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A generalized Hyers-Ulam stability of a Pexiderized logarithmic functional equation in restricted domains

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Abstract

Let $\mathbb{R}_+$ and B be the set of positive real numbers and a Banach space, respectively, f, g, $h : \mathbb{R}_+ \to B$ and $\psi : \mathbb{R}_+^2 \to \mathbb{R}$ be a nonnegative function of some special forms. Generalizing the stability theorem for a Jensen-type logarithmic functional equation, we prove the Hyers-Ulam stability of the Pexiderized logarithmic functional inequality

$$||f(xy) - g(x) - h(y)|| \le \psi(x, y)$$

in restricted domains of the form $\{(x, y) : x^k y^s \ge d\}$ for fixed $k, s \in \mathbb{R}$, $d > 0$. We also discuss an L^∞-version of the Hyers-Ulam stability of the inequality. **2000 MSC**: 39B22.

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1. Introduction

The Hyers-Ulam stability problems of functional equations go back to 1940 when Ulam proposed a question concerning the approximate homomorphisms from a group to a metric group (see [1]). A partial answer was given by Hyers [2,3] under the assumption that the target space of the involved mappings is a Banach space. After the result of Hyers, Aoki [4] and Bourgin [5,6] treated with this problem, however, there were no other results on this problem until 1978 when Rassias [7] treated again with the inequality of Aoki [4]. Following the Rassias' result a great number of articles on the subject have been published concerning numerous functional equations in various directions [8-19]. Among the results, the stability problem in a restricted domain was investigated by Skof, who proved the stability problem of the Cauchy functional equation in a restricted domain [20]. Developing this result, Jung, Rassias and Rassias considered the stability problems in restricted domains for the Jensen functional equation [21,22] and Jensen-type functional equations [23]. We also refer the reader to [24-29] for some interesting results on functional equations and their Hyers-Ulam stabilities in restricted conditions. In this article, generalizing the result in [8], we consider the Hyers-Ulam stability of the Pexiderized Jensen functional equation

$$||f(xy) - g(x) - h(y)|| \le \psi(x, y) \tag{1.1}$$

in the restricted domains $U_{k,s,d} = \{(x, y): x > 0, y > 0, x^k y^s \ge d\}$ for fixed $k, s \in \mathbb{R}$ and $d > 0$, where $\psi(x, y) = \varphi(xy)$, $\varphi(x)$ or $\varphi(y)$. Making use of the result, we prove the asymptotic behavior of f, g and h satisfying

$$||f(xy) - g(x) - h(y)|| \to 0 \tag{1.2}$$

as $x^k y^s \to \infty$. Finally, we discuss the Hyers-Ulam stability of the inequality

$$||f(xy) - g(x) - h(y)||_{L^\infty(U_{k,s,d})} \le \varepsilon \tag{1.3}$$

and its asymptotic behavior.

2. Stability in classical sense

We call L: $\mathbb{R}_+ \to B$ *a logarithmic function* provided that

$$L(xy) - L(x) - L(y) = 0$$

for all x, $y > 0$. Let $\varphi : \mathbb{R}_+ \to [0, \infty)$. We assume that

$$\Phi(x) := \sum_{k=1}^{\infty} 2^{-k}\left(\phi(x^{2^k}) + 2\phi(x^{2^{k-1}}) + \phi(1)\right) < \infty$$

for all $x > 0$. As a direct consequence of Aoki [4] or Bourgin [5,6], we obtain the generalized Hyers-Ulam stability for the logarithmic functional equation, viewing $\langle \mathbb{R}_+, \times \rangle$ as a multiplicative group.

Theorem A. Suppose that $f : \mathbb{R}_+ \to B$ satisfies

$$||f(xy) - f(x) - f(y)|| \le \phi(xy) + \phi(x) + \phi(y) + \phi(1)$$

for all x, $y > 0$. Then, there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ satisfying

$$||f(x) - L(x)|| \le \Phi(x)$$

for all $x > 0$.

In this section, we first consider the logarithmic functional inequality (1.1) in the restricted domain

$$U_{k,s,d} = \{(x, y) : x > 0, y > 0, x^k y^s \ge d\}$$

for fixed k, $s \in \mathbb{R}$ and $d > 0$.

Theorem 2.1. *Let $d > 0$, k, $s \in \mathbb{R}$, $k \ne s$. Suppose that f, g, $h : \mathbb{R}_+ \to B$ satisfy*

$$||f(xy) - g(x) - h(y)|| \le \phi(xy) \tag{2.1}$$

for all $x,y \in U_{k,s,d}$. Then, there exists a unique logarithmic function $L_1 : \mathbb{R}_+ \to B$ such that

$$||f(x) - L_1(x) - f(1)|| \le \Phi(x) \tag{2.2}$$

for all $x \in \mathbb{R}_+$.

Proof. For given $x,y \in \mathbb{R}_+$, choosing a $z > 0$ such that $x^k y^s z^{s-k} \ge d$, $x^k z^{s-k} \ge d$, $y^s z^{s-k} \ge d$ and $z^{s-k} \ge d$, we have

$$\begin{aligned}||f(xy) - f(x) - f(y) + f(1)|| &\leq ||f(xy) - g(xz^{-1}) - h(yz)|| \\ &\quad + || - f(x) + g(xz^{-1}) + h(z)|| \\ &\quad + || - f(y) + g(z^{-1}) + h(yz)|| \\ &\quad + ||f(1) - g(z^{-1}) - h(z)|| \\ &\leq \phi(xy) + \phi(x) + \phi(y) + \phi(1).\end{aligned} \tag{2.3}$$

Now, by Theorem A, we get the result.

Corollary 2.2. *Let* $\epsilon, d > 0$, k, $s \in \mathbb{R}$, $k \neq s$. *Suppose that f, g, h* : $\mathbb{R}_+ \to B$ *satisfy*

$$||f(xy) - g(x) - h(y)|| \leq \varepsilon \tag{2.4}$$

for all $x, y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* L_1: $\mathbb{R}_+ \to B$ *such that*

$$||f(x) - L_1(x) - f(1)|| \leq 4\varepsilon \tag{2.5}$$

for all $x \in \mathbb{R}_+$.

Remark 2.1. Note that the Corollary 2.2 fails if $k = s$. Indeed, let L: $\mathbb{R}_+ \to B$ be a nonzero logarithmic function. Define $g(x) = h(x) = L(x)$ for all $x > 0$ and

$$f(x) = \begin{cases} L(x), & x \geq d^{1/s}, \\ 0, & 0 < x < d^{1/s}. \end{cases}$$

Then, it is easy to see that the inequality (2.4) holds for all x, $y > 0$, with $xy \geq d^{1/s}$. Assume that there exists a logarithmic function L_1 satisfying (2.5). Then, we have

$$||L_1(x)|| \leq |f(1)| + 4\varepsilon = 4\varepsilon \tag{2.6}$$

for all $0 < x < d^{1/s}$. The inequality (2.6) implies $L_1 = 0$. Indeed, if $L_1(x_0) \neq 0$ for some $x_0 > 0$, then we have $L_1(1/x_0) = -L_1(x_0) \neq 0$. Thus, we may assume that $0 < x_0 < 1$. Now, we encounter the contradiction

$$|nL_1(x_0)| = |L_1(x_0^n)| \leq 4\varepsilon$$

for all large integers n. Thus, $L_1 = 0$ and the inequality (2.5) implies

$$||L(x)|| \leq 4\varepsilon \tag{2.7}$$

for all $x \geq d^{1/s}$. Similarly, using (2.7), we can show that $L = 0$, which contracts to the choice of L.

As a direct consequence of Corollary 2.2, we have the following.

Corollary 2.3. [8]*Let p,q,P,Q be nonzero real numbers and* $\varepsilon, d > 0, k, s \in \mathbb{R}, \frac{k}{p} \neq \frac{s}{q}$. *Suppose that* $f : \mathbb{R}_+ \to B$ *satisfies*

$$||f(x^p y^q) - Pf(x) - Qf(y)|| \leq \varepsilon \tag{2.8}$$

for all $x, y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* $L : \mathbb{R}_+ \to B$ *such that*

$$||f(x) - L(x) - f(1)|| \leq 4\varepsilon \tag{2.9}$$

for all $x \in \mathbb{R}_+$.

Proof. Replacing x by $x^{\frac{1}{p}}$, y by $y^{\frac{1}{q}}$ in (2.8), we have

$$||f(xy) - Pf(x^{\frac{1}{p}}) - Qf(y^{\frac{1}{q}})|| \le \varepsilon$$

for all $x,y > 0$, with $x^{\frac{k}{p}}y^{\frac{s}{q}} \ge d$. Letting $g(x) = Pf(x^{\frac{1}{p}}),\ h(y) = Qf(y^{\frac{1}{q}})$, applying Corollary 2.2 and letting $L(x) = L_1(x)$, we get the result.

Theorem 2.4. *Let* $d > 0$, $s \neq 0$. *Suppose that* f, g, h: $\mathbb{R}_+ \to B$ *satisfy*

$$||f(xy) - g(x) - h(y)|| \le \phi(x) \tag{2.10}$$

for all $x,y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* $L_2 : \mathbb{R}_+ \to B$ *such that*

$$||g(x) - L_2(x) - g(1)|| \le \Phi(x) \tag{2.11}$$

for all $x \in \mathbb{R}_+$.

Proof. For given $x,y \in \mathbb{R}_+$, choosing a $z > 0$ such that $x^ky^kz^s \ge d$, $x^ky^sz^s \ge d$, $y^kz^s \ge d$ and $y^sz^s \ge d$, we have

$$\begin{aligned} ||g(xy) - g(x) - g(y) + g(1)|| \le\ & || - f(xyz) + g(xy) + h(z)|| \\ & + ||f(xyz) - g(x) - h(yz)|| \\ & + ||f(yz) - g(y) - h(z)|| \\ & + || - f(yz) + g(1) + h(yz)|| \\ \le\ & \phi(xy) + \phi(x) + \phi(y) + \phi(1). \end{aligned} \tag{2.12}$$

Now, by Theorem A, we get the result.

Corollary 2.5. *Let* ϵ, $d > 0$, $s \neq 0$. *Suppose that* $f, g, h : \mathbb{R}_+ \to B$ *satisfy*

$$||f(xy) - g(x) - h(y)|| \le \varepsilon \tag{2.13}$$

for all x, $y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* L_2: $\mathbb{R}_+ \to B$ *such that*

$$||g(x) - L_2(x) - g(1)|| \le 4\varepsilon \tag{2.14}$$

for all $x \in \mathbb{R}_+$.

Remark 2.2. Similarly as in Corollary 2.2, the above result fails if $s = 0$. Let $L : \mathbb{R}_+ \to B$ be a nonzero logarithmic function. Define $f(x) = h(x) = L(x)$ for all $x > 0$ and

$$g(x) = \begin{cases} L(x), & x \ge d^{1/k}, \\ 0, & 0 < x < d^{1/s}. \end{cases}$$

Then, the inequality (2.13) holds for all x, $y > 0$, with $x^k \ge d$ but (2.14) does not hold for any logarithmic function L_2.

As a direct consequence of Corollary 2.5, we have the following.

Corollary 2.6. [8]*Let* p, q, P, Q *be nonzero real numbers and* ϵ, $d > 0$, k, $s \in \mathbb{R}$ *with* $s \neq 0$. *Suppose that* $f : \mathbb{R}_+ \to B$ *satisfies*

$$||f(x^py^q) - Pf(x) - Qf(y)|| \le \varepsilon \tag{2.15}$$

for all $x,y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* $L : \mathbb{R}_+ \to B$ *such that*

$$||f(x) - L(x) - f(1)|| \leq \frac{4\varepsilon}{|P|} \tag{2.16}$$

for all $x \in \mathbb{R}_+$.

Proof. Replacing x by $x^{\frac{1}{p}}$, y by $y^{\frac{1}{q}}$ in (2.15), we have

$$||f(xy) - Pf(x^{\frac{1}{p}}) - Qf(y^{\frac{1}{q}})|| \leq \varepsilon$$

for all $x,y > 0$, with $x^{\frac{k}{p}} y^{\frac{s}{q}} \geq d$. Letting $g(x) = Pf(x^{\frac{1}{p}}),\ h(y) = Qf(y^{\frac{1}{q}})$, applying Corollary 2.5 and dividing the result by $|P|$, we get the result with $L(x) = \frac{1}{P}L_2(x^p)$.

Theorem 2.7. *Let* $d > 0$, $k \neq 0$. *Suppose that f, g, h* : $\mathbb{R}_+ \to B$ *satisfy*

$$||f(xy) - g(x) - h(y)|| \leq \phi(y) \tag{2.17}$$

for all $x,y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* $L_3 : \mathbb{R}_+ \to B$ *such that*

$$||h(x) - L_3(x) - h(1)|| \leq \Phi(x) \tag{2.18}$$

for all $x \in \mathbb{R}_+$.

Proof. For given $x,y \in \mathbb{R}_+$, choosing a $z > 0$ such that $x^s y^s z^k \geq d$, $x^k y^s z^k \geq d$, $x^s z^k \geq d$ and $x^k z^k \geq d$, we have

$$\begin{aligned} ||h(xy) - h(x) - h(y) + h(1)|| \leq{}& || - f(xyz) + g(z) + h(xy)|| \\ &+ ||f(xyz) - g(xz) - h(y)|| \\ &+ ||f(zx) - g(z) - h(x)|| \\ &+ || - f(xz) + g(xz) + h(1)|| \\ \leq{}& \phi(xy) + \phi(x) + \phi(y) + \phi(1). \end{aligned} \tag{2.19}$$

Now, by Theorem A, we get the result.

Corollary 2.8. *Let* ϵ, $d > 0$, $k \neq 0$. *Suppose that f, g, h*: $\mathbb{R}_+ \to B$ *satisfy*

$$||f(xy) - g(x) - h(y)|| \leq \varepsilon \tag{2.20}$$

for all $x,y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* $L_3 : \mathbb{R}_+ \to B$ *such that*

$$||h(x) - L_3(x) - h(1)|| \leq 4\varepsilon \tag{2.21}$$

for all $x \in \mathbb{R}_+$.

Remark 2.3. Similarly, as in Remark 2.2, we can show that the above result fails if $k = 0$. Also, as a direct consequence of the result, we have the following.

Corollary 2.9. [8]*Let p, q, P, Q be nonzero real numbers and* ϵ, $d > 0$, $k, s \in \mathbb{R}$ *with* $k \neq 0$. *Suppose that* $f : \mathbb{R}_+ \to B$ *satisfies*

$$||f(x^p y^q) - Pf(x) - Qf(y)|| \leq \varepsilon \tag{2.22}$$

for all $x,y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function L*: $\mathbb{R}_+ \to B$ *such that*

$$||f(x) - L(x) - f(1)|| \le \frac{4\varepsilon}{|Q|} \tag{2.23}$$

for all $x \in \mathbb{R}_+$.

Theorem 2.10. *Let* ϵ, $d > 0$, k, $s \neq 0$, $k \neq s$. *Suppose that* $f, g, h : \mathbb{R}_+ \to B$ *satisfy*

$$||f(xy) - g(x) - h(y)|| \le \varepsilon \tag{2.24}$$

for all $x,y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* $L : \mathbb{R}_+ \to B$ *such that*

$$\begin{aligned} &||f(x) - L(x) - f(1)|| \le 4\varepsilon, \\ &||g(x) - L(x) - g(1)|| \le 4\varepsilon, \\ &||h(x) - L(x) - h(1)|| \le 4\varepsilon \end{aligned}$$

for all $x \in \mathbb{R}_+$.

Proof. In view of Corollaries 2.2, 2.5 and 2.8, it suffices to prove that $L_1 = L_2 = L_3$. For given $x,y > 0$, choose a $z > 0$ such that $x^k y^s z^{s-k} \ge d$, $z^{s-k} \ge d$. Then, in view of (2.24), we have

$$||f(xy) - g(xz^{-1}) - h(yz)|| \le \varepsilon, \tag{2.25}$$

$$|| - f(1) + g(z^{-1}) + h(z)|| \le \varepsilon. \tag{2.26}$$

Using the inequalities (2.10) and (2.15), we have

$$||g(xz^{-1}) - g(x) - g(z^{-1}) + g(1)|| \le 4\varepsilon, \tag{2.27}$$

$$||h(yz) - h(z) - h(y) + h(1)|| \le 4\varepsilon \tag{2.28}$$

for all $x,y,z > 0$. From (2.25)-(2.28), using the triangle inequality, we have

$$||f(xy) - g(x) - h(y) - f(1) + g(1) + h(1)|| \le 10\varepsilon \tag{2.29}$$

for all $x,y > 0$. From the inequalities (2.5), (2.14), (2.21), (2.29) using the triangle inequality, we have

$$||L_1(xy) - L_2(x) - L_3(y)|| \le 22\varepsilon. \tag{2.30}$$

Putting $y = 1$ and $x = 1$ in (2.30) separately, and using the fact that for all $x > 0$, $n \in \mathbb{N}$, $L_j(x^n) = nL_j(x)$, $j = 1,2,3$, we can show that $L_1 = L_2$ and $L_1 = L_3$. This completes the proof.

As a direct consequence of Theorem 2.10, we have the following.

Corollary 2.11. [8]*Let* p, q, P ,Q *be nonzero real numbers and* ϵ, $d > 0$, $k, s \in \mathbb{R}$ *with* $k \neq 0$, $s \neq 0$ *and* $k \neq s$. *Suppose that* $f : \mathbb{R}_+ \to B$ *satisfies*

$$||f(x^p y^q) - Pf(x) - Qf(y)|| \le \varepsilon \tag{2.31}$$

for all $x,y \in U_{k,s,d}$. *Then, there exists a unique logarithmic function* L: $\mathbb{R}_+ \to B$ *such that*

$$||f(x) - L(x) - f(1)|| \le \min\left\{4\varepsilon, \frac{4\varepsilon}{|P|}, \frac{4\varepsilon}{|Q|}\right\} \tag{2.32}$$

for all $x \in \mathbb{R}_+$.

3. Asymptotic behaviors

In this section, we consider asymptotic behaviors of f,g, h satisfying (1.2).

Theorem 3.1. *Let* $k, s \in \mathbb{R}$, $k \neq s$. *Suppose that* $f, g, h : \mathbb{R}_+ \to B$ *satisfy the asymptotic condition*

$$||f(xy) - g(x) - h(y)|| \to 0 \tag{3.1}$$

as $x^k y^s \to \infty$. *Then, there exists a unique logarithmic function* $L : \mathbb{R}_+ \to B$ *such that*

$$f(x) = L(x) + f(1) \tag{3.2}$$

for all $x \in \mathbb{R}_+$.

Proof. By the condition (3.1), for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$||f(xy) - g(x) - h(y)|| \leq \frac{1}{n} \tag{3.3}$$

for all x, $y > 0$, with $x^k y^s \geq d_n$. By Corollary 2.2, there exists a unique logarithmic function $L_n : \mathbb{R}_+ \to B$ such that

$$||f(x) - L_n(x) - f(1)|| \leq \frac{4}{n} \tag{3.4}$$

for all $x \in \mathbb{R}_+$. Replacing n by m in (3.4) and using the triangle inequality we have

$$||L_n(x) - L_m(x)|| \leq \frac{4}{n} + \frac{4}{m} \leq 8 \tag{3.5}$$

for all $x \in \mathbb{R}_+$. Now, for all $x > 0$ and all rational numbers $r > 0$, we have

$$||L_n(x) - L_m(x)|| = \frac{1}{r}||L_n(x^r) - L_m(x^r)|| \leq \frac{8}{r}. \tag{3.6}$$

Letting $r \to \infty$ in (3.6), we have $L_n = L_m$. Letting $n \to \infty$ in (3.4), we get the result.

Using Corollary 2.5, we obtain the following.

Theorem 3.2. *Let* $s \neq 0$. *Suppose that* $f, g, h : \mathbb{R}_+ \to B$ *satisfy the asymptotic condition*

$$||f(xy) - g(x) - h(y)|| \to 0 \tag{3.7}$$

as $x^k y^s \to \infty$. *Then, there exists a unique logarithmic function* L: $\mathbb{R}_+ \to B$ *such that*

$$g(x) = L(x) + g(1) \tag{3.8}$$

for all $x \in \mathbb{R}_+$.

Using Corollary 2.8, we obtain the following.

Theorem 3.3. *Let* $k \neq 0$. *Suppose that* $f, g, h : \mathbb{R}_+ \to B$ *satisfies the asymptotic condition*

$$||f(xy) - g(x) - h(y)|| \to 0 \tag{3.9}$$

as $x^k y^s \to \infty$. *Then, there exists a unique logarithmic function* $L : \mathbb{R}_+ \to B$ *such that*

$$h(x) = L(x) + h(1) \tag{3.10}$$

for all $x \in \mathbb{R}_+$.

Theorem 3.4. *Let k, $s \neq 0$ and $k \neq s$. Suppose that $f, g, h : \mathbb{R}_+ \to B$ satisfy the asymptotic condition*

$$||f(xy) - g(x) - h(y)|| \to 0 \tag{3.11}$$

as $x^k y^s \to \infty$. Then, there exists a unique logarithmic function $L : \mathbb{R}_+ \to B$ and c_1, $c_2 \in B$ such that

$$f(x) = L(x) + c_1 + c_2,$$
$$g(x) = L(x) + c_1,$$
$$h(x) = L(x) + c_2$$

for all $x \in \mathbb{R}_+$.

Proof. By the condition (3.11), for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$||f(xy) - g(x) - h(y)|| \leq \frac{1}{n} \tag{3.12}$$

for all x, $y > 0$, with $x^k y^s \geq d_n$. By Theorem 2.10, there exists a unique logarithmic function $L_n : \mathbb{R}_+ \to B$ such that

$$||f(x) - L_n(x) - f(1)|| \leq \frac{4}{n}, \tag{3.13}$$

$$||g(x) - L_n(x) - g(1)|| \leq \frac{4}{n}, \tag{3.14}$$

$$||h(x) - L_n(x) - h(1)|| \leq \frac{4}{n} \tag{3.15}$$

for all $x \in \mathbb{R}_+$. Similarly, as in the proof of Theorem 3.1, we have $L_n = L_m$ for all n, $m \in \mathbb{N}$. Letting $n \to \infty$ in (3.13)-(3.15), and using (3.11), we get the result.

4. Stability in L^∞-sense and its asymptotic behavior

Let f, g, h be locally integrable functions on $\mathbb{R}^+$. In this section, we consider the L^∞-version of Hyers-Ulam stability of the inequality

$$||f(xy) - g(x) - h(y)||_{L^\infty(U_{k,s,d})} \leq \varepsilon, \tag{4.1}$$

where $k \neq 0$, $s \neq 0$, $k \neq s$, $d > 0$ are fixed and $U_{k,s,d} = \{(x, y): x^k y^s \geq d\}$. Let ω on $\mathbb{C}$ be a nonnegative infinitely differentiable function satisfying the conditions

$$\mathrm{supp}\omega \subset \{\, x : |x| \leq 1\}$$

and

$$\int \omega(x)dx = 1.$$

Let $\omega_t(x) := t^{-1}\omega(x/t)$, $t > 0$ and f be a locally integrable function. Then, for each $t > 0$, $f * \omega_t(x) = \int f(y)\omega_t(x - y)\, dy$ is a smooth function of $x \in \mathbb{C}$ and $f * \omega_t(x) \to f(x)$ for almost every $x \in \mathbb{C}$ as $t \to 0^+$. Now, we are in a position to prove the Hyers-Ulam stability of the inequality (3.1).

Theorem 4.1. *Let f, g, h be locally integrable functions satisfying* (3.1). *Then, there exist* $c_1, c_2, c_3, a \in \mathbb{C}$ *such that*

$$||f(x) - c_1 - a \ln x||_{L^\infty(\mathbb{R}_+)} \leq 4\varepsilon,$$
$$||g(x) - c_2 - a \ln x||_{L^\infty(\mathbb{R}_+)} \leq 4\varepsilon,$$
$$||h(x) - c_3 - a \ln x||_{L^\infty(\mathbb{R}_+)} \leq 4\varepsilon.$$

Proof. Using the change of variables x by 2^x and y by 2^y in (4.1), we have

$$||f(2^{x+y}) - g(2^x) - h(2^y)||_{L^\infty(U_d)} \leq \varepsilon, \tag{4.2}$$

where $U_d = \{(x, y) : kx + sy \geq \log_2^d := d_1\}$. Now, let

$$F(x) = f(2^x), \quad G(x) = g(2^x), \quad H(x) = h(2^x). \tag{4.3}$$

Then, we have

$$||F(x + y) - G(x) - H(y)||_{L^\infty(U_d)} \leq \varepsilon. \tag{4.4}$$

Convolving $\omega_t(x)\omega_s(y)$ in (4.4) as in the proof of [8, Theorem 3.1], we have

$$|F * \omega_t * \omega_s(x + y) - G * \omega_t(x) - H * \omega_s(y)| \leq \varepsilon \tag{4.5}$$

holds for all $kx + sy \geq d_2 := d_1 + \sqrt{k^2 + s^2}$ and $0 < t < 1$, $0 < s < 1$. Using the same method as in [9, Theorem 4.3], we get the result.

Now, we discuss an asymptotic behavior of the inequality (4.1).

Theorem 4.2. *Let* $f, g, h : \mathbb{R}_+ \to \mathbb{C}$, $j = 1, 2, 3$, *be locally integrable functions satisfying*

$$||f(xy) - g(x) - h(y)||_{L^\infty(U_{k,s,d})} \to 0 \tag{4.6}$$

as $d \to \infty$. *Then, there exist* $a, c_1, c_2, c_3 \in \mathbb{C}$ *such that*

$$||f(x) - c_1 - a \ln x||_{L^\infty(\mathbb{R}_+)} = 0,$$
$$||g(x) - c_2 - a \ln x||_{L^\infty(\mathbb{R}_+)} = 0,$$
$$||h(x) - c_3 - a \ln x||_{L^\infty(\mathbb{R}_+)} = 0.$$

Proof. By the condition (4.6), for any positive integer n, there exists $d_n > 0$ such that

$$||f(xy) - g(x) - h(y)||_{L^\infty(U_{k,s,d_n})} \leq \frac{1}{n} \tag{4.7}$$

for all $x, y \in U_{k,s,d_n}$. Now, by Theorem 4.1, there exist $a, c_1, c_2, c_3 \in \mathbb{C}$ (which are independent of n) such that

$$||f(x) - c_1 - a \ln x||_{L^\infty(\mathbb{R}_+)} \leq \frac{4}{n}, \tag{4.8}$$

$$||g(x) - c_2 - a \ln x——_{L^\infty(\mathbb{R}_+)} \leq \frac{4}{n}, \tag{4.9}$$

$$||h(x) - c_3 - a \ln x||_{L^\infty(\mathbb{R}_+)} \leq \frac{4}{n}. \tag{4.10}$$

Letting $n \to \infty$ in (4.8)-(4.10), we get the result.

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Competing interests
The author declares that they have no competing interests.

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