\beta,\mu} f \in \Sigma_p C^*(\eta,\gamma;\phi,\psi) \right\}.$$

We also note that

$$f(z) \in \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi) \quad \Leftrightarrow \quad -\frac{zf'(z)}{p} \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi), \tag{1.12}$$

and

$$f(z) \in \Sigma_p C^{\alpha*}_{\beta,\mu}(\eta,\gamma;\phi,\psi) \quad \Leftrightarrow \quad -\frac{zf'(z)}{p} \in \Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi). \tag{1.13}$$

In particular, we set

$$\Sigma_p S^{\alpha}_{\beta,\mu}\left(\eta; \frac{1+Az}{1+Bz}\right) = \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;A,B) \quad (-1 < B < A \le 1)$$

and

$$\Sigma_p K^{\alpha}_{\beta,\mu}\left(\eta; \frac{1+Az}{1+Bz}\right) = \Sigma_p K^{\alpha}_{\beta,\mu}(\eta; A, B) \quad (-1 < B < A \leq 1).$$

In this paper, we investigate several inclusion properties of the classes $\Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$, $\Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi)$, $\Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$, and $\Sigma_p C^{\alpha*}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$ associated with the operator $H^{\alpha}_{p,\beta,\mu}$. Some applications involving integral operators are also considered.

In order to establish our main results, we need the following lemmas.

Lemma 1 [4] *Let ς and υ be complex constants and let $h(z)$ be convex (univalent) in U with $h(0) = 1$ and* $\operatorname{Re}\{\varsigma h(z) + \upsilon\} > 0$. *If*

$$q(z) = 1 + q_1 z + \cdots \tag{1.14}$$

is analytic in U, then

$$q(z) + \frac{zq'(z)}{\varsigma q(z) + \upsilon} \prec h(z) \quad (z \in U),$$

implies

$$q(z) \prec h(z) \quad (z \in U).$$

Lemma 2 [7] *Let $h(z)$ be convex (univalent) in U and $\psi(z)$ be analytic in U with* $\operatorname{Re}\{\psi(z)\} \geq 0$. *If q is analytic in U and $q(0) = h(0)$, then*

$$q(z) + \psi(z) zq'(z) \prec h(z) \quad (z \in U)$$

implies

$$q(z) \prec h(z) \quad (z \in U).$$

2 Some inclusion results

In this section, we give some inclusion properties for meromorphic function classes, which are associated with the operator $H^{\alpha}_{p,\beta,\mu}$. Unless otherwise mentioned, we assume that $\alpha \geq 1$, $\beta > -1$, $\mu > 0$, $0 \leq \gamma$, $\eta < p$ and $p \in \mathbb{N}$.

Theorem 1 *For $f(z) \in \Sigma_p$, $\phi \in M$ with*

$$\max_{z\in U}\left(\operatorname{Re}\{\phi(z)\}\right) < \min_{z\in U}\left[\frac{p+\mu-\eta}{p-\eta}, \frac{\alpha+\beta+p-\eta}{p-\eta}\right],$$

then we have

$$\Sigma_p S^{\alpha}_{\beta,\mu+1}(\eta,\phi) \subset \Sigma_p S^{\alpha}_{\beta,\mu}(\eta,\phi) \subset \Sigma_p S^{\alpha-1}_{\beta,\mu}(\eta,\phi). \tag{2.1}$$

Proof We will first show that

$$\Sigma_p S^{\alpha}_{\beta,\mu+1}(\eta,\phi) \subset \Sigma_p S^{\alpha}_{\beta,\mu}(\eta,\phi). \tag{2.2}$$

Let $f \in \Sigma_p S^{\alpha}_{\beta,\mu+1}(\eta;\phi)$ and put

$$q(z) = \frac{1}{p-\eta}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu}f(z))'}{H^{\alpha}_{p,\beta,\mu}f(z)} - \eta\right), \tag{2.3}$$

where $q(z)$ is analytic in U with $q(0) = 1$. Applying (1.10) in (2.3), we have

$$-\mu\frac{H^{\alpha}_{p,\beta,\mu+1}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)} = (p-\eta)q(z) + \eta - (p+\mu). \tag{2.4}$$

Differentiating (2.4) logarithmically with respect to z, we have

$$\frac{1}{p-\eta}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu+1}f(z))'}{H^{\alpha}_{p,\beta,\mu+1}f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p+\mu)-\eta-(p-\eta)q(z)} \quad (z \in U). \tag{2.5}$$

Since

$$\max_{z\in U}\big(\mathrm{Re}\{\phi(z)\}\big) < \min_{z\in U}\frac{p+\mu-\eta}{p-\eta},$$

we see that

$$\mathrm{Re}\{(p+\mu)-\eta-(p-\eta)\phi(z)\} > 0 \quad (z \in U).$$

Applying Lemma 1 to (2.5), it follows that $q \prec \phi$, that is, that $f \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$. The proof of the second part will follow by using arguments similar to those used in the first part with $f \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$ and using the identity (1.9) instead of (1.10). This completes the proof of Theorem 1. □

Theorem 2 *For $f(z) \in \Sigma_p$, $\phi \in M$ with*

$$\max_{z\in U}\big(\mathrm{Re}\{\phi(z)\}\big) < \min_{z\in U}\left[\frac{p+\mu-\eta}{p-\eta}, \frac{\alpha+\beta+p-\eta}{p-\eta}\right],$$
$$\Sigma_p K^{\alpha}_{\beta,\mu+1}(\eta;\phi) \subset \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi) \subset \Sigma_p K^{\alpha-1}_{\beta,\mu}(\eta;\phi) \quad (0 \le \eta < p; \phi \in M).$$

Proof Applying (1.10) and using Theorem 1, we have

$$\begin{aligned} f(z) \in \Sigma_p K^{\alpha}_{\beta,\mu+1}(\eta;\phi) \quad &\Leftrightarrow \quad H^{\alpha}_{p,\beta,\mu+1}f(z) \in \Sigma_p K(\eta;\phi) \\ &\Leftrightarrow \quad -\frac{z(H^{\alpha}_{p,\beta,\mu+1}f(z))'}{p} \in \Sigma_p S(\eta;\phi) \\ &\Leftrightarrow \quad H^{\alpha}_{p,\beta,\mu+1}\left(-\frac{zf'(z)}{p}\right) \in \Sigma_p S(\eta;\phi) \\ &\Leftrightarrow \quad -\frac{zf'(z)}{p} \in \Sigma_p S^{\alpha}_{\beta,\mu+1}(\eta;\phi) \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad -\frac{zf'(z)}{p} \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi) \\
&\Leftrightarrow \quad H^{\alpha}_{p,\beta,\mu}\left(-\frac{zf'(z)}{p}\right) \in \Sigma_p S(\eta;\phi) \\
&\Leftrightarrow \quad -\frac{z(H^{\alpha}_{p,\beta,\mu}f(z))'}{p} \in \Sigma_p S(\eta;\phi) \\
&\Leftrightarrow \quad H^{\alpha}_{p,\beta,\mu}f(z) \in \Sigma_p K(\eta;\phi) \\
&\Leftrightarrow \quad f(z) \in \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi).
\end{aligned}$$

Also,

$$\begin{aligned}
f(z) \in \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi) \quad &\Leftrightarrow \quad -\frac{zf'(z)}{p} \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi) \\
&\Rightarrow \quad -\frac{zf'(z)}{p} \in \Sigma_p S^{\alpha-1}_{\beta,\mu}(\eta;\phi) \\
&\Leftrightarrow \quad -\frac{z(H^{\alpha-1}_{p,\beta,\mu}f(z))'}{p} \in \Sigma_p S(\eta;\phi) \\
&\Leftrightarrow \quad H^{\alpha-1}_{p,\beta,\mu}f(z) \in \Sigma_p K(\eta;\phi) \\
&\Leftrightarrow \quad f(z) \in \Sigma_p K^{\alpha-1}_{\beta,\mu}(\eta;\phi).
\end{aligned}$$

This completes the proof of Theorem 2. □

Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 < B < A \leq 1; z \in U)$$

in Theorem 1 and Theorem 2, we have the following corollary.

Corollary 1 *Let* $f(z) \in \Sigma_p$ *and*

$$\frac{1+A}{1+B} < \min_{z\in U}\left(\frac{p+\mu-\eta}{p-\eta}, \frac{\alpha+\beta+p-\eta}{p-\eta}\right) \quad (-1 < B < A \leq 1).$$

Then we have

$$\Sigma_p S^{\alpha}_{\beta,\mu+1}(\eta;A,B) \subset \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;A,B) \subset \Sigma_p S^{\alpha-1}_{\beta,\mu}(\eta;A,B)$$

and

$$\Sigma_p K^{\alpha}_{\beta,\mu+1}(\eta;A,B) \subset \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;A,B) \subset \Sigma_p K^{\alpha-1}_{\beta,\mu}(\eta;A,B).$$

Now, using Lemma 2, we obtain similar inclusion relations for the subclass $\Sigma_p C^{\alpha}_{\beta,\mu}(\eta;\gamma;\phi,\psi)$.

Theorem 3 *Let* $f(z) \in \Sigma_p$ *and*

$$\max_{z\in U} \operatorname{Re}\{\phi(z)\} < \min_{z\in U}\left(\frac{p+\mu-\eta}{p-\eta}, \frac{\alpha+\beta+p-\eta}{p-\eta}\right).$$

Then we have

$$\Sigma_p C^{\alpha}_{\beta,\mu+1}(\eta;\gamma;\phi,\psi) \subset \Sigma_p C^{\alpha}_{\beta,\mu}(\eta;\gamma;\phi,\psi) \subset \Sigma_p C^{\alpha-1}_{\beta,\mu}(\eta;\gamma;\phi,\psi)$$
$$(0 \le \eta,\gamma < p;\phi,\psi \in M).$$

Proof First, we will prove that

$$\Sigma_p C^{\alpha}_{\beta,\mu+1}(\eta;\gamma;\phi,\psi) \subset \Sigma_p C^{\alpha}_{\beta,\mu}(\eta;\gamma;\phi,\psi).$$

Let $f \in \Sigma_p C^{\alpha}_{\beta,\mu+1}(\eta;\gamma;\phi,\psi)$. Then, from the definition of the class $\Sigma_p C^{\alpha}_{\beta,\mu}(\eta;\gamma;\phi,\psi)$, there exists a function $g \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$ such that

$$\frac{1}{p-\gamma}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu+1}f(z))'}{H^{\alpha}_{p,\beta,\mu+1}g(z)}-\gamma\right) \prec \psi(z) \quad (z \in U). \tag{2.6}$$

Now, let

$$q(z) = \frac{1}{p-\gamma}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu}f(z))'}{H^{\alpha}_{p,\beta,\mu}g(z)}-\gamma\right), \tag{2.7}$$

where $q(z)$ is analytic in U with $q(0) = 1$. Applying (1.10) in (2.6), we have

$$\begin{aligned}
&\frac{1}{p-\gamma}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu+1}f(z))'}{H^{\alpha}_{p,\beta,\mu+1}g(z)}-\gamma\right)\\
&\quad=\frac{1}{p-\gamma}\left(\frac{H^{\alpha}_{p,\beta,\mu+1}(\frac{-zf'(z)}{p})}{H^{\alpha}_{p,\beta,\mu+1}g(z)}-\gamma\right)\\
&\quad=\frac{1}{p-\gamma}\left(\frac{z(H^{\alpha}_{p,\beta,\mu}(-\frac{zf'(z)}{p}))'+(p+\mu)H^{\alpha}_{p,\beta,\mu}(-\frac{zf'(z)}{p})}{z(H^{\alpha}_{p,\beta,\mu}g(z))'+(p+\mu)H^{\alpha}_{p,\beta,\mu}g(z)}-\gamma\right)\\
&\quad=\frac{1}{p-\gamma}\left(\frac{\frac{z(H^{\alpha}_{p,\beta,\mu}(-\frac{zf'(z)}{p}))'}{H^{\alpha}_{p,\beta,\mu}g(z)}+(p+\mu)\frac{H^{\alpha}_{p,\beta,\mu}(-\frac{zf'(z)}{p})}{H^{\alpha}_{p,\beta,\mu}g(z)}}{\frac{z(H^{\alpha}_{p,\beta,\mu}g(z))'}{H^{\alpha}_{p,\beta,\mu}g(z)}+(p+\mu)}-\gamma\right).
\end{aligned} \tag{2.8}$$

Since, by Theorem 1,

$$g(z) \in \Sigma_p S^{\alpha}_{\beta,\mu+1}(\eta;\phi) \subset \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi),$$

set

$$h(z) = \frac{1}{p-\eta}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu}g(z))'}{H^{\alpha}_{p,\beta,\mu}g(z)}-\eta\right),$$

where $h \prec \phi$ in U, and $\phi \in M$. Then, using (2.7) and (2.8), we have

$$H^{\alpha}_{p,\beta,\mu}\left(-\frac{zf'(z)}{p}\right) = \left[(p-\gamma)q(z)+\gamma\right]H^{\alpha}_{p,\beta,\mu}g(z) \tag{2.9}$$

and

$$\frac{1}{p-\gamma}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu+1}f(z))'}{H^{\alpha}_{p,\beta,\mu+1}g(z)}-\gamma\right) = \frac{1}{p-\gamma}\left(\frac{\frac{z(H^{\alpha}_{p,\beta,\mu}(-\frac{zf'(z)}{p}))'}{H^{\alpha}_{p,\beta,\mu}g(z)}+(p+\mu)[(p-\gamma)q(z)+\gamma]}{p+\mu-\eta-(p-\eta)h(z)}-\gamma\right). \tag{2.10}$$

Differentiating both sides of (2.9) with respect to z and multiplying by z, we have

$$\frac{z(H^{\alpha}_{p,\beta,\mu}(-\frac{zf'(z)}{p}))'}{H^{\alpha}_{p,\beta,\mu}g(z)} = (p-\gamma)zq'(z)-\big[(p-\gamma)q(z)+\gamma\big]\big[(p-\eta)h(z)+\eta\big]. \tag{2.11}$$

Making use of (2.6), (2.10), and (2.11), we have

$$\frac{1}{p-\gamma}\left(-\frac{z(H^{\alpha}_{p,\beta,\mu+1}f(z))'}{H^{\alpha}_{p,\beta,\mu+1}g(z)}-\gamma\right) = q(z)+\frac{zq'(z)}{p+\mu-\eta-(p-\eta)h(z)} \prec \psi(z), \quad z\in U. \tag{2.12}$$

Since $h \prec \phi$ in U, and

$$\max_{z\in U}\operatorname{Re}\big\{h(z)\big\} < \frac{p+\mu-\eta}{p-\eta},$$

then

$$\operatorname{Re}\big\{p+\mu-\eta-(p-\eta)h(z)\big\} > 0 \quad (z\in U). \tag{2.13}$$

Hence, putting

$$\chi(z) = \frac{1}{\{p+\mu-\eta-(p-\eta)h(z)\}},$$

in Eq. (2.12) and applying Lemma 2, we can show that $q \prec \psi$, that is, that $f \in \Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$. The second part can be proved by using similar arguments and using (1.9). This completes the proof of Theorem 3. □

3 Inclusion properties involving the integral operator $F_{p,\delta}$

Now, we consider the generalized Libera integral operator $F_{p,\delta}(f)$ (see [6] and [9]), defined by

$$F_{p,\delta}(f)(z) = \frac{\delta}{z^{\delta+p}}\int_0^z t^{\delta+p-1}f(t)\,dt = z^{-p}+\sum_{k=1}^{\infty}\frac{\delta}{\delta+k}a_{k-p}z^{k-p} \quad (\delta > -p). \tag{3.1}$$

From (3.1), we have

$$z\big(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(f)(z)\big)' = \delta H^{\sigma}_{p,\beta,\mu}f(z)-(\delta+p)H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(f)(z). \tag{3.2}$$

Theorem 4 *Let $\phi \in M$ with*

$$\max_{z\in U}\left(\mathrm{Re}\{\phi(z)\}\right) < \frac{\delta + p - \eta}{p - \eta} \quad (\delta > -p).$$

If $f \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$, then $F_{p,\delta}(f) \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$.

Proof Let $f \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$ and put

$$h(z) = \frac{1}{p-\eta}\left(-\frac{z(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(f)(z))'}{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(f)(z)} - \eta\right), \tag{3.3}$$

where h is analytic in U with $h(0) = 1$. Then, by using (3.2) and (3.3), we have

$$-\delta\frac{H^{\sigma}_{p,\beta,\mu}f(z)}{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(f)(z)} = (p-\eta)h(z) + \eta - (p+\delta). \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z, we have

$$\frac{1}{p-\eta}\left(-\frac{z(H^{\sigma}_{p,\beta,\mu}f(z))'}{H^{\sigma}_{p,\beta,\mu}f(z)} - \eta\right) = h(z) + \frac{zh'(z)}{p+\delta-\eta-(p-\eta)h(z)} \quad (z \in U).$$

Applying Lemma 1, we conclude that $h \prec \phi$ $(z \in U)$, which implies that $F_{p,\delta}(f) \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$. □

Theorem 5 *Let $\phi \in M$ with*

$$\max_{z\in U}\left(\mathrm{Re}\{\phi(z)\}\right) < \frac{\delta + p - \eta}{p - \eta} \quad (\delta > -p).$$

If $f \in \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi)$, then $F_{p,\delta}(f) \in \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi)$.

Proof Applying Theorem 4 and (1.12), we have

$$\begin{aligned} f(z) \in \Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi) \quad &\Leftrightarrow \quad -\frac{zf'(z)}{p} \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi) \\ &\Rightarrow \quad F_{p,\delta}\left(-\frac{zf'(z)}{p}\right)(z) \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi) \\ &\Leftrightarrow \quad -\frac{z}{p}F'_{p,\delta}(f)(z) \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi) \\ &\Leftrightarrow \quad F_{p,\delta}(f)(z) \in K^{\sigma}_{p,\lambda}(\eta;\phi). \end{aligned}$$

This completes the proof of Theorem 5. □

From Theorem 4 and Theorem 5, we have the following corollary.

Corollary 2 *Suppose that*

$$\frac{1+A}{1+B} < \frac{\delta + p - \eta}{p - \eta} \quad (\delta > -p; -1 < B < A \leq 1).$$

Then, for the classes $\Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$ and $\Sigma_p K^{\alpha}_{\beta,\mu}(\eta;\phi)$, the following inclusion relations hold true:

$$f \in \Sigma_p S^{\alpha}_{\beta,\mu}(A,B) \quad \Rightarrow \quad F_{p,\delta}(f) \in \Sigma_p S^{\alpha}_{\beta,\mu}(A,B)$$

and

$$f \in \Sigma_p K^{\alpha}_{\beta,\mu}(A,B) \quad \Rightarrow \quad F_{p,\delta}(f) \in \Sigma_p K^{\alpha}_{\beta,\mu}(A,B).$$

Theorem 6 *Let $\phi, \psi \in M$ with*

$$\max_{z\in U} \operatorname{Re}\{\phi(z)\} < \frac{\delta + p - \eta}{p - \eta} \quad (\delta > -p).$$

If $f \in \Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$, then $F_{p,\delta}(f) \in \Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$.

Proof Let $f \in \Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$. Then, from the definition of the class $\Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$, there exists a function $g \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$ such that

$$\frac{1}{p-\gamma}\left(-\frac{z(H^{\sigma}_{p,\beta,\mu}f(z))'}{H^{\sigma}_{p,\beta,\mu}g(z)} - \gamma\right) \prec \psi(z) \quad (z \in U). \tag{3.5}$$

Now, let

$$h(z) = \frac{1}{p-\gamma}\left(-\frac{z(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(f)(z))'}{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z)} - \gamma\right), \tag{3.6}$$

where $h(z)$ is analytic in U with $h(0) = 1$. Applying (3.2) in (3.6), we have

$$\begin{aligned}
&\frac{1}{p-\gamma}\left(-\frac{z(H^{\sigma}_{p,\beta,\mu}f(z))'}{H^{\sigma}_{p,\beta,\mu}g(z)} - \gamma\right) \\
&\quad = \frac{1}{p-\gamma}\left(\frac{(H^{\sigma}_{p,\beta,\mu}(-\frac{zf'(z)}{p})(z))'}{H^{\sigma}_{p,\beta,\mu}g(z)} - \gamma\right) \\
&\quad = \frac{1}{p-\gamma}\left\{\frac{z(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(-\frac{zf'(z)}{p})(z))' + (p+\delta)H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(-\frac{zf'(z)}{p})}{z(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z))' + (p+\delta)H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z)} - \gamma\right\} \\
&\quad = \frac{1}{p-\gamma}\left\{\frac{\frac{z(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(-\frac{zf'(z)}{p})(z))'}{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z)} + (p+\delta)\frac{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(-\frac{zf'(z)}{p})}{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z)}}{\frac{z(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z))'}{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z)} + p + \delta} - \gamma\right\}.
\end{aligned} \tag{3.7}$$

Since $g \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$, then by Theorem 4, we have $F_{p,\delta}(g)(z) \in \Sigma_p S^{\alpha}_{\beta,\mu}(\eta;\phi)$. Let

$$q(z) = \frac{1}{p-\eta}\left(-\frac{z(H^{\sigma}_{p,\beta,\mu}F_{p,\delta}(g)(z))'}{H^{\sigma}_{p,\beta,\mu}F_{p,\delta}g(z)} - \eta\right), \tag{3.8}$$

where $q \prec \phi$ in U. Then, using the same techniques as in the proof of Theorem 3 and using (3.5) and (3.7), we have

$$\frac{1}{p-\gamma}\left(-\frac{z(H^{\sigma}_{p,\beta,\mu}f(z))'}{H^{\sigma}_{p,\beta,\mu}g(z)}-\gamma\right)=h(z)+\frac{zh'(z)}{\delta+p-\eta-(p-\eta)q(z)}\prec\psi(z). \tag{3.9}$$

Since $\operatorname{Re}\{\frac{1}{\delta+p-\eta-(p-\eta)q(z)}\}>0$, then applying Lemma 2, we find that $h \prec \psi$, which yields $F_{p,\delta}(f)(z) \in \Sigma_p C^{\alpha}_{\beta,\mu}(\eta,\gamma;\phi,\psi)$. This completes the proof of Theorem 6. □

Remark Putting $\mu = 1$ in the above results, we obtain the results corresponding to the operator $H^{\alpha}_{p,\beta,1}$ defined by (1.11).

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author read and approved the .nal manuscript.

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