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Generalized set-valued variational-like inclusions involving $H(\cdot,\cdot)$-η-cocoercive operator in Banach spaces

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Abstract

The aim of this paper is to introduce a new $H(\cdot,\cdot)$-η-cocoercive operator and its resolvent operator. We study some of the properties of $H(\cdot,\cdot)$-η-cocoercive operator and prove the Lipschitz continuity of resolvent operator associated with $H(\cdot,\cdot)$-η-cocoercive operator. Finally, we apply the techniques of resolvent operator to solve a generalized set-valued variational-like inclusion problem in Banach spaces. Our results are new and generalize many known results existing in the literature. Some examples are given in support of definition of $H(\cdot,\cdot)$-η-cocoercive operator.

MSC: 47H19; 49J40

Keywords: $H(\cdot,\cdot)$-η-cocoercive; Lipschitz continuity; algorithm; variational-like inclusion

1 Introduction

Variational inclusion problems are interesting and intensively studied classes of mathematical problems and have wide applications in the field of optimization and control, economics and transportation equilibrium, and engineering sciences, etc., see for example [1–7]. Several authors used the resolvent operator technique to propose and analyze the iterative algorithms for computing the approximate solutions of different kinds of variational inclusions. Fang and Huang [8] studied variational inclusions by introducing a class of generalized monotone operators, called H-monotone operators and defined the associated resolvent operator. Fang and Huang [9] further extended the notion of H-monotone operators to Banach spaces, called H-accretive operators. Recently, Zou and Huang [10] introduced and studied $H(\cdot,\cdot)$-accretive operators and apply them to solve a variational inclusion problem in Banach spaces. After that Xu and Wang [11] introduced and studied $(H(\cdot,\cdot)$-$\eta)$-monotone operators. Very recently, Ahmad *et al.* [12] introduced $H(\cdot,\cdot)$-cocoercive operators and apply them to solve a set-valued variational inclusion problem in Hilbert spaces.

By taking into account the fact that η-cocoercivity is an intermediate concept that lies between η-strong monotonicity and η-monotonicity, in this paper, we introduce $H(\cdot,\cdot)$-η-cocoercive operator and its resolvent operator. We then apply these new concepts to solve a generalized set-valued variational-like inclusion problem in Banach spaces.

2 Preliminaries

Throughout the paper, we assume that X is a real Banach space, X^* is the topological dual space of X, $CB(X)$ is the family of all nonempty closed and bounded subsets of X, $\mathcal{D}(\cdot,\cdot)$ is the Hausdörff metric on $CB(X)$ defined by

$$\mathcal{D}(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\Big\},$$

$\langle\cdot,\cdot\rangle$ is the dual pair between X and X^*.

Definition 2.1 A continuous and strictly increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$ is called a gauge function.

Definition 2.2 Given a gauge function φ, the mapping $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \big\{u^* \in X^* : \langle x, u^*\rangle = \|x\|\,\|u^*\|; \|u^*\| = \varphi(\|x\|)\big\}$$

is called the duality mapping with gauge function φ, where X is any normed space.

In particular if $\varphi(t) = t$, the duality mapping $J = J_\varphi$ is called the normalized duality mapping.

Lemma 2.1 [13] *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\big\langle y, j(x+y)\big\rangle,$$

for all $j(x+y) \in J(x+y)$.

Definition 2.3 Let $A : X \to X$ and $\eta : X \times X \to X$ be two mappings and let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then A is called

(i) η-cocoercive, if there exists a constant $\mu_1 > 0$ such that

$$\big\langle Ax - Ay, j\big(\eta(x,y)\big)\big\rangle \ge \mu_1\|Ax - Ay\|^2, \quad \forall x, y \in X, j\big(\eta(x,y)\big) \in J\big(\eta(x,y)\big);$$

(ii) η-accretive, if

$$\big\langle Ax - Ay, j\big(\eta(x,y)\big)\big\rangle \ge 0, \quad \forall x, y \in X, j\big(\eta(x,y)\big) \in J\big(\eta(x,y)\big);$$

(iii) η-strongly accretive, if there exists a constant $\beta_1 > 0$ such that

$$\big\langle Ax - Ay, j\big(\eta(x,y)\big)\big\rangle \ge \beta_1\|x - y\|^2, \quad \forall x, y \in X, j\big(\eta(x,y)\big) \in J\big(\eta(x,y)\big);$$

(iv) η-relaxed cocoercive, if there exists a constant $\gamma_1 > 0$ such that

$$\big\langle Ax - Ay, j\big(\eta(x,y)\big)\big\rangle \ge (-\gamma_1)\|Ax - Ay\|^2, \quad \forall x, y \in X, j\big(\eta(x,y)\big) \in J\big(\eta(x,y)\big);$$

(v) Lipschitz continuous, if there exists a constant $\lambda_A > 0$ such that

$$\|Ax - Ay\| \le \lambda_A\|x - y\|, \quad \forall x, y \in X;$$

(vi) α-expansive, if there exists a constant $\alpha > 0$ such that

$$\|Ax - Ay\| \geq \alpha \|x - y\|, \quad \forall x, y \in X;$$

(vii) η is said to be Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in X.$$

If $X = H$, a Hilbert space, then definitions (i) to (iv) reduce to the definitions of η-cocoercive, η-monotone, η-strongly monotone and η-relaxed cocoercive, respectively, introduced by Ansari and Yao [3].

If in addition, $\eta(x, y) = x - y$, for all $x, y \in X$, then definitions (i) to (iv) reduce to the definitions of cocoercivity [14], monotonicity, strong monotonicity [15] and relaxed cocoercive, respectively.

Definition 2.4 Let $A, B : X \to X$, $H : X \times X \to X$, $\eta : X \times X \to X$ be three single-valued mappings and $J : X \to 2^{X^*}$ be a normalized duality mapping. Then

(i) $H(A, \cdot)$ is said to be η-cocoercive with respect to A, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), j(\eta(x, y))\rangle \geq \mu \|Ax - Ay\|^2,$$

$\forall x, y \in X$, $j(\eta(x, y)) \in J(\eta(x, y))$;

(ii) $H(\cdot, B)$ is said to be η-relaxed cocoercive with respect to B, if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By), j(\eta(x, y))\rangle \geq (-\gamma)\|Bx - By\|^2,$$

$\forall x, y \in X$, $j(\eta(x, y)) \in J(\eta(x, y))$;

(iii) $H(A, \cdot)$ is said to be r_1-Lipschitz continuous with respect to A, if there exists a constant $r_1 > 0$ such that

$$\|H(Ax, u) - H(Ay, u)\| \leq r_1 \|x - y\|, \quad \forall x, y \in X;$$

(iv) $H(\cdot, B)$ is said to be r_2-Lipschitz continuous with respect to B, if there exists a constant $r_2 > 0$ such that

$$\|H(u, Bx) - H(u, By)\| \leq r_2 \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.5 A set-valued mapping $M : X \to 2^X$ is said to be η-cocoercive, if there exists a constant $\mu_2 > 0$ such that

$$\langle u - v, j(\eta(x, y))\rangle \geq \mu_2 \|u - v\|^2, \quad \forall x, y \in X, u \in Mx, v \in My, j(\eta(x, y)) \in J(\eta(x, y)).$$

Definition 2.6 A mapping $T : X \to CB(X)$ is said to be $\mathcal{D}$-Lipschitz continuous, if there exists a constant $\lambda_T > 0$ such that

$$\mathcal{D}(Tx, Ty) \leq \lambda_T \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.7 Let $T, Q : X \to CB(X)$ be the mappings. A mapping $N : X \times X \to X$ is said to be

(i) Lipschitz continuous in the first argument with respect to T, if there exists a constant $t_1 > 0$ such that

$$\left\| N(w_1, \cdot) - N(w_2, \cdot) \right\| \le t_1 \|w_1 - w_2\|, \quad \forall u_1, u_2 \in X, w_1 \in T(u_1), w_2 \in T(u_2);$$

(ii) Lipschitz continuous in the second argument with respect to Q, if there exists a constant $t_2 > 0$ such that

$$\left\| N(\cdot, v_1) - N(\cdot, v_2) \right\| \le t_2 \|v_1 - v_2\|, \quad \forall u_1, u_2 \in X, v_1 \in Q(u_1), v_2 \in Q(u_2);$$

(iii) η-relaxed Lipschitz in the first argument with respect to T, if there exists a constant $\tau_1 > 0$ such that

$$\left\langle N(w_1, \cdot) - N(w_2, \cdot), j\big(\eta(u_1, u_2)\big) \right\rangle \le -\tau_1 \|u_1 - u_2\|^2,$$

$\forall u_1, u_2 \in X$, $w_1 \in T(u_1)$, $w_2 \in T(u_2)$, $j(\eta(u_1, u_2)) \in J(\eta(u_1, u_2))$;

(iv) η-relaxed Lipschitz in the second argument with respect to Q, if there exists a constant $\tau_2 > 0$ such that

$$\left\langle N(\cdot, v_1) - N(\cdot, v_2), j\big(\eta(u_1, u_2)\big) \right\rangle \le -\tau_2 \|u_1 - u_2\|^2,$$

$\forall u_1, u_2 \in X$, $v_1 \in Q(u_1)$, $v_2 \in Q(u_2)$, $j(\eta(u_1, u_2)) \in J(\eta(u_1, u_2))$.

3 $H(\cdot,\cdot)$-η-cocoercive operator

In this section, we define a new $H(\cdot,\cdot)$-η-cocoercive operator and show some of its properties.

Definition 3.1 Let $A, B : X \to X$, $H : X \times X \to X$, $\eta : X \times X \to X$ be four single-valued mappings. Let $M : X \to 2^X$ be a set-valued mapping. M is said to be $H(\cdot,\cdot)$-η-cocoercive operator with respect to A and B, if M is η-cocoercive and $(H(A, B) + \lambda M)(X) = X$, for every $\lambda > 0$.

Example 3.1 Let $X = \mathbb{R}$ and $A, B : \mathbb{R} \to \mathbb{R}$ be defined by

$$A(x) = x^3 \quad \text{and} \quad B(x) = \sin x, \quad \forall x \in \mathbb{R}.$$

Let $H(A, B), \eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $H(Ax, Bx) = A(x) - B(x)$, $\forall x \in \mathbb{R}$ and

$$\eta(x, y) = \begin{cases} (x - y), & \text{if } |xy| < \frac{1}{3}, \\ e^{|xy|}(x - y), & \text{if } \frac{1}{3} \le |xy| < \frac{1}{2}, \\ 2(x - y), & \text{if } \frac{1}{2} \le |xy|. \end{cases}$$

Let $M : \mathbb{R} \to 2^{\mathbb{R}}$ be a set-valued mapping defined by

$$M(x) = \left\{ x, \frac{x}{2}, \frac{x}{3}, \dots, \frac{x}{n} \right\}, \quad \text{for any fixed natural number } n.$$

Then

(i) $\langle u - v, \eta(x,y)\rangle \geq 2n\|u - v\|^2$, $\forall x, y \in \mathbb{R}$, $u \in M(x)$, $v \in M(y)$.

(ii) $(H(A,B) + \lambda M)(\mathbb{R}) = \mathbb{R}$.

Thus M is $H(\cdot,\cdot)$-η-cocoercive with respect to A and B.

Now, we show that the mapping M need not be $H(\cdot,\cdot)$-η-cocoercive with respect to A and B.

Let $X = C[0,1]$ be space of all real valued continuous functions defined over closed interval $[0,1]$ with the norm

$$\|f\| = \max_{t\in[0,1]}\big|f(t)\big|.$$

Let $A, B : X \to X$ be defined by

$$A(f) = e^f \quad \text{and} \quad B(g) = e^{-g}, \quad \forall f, g \in X.$$

Let $H(A,B) : X \times X \to X$ be defined as

$$H\big(A(f), B(g)\big) = A(f) + B(g), \quad \forall f, g \in X.$$

Suppose that $M = I$, where I is the identity mapping. Then for $\lambda = 1$, we have

$$\big\|\big(H(A,B) + M\big)(f)\big\| = \big\|A(f) + B(f) + f\big\| = \max_{t\in[0,1]}\big|e^{f(t)} + e^{-f(t)} + f(t)\big| > 0,$$

which means that $0 \notin (H(A,B) + M)(X)$ and thus M is not $H(\cdot,\cdot)$-η-cocoercive with respect to A and B.

Proposition 3.1 *Let $H(A,B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A is α-expansive and B is β-Lipschitz continuous and $\mu > \gamma$, $\alpha > \beta$. Let $M : X \to 2^X$ be $H(\cdot,\cdot)$-η-cocoercive operator. If the following inequality*

$$\big\langle x - y, j\big(\eta(u,v)\big)\big\rangle \geq 0$$
$$\textit{holds for all } (v,y) \in \operatorname{Graph}(M), j\big(\eta(u,v)\big) \in J\big(\eta(u,v)\big),$$

then $x \in Mu$, where $\operatorname{Graph}(M) = \{(u,x) \in X \times X : x \in Mu\}$.

Proof Suppose that there exists some (u_0, x_0) such that

$$\big\langle x_0 - y, j\big(\eta(u_0,v)\big)\big\rangle \geq 0 \quad \text{holds for all } (v,y) \in \operatorname{Graph}(M). \tag{3.1}$$

Since M is $H(\cdot,\cdot)$-η-cocoercive with respect to A and B, we know that $(H(A,B) + \lambda M)(X) = X$ holds for every $\lambda > 0$ and so there exists $(u_1, x_1) \in \operatorname{Graph}(M)$ such that

$$H(Au_1, Bu_1) + \lambda x_1 = H(Au_0, Bu_0) + \lambda x_0 \in X. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&0 \le \langle \lambda x_0 + H(Au_0, Bu_0) - \lambda x_1 - H(Au_1, Bu_1), j(\eta(u_0, u_1))\rangle, \\
&0 \le \lambda\langle x_0 - x_1, j(\eta(u_0, u_1))\rangle = -\langle H(Au_0, Bu_0) - H(Au_1, Bu_1), j(\eta(u_0, u_1))\rangle \\
&\quad = -\langle H(Au_0, Bu_0) - H(Au_1, Bu_0), j(\eta(u_0, u_1))\rangle \\
&\qquad - \langle H(Au_1, Bu_0) - H(Au_1, Bu_1), j(\eta(u_0, u_1))\rangle \\
&\quad \le -\mu\|Au_0 - Au_1\|^2 + \gamma\|Bu_0 - Bu_1\|^2 \\
&\quad \le -\mu\alpha^2\|u_0 - u_1\|^2 + \gamma\beta^2\|u_0 - u_1\|^2 \\
&\quad = -(\mu\alpha^2 - \gamma\beta^2)\|u_0 - u_1\|^2 \le 0,
\end{aligned}$$

which gives $u_1 = u_0$, since $\mu > \gamma$ and $\alpha > \beta$. By (3.2), we have $x_1 = x_0$. Hence $(u_0, x_0) = (u_1, x_1) \in \mathrm{Graph}(M)$ and so $x_0 \in Mu_0$. □

Theorem 3.1 *Let $H(A,B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-η-cocoercive operator with respect to A and B. Then the operator $(H(A,B) + \lambda M)^{-1}$ is single-valued.*

Proof For any given $u \in X$, let $x, y \in (H(A,B) + \lambda M)^{-1}(u)$. It follows that

$$\begin{aligned}
&-H(Ax, Bx) + u \in \lambda Mx \quad \text{and} \\
&-H(Ay, By) + u \in \lambda My.
\end{aligned}$$

As M is η-cocoercive (thus η-accretive), we have

$$\begin{aligned}
0 &\le \langle -H(Ax, Bx) + u - (-H(Ay, By) + u), j(\eta(x, y))\rangle \\
&= -\langle H(Ax, Bx) - H(Ay, By), j(\eta(x, y))\rangle \\
&= -\langle H(Ax, Bx) - H(Ay, Bx) + H(Ay, Bx) - H(Ay, By), j(\eta(x, y))\rangle \\
&= -\langle H(Ax, Bx) - H(Ay, Bx), j(\eta(x, y))\rangle \\
&\quad - \langle H(Ay, Bx) - H(Ay, By), j(\eta(x, y))\rangle.
\end{aligned} \tag{3.3}$$

Since H is η-cocoercive with respect to A with constant μ and η-relaxed cocoercive with respect to B with constant γ, A is α-expansive and B is β-Lipschitz continuous, thus (3.3) becomes

$$0 \le -\mu\alpha^2\|x - y\|^2 + \gamma\beta^2\|x - y\|^2 = -(\mu\alpha^2 - \gamma\beta^2)\|x - y\|^2 \le 0, \tag{3.4}$$

since $\mu > \gamma$ and $\alpha > \beta$. Thus, we have $x = y$ and so $(H(A,B) + \lambda M)^{-1}$ is single-valued. □

Definition 3.2 Let $H(A,B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A is α-expansive and B is

β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let M be $H(\cdot,\cdot)$-η-cocoercive operator with respect to A and B. Then the resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)-\eta} : X \to X$ is defined by

$$R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) = \big(H(A,B) + \lambda M\big)^{-1}(u), \quad \forall u \in X. \tag{3.5}$$

Now, we show the Lipschitz continuity of the resolvent operator defined by (3.5) and calculate its Lipschitz constant.

Theorem 3.2 *Let $H(A,B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A is α-expansive, B is β-Lipschitz continuous and η is τ-Lipschitz continuous and $\mu > \gamma$, $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-η-cocoercive operator with respect to A and B. Then the resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)-\eta} : X \to X$ is $\frac{\tau}{\mu\alpha^2-\gamma\beta^2}$-Lipschitz continuous, that is*

$$\big\| R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) - R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\big\| \le \frac{\tau}{\mu\alpha^2 - \gamma\beta^2}\|u - v\|, \quad \forall u,v \in X.$$

Proof Let u and v be any given points in X. It follows from (3.5) that

$$\begin{aligned} &R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) = \big(H(A,B) + \lambda M\big)^{-1}(u), \quad \text{and} \\ &R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v) = \big(H(A,B) + \lambda M\big)^{-1}(v). \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{1}{\lambda}\Big(u - H\big(A\big(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u)\big), B\big(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u)\big)\big)\Big) \in M\big(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u)\big), \quad \text{and} \\ &\frac{1}{\lambda}\Big(v - H\big(A\big(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\big), B\big(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\big)\big)\Big) \in M\big(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\big). \end{aligned}$$

For the sake of convenience, we take

$$Pu = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) \quad \text{and} \quad Pv = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v).$$

Since M is η-cocoercive (thus η-accretive), we have

$$\begin{aligned} &\frac{1}{\lambda}\big\langle u - H\big(A(Pu), B(Pu)\big) - \big(v - H\big(A(Pv), B(Pv)\big)\big), j\big(\eta(Pu,Pv)\big)\big\rangle \ge 0, \\ &\quad = \frac{1}{\lambda}\big\langle u - v - \big(H\big(A(Pu), B(Pu)\big) - H\big(A(Pv), B(Pv)\big)\big), j\big(\eta(Pu,Pv)\big)\big\rangle \ge 0, \end{aligned}$$

which implies that

$$\big\langle u - v, j\big(\eta(Pu,Pv)\big)\big\rangle \ge \big\langle H\big(A(Pu), B(Pu)\big) - H\big(A(Pv), B(Pv)\big), j\big(\eta(Pu,Pv)\big)\big\rangle.$$

It follows that

$$\begin{aligned} \|u - v\|\,\big\|\eta(Pu,Pv)\big\| &\ge \big\langle u - v, j\big(\eta(Pu,Pv)\big)\big\rangle \\ &\ge \big\langle H\big(A(Pu), B(Pu)\big) - H\big(A(Pv), B(Pv)\big), j\big(\eta(Pu,Pv)\big)\big\rangle \end{aligned}$$

$$
\begin{aligned}
&= \big\langle H\big(A(Pu),B(Pu)\big) - H\big(A(Pv),B(Pu)\big) \\
&\quad + H\big(A(Pv),B(Pu)\big) - H\big(A(Pv),B(Pv)\big), j\big(\eta(Pu,Pv)\big)\big\rangle \\
&= \big\langle H\big(A(Pu),B(Pu)\big) - H\big(A(Pv),B(Pu)\big), j\big(\eta(Pu,Pv)\big)\big\rangle \\
&\quad + \big\langle H\big(A(Pv),B(Pu)\big) - H\big(A(Pv),B(Pv)\big), j\big(\eta(Pu,Pv)\big)\big\rangle \\
&\geq \mu \big\|A(Pu) - A(Pv)\big\|^2 - \gamma \big\|B(Pu) - B(Pv)\big\|^2 \\
&\geq \mu\alpha^2 \|Pu - Pv\|^2 - \gamma\beta^2 \|Pu - Pv\|^2
\end{aligned}
$$

and so

$$\|u - v\| \big\|\eta(Pu,Pv)\big\| \geq \big(\mu\alpha^2 - \gamma\beta^2\big)\|Pu - Pv\|^2$$

i.e.

$$
\begin{aligned}
&\big(\mu\alpha^2 - \gamma\beta^2\big)\|Pu - Pv\|^2 \leq \|u - v\| \big\|\eta(Pu,Pv)\big\| \leq \tau \|u - v\| \|Pu - Pv\| \\
&\|Pu - Pv\| \leq \frac{\tau}{\mu\alpha^2 - \gamma\beta^2} \|u - v\|, \quad \forall u, v \in X,
\end{aligned}
$$

i.e.

$$\big\|R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) - R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\big\| \leq \frac{\tau}{\mu\alpha^2 - \gamma\beta^2} \|u - v\|, \quad \forall u, v \in X.$$

This completes the proof. □

4 Existence result for generalized set-valued variational-like inclusion problem

In this section, we apply $H(\cdot,\cdot)$-η-cocoercive operators to find a solution of generalized set-valued variational-like inclusion problem.

Let $N : X \times X \to X$, $\eta : X \times X \to X$, $H : X \times X \to X$, $A, B : X \to X$ be the single-valued mappings and $T, Q : X \to CB(X)$, $M : X \to 2^X$ be the set-valued mappings such that M is $H(\cdot,\cdot)$-η-cocoercive with respect to A and B. Then we consider the following problem. Find $u \in X$, $w \in T(u)$, $v \in Q(u)$ such that

$$0 \in N(w, v) + M(u). \tag{4.1}$$

We call problem (4.1), a generalized set-valued variational-like inclusion problem.

Let X be a Hilbert space. If $Q \equiv 0$ and $\eta(u,v) = u - v$, $\forall u, v \in X$ and $N(\cdot,\cdot) = T(\cdot)$, where $T : X \to CB(X)$ be a set-valued mapping. Then problem (4.1) reduces to the problem of finding $u \in X$, $w \in T(u)$ such that

$$0 \in w + M(u). \tag{4.2}$$

A problem similar to (4.2) is studied by Ahmad *et al.* [12] by applying $H(\cdot,\cdot)$-cocoercive operators.

If $H(\cdot,\cdot) = H(\cdot)$ and M is H-accretive mapping, then problem (4.1) is introduced and studied by Chang *et al.* [5]. It is clear that for suitable choices of operators involved in the formulation of (4.1), one can obtain many variational inclusions studied in literature.

Lemma 4.1 *The* (u,w,v), *where* $u \in X$, $w \in T(u)$, $v \in Q(u)$ *is a solution of problem* (4.1) *if and only if* (u,w,v) *is a solution of the following equation*

$$u = R^{H(\cdot,\cdot)-\eta}_{\lambda,M}\big[H\big(A(u),B(u)\big) - \lambda N(w,v)\big], \tag{4.3}$$

where $\lambda > 0$ *is a constant.*

Proof Proof is straightforward by the use of definition of resolvent operator. □

Based on Lemma 4.1, we define the following algorithm for approximating a solution of generalized set-valued variational-like inclusion problem (4.1).

Algorithm 4.1 For any $u_0 \in X$, $w_0 \in T(u_0)$, $v_0 \in Q(u_0)$, compute the sequences $\{u_n\}$, $\{w_n\}$, and $\{v_n\}$ by the following iterative scheme:

$$u_{n+1} = R^{H(\cdot,\cdot)-\eta}_{\lambda,M}\big[H\big(A(u_n),B(u_n)\big) - \lambda N(w_n,v_n)\big], \tag{4.4}$$

$$w_n \in T(u_n), \|w_n - w_{n+1}\| \le \mathcal{D}\big(T(u_n),T(u_{n+1})\big), \tag{4.5}$$

$$v_n \in Q(u_n), \|v_n - v_{n+1}\| \le \mathcal{D}\big(Q(u_n),Q(u_{n+1})\big), \tag{4.6}$$

for all $n = 0,1,2,\ldots$ and $\lambda > 0$ is a constant.

Theorem 4.1 *Let* X *be a real Banach space. Let* $A,B : X \to X$, $H : X \times X \to X$, $N : X \times X \to X$, $\eta : X \times X \to X$ *be the single-valued mappings. Suppose that* $T,Q : X \to CB(X)$ *and* $M : X \to 2^X$ *are set-valued mappings such that* M *is* $H(\cdot,\cdot)$-η*-cocoercive operator with respect to* A *and* B. *Let*

(i) T *is* $\mathcal{D}$*-Lipschitz continuous with constant* λ_T *and* Q *is* $\mathcal{D}$*-Lipschitz continuous with constant* λ_Q;

(ii) $H(A,B)$ *is* η*-cocoercive with respect to* A *with constant* $\mu > 0$ *and* η*-relaxed cocoercive with respect to* B *with constant* $\gamma > 0$;

(iii) A *is* α*-expansive and* B *is* β*-Lipschitz continuous*;

(iv) $H(A,B)$ *is* r_1*-Lipschitz continuous with respect to* A *and* r_2*-Lipschitz continuous with respect to* B;

(v) N *is* t_1*-Lipschitz continuous with respect to* T *in the first argument and* t_2*-Lipschitz continuous with respect to* Q *in the second argument*;

(vi) η *is* τ*-Lipschitz continuous*;

(vii) N *is* η*-relaxed Lipschitz continuous with respect to* T *in the first argument and* η*-relaxed Lipschitz continuous with respect to* Q *in the second argument with constants* τ_1 *and* τ_2, *respectively.*

Suppose that the following condition is satisfied:

$$\sqrt{r_1^2 - 2\lambda(t_1\lambda_T + t_2\lambda_Q)\big[r_1 + \lambda(t_1\lambda_T + t_2\lambda_Q) + \tau\big] - 2\lambda(\tau_1 + \tau_2)} < \frac{\mu\alpha^2 - \gamma\beta^2}{\tau} - r^2, \tag{4.7}$$

$$\mu\alpha^2 - \gamma\beta^2 > \tau r^2, \mu > \gamma, \alpha > \beta.$$

Then there exist $u \in X$, $w \in T(u)$ *and* $v \in Q(u)$ *satisfying the generalized set-valued variational-like inclusion problem* (4.1) *and the iterative sequences* $\{u_n\}$, $\{w_n\}$ *and* $\{v_n\}$ *generated by Algorithm* 4.1 *converge strongly to* u, w *and* v, *respectively.*

Proof Since T is $\mathcal{D}$-Lipschitz continuous with constant λ_T and Q is $\mathcal{D}$-Lipschitz continuous with constant λ_Q, it follows from Algorithm 4.1 that

$$\|w_n - w_{n+1}\| \le \mathcal{D}\big(T(u_n), T(u_{n+1})\big) \le \lambda_T \|u_n - u_{n+1}\|, \quad \text{and} \tag{4.8}$$

$$\|v_n - v_{n+1}\| \le \mathcal{D}\big(Q(u_n), Q(u_{n+1})\big) \le \lambda_Q \|u_n - u_{n+1}\|. \tag{4.9}$$

By using Algorithm 4.1 and Lipschitz continuity of resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)-\eta}$, we have

$$\begin{aligned}\|u_{n+1} - u_n\| &= \big\|R_{\lambda,M}^{H(\cdot,\cdot)-\eta}\big[H(Au_n, Bu_n) - \lambda N(w_n, v_n)\big] \\ &\quad - R_{\lambda,M}^{H(\cdot,\cdot)-\eta}\big[H(Au_{n-1}, Bu_{n-1}) - \lambda N(w_{n-1}, v_{n-1})\big]\big\| \\ &\le \frac{\tau}{\mu\alpha^2 - \gamma\beta^2}\big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_{n-1}) \\ &\quad - \lambda\big[N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big]\big\| \\ &\le \frac{\tau}{\mu\alpha^2 - \gamma\beta^2}\big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n) \\ &\quad - \lambda\big[N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big]\big\| \\ &\quad + \frac{\tau}{\mu\alpha^2 - \gamma\beta^2}\big\|H(Au_{n-1}, Bu_n) - H(Au_{n-1}, Bu_{n-1})\big\|.\end{aligned} \tag{4.10}$$

Using Lemma 2.1, we have

$$\begin{aligned}&\big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n) - \lambda\big[N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big]\big\|^2 \\ &\quad\le \big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n)\big\|^2 - 2\lambda\big\langle N(w_n, v_n) - N(w_{n-1}, v_{n-1}), \\ &\qquad j\big[H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n) - \lambda\big(N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big)\big]\big\rangle \\ &\quad= \big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n)\big\|^2 - 2\lambda\big\langle N(w_n, v_n) - N(w_{n-1}, v_{n-1}), \\ &\qquad j\big[H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n) - \lambda\big(N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big)\big] \\ &\qquad + j\big(\eta(u_n, u_{n-1})\big)\big\rangle + 2\lambda\big\langle N(w_n, v_n) - N(w_{n-1}, v_{n-1}), j\big(\eta(u_n, u_{n-1})\big)\big\rangle \\ &\quad\le \big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n)\big\|^2 - 2\lambda\big\|N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big\| \\ &\qquad \times \big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n)\big\| + \lambda\big\|N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big\| \\ &\qquad + \big\|\eta(u_n, u_{n-1})\big\|] + 2\lambda\big\langle N(w_n, v_n) - N(w_{n-1}, v_{n-1}), j\big(\eta(u_n, u_{n-1})\big)\big\rangle.\end{aligned} \tag{4.11}$$

As $H(\cdot,\cdot)$ is r_1-Lipschitz continuous with respect to A, we have

$$\big\|H(Au_n, Bu_n) - H(Au_{n-1}, Bu_n)\big\| \le r_1\|u_n - u_{n-1}\|. \tag{4.12}$$

Since N is t_1-Lipschitz continuous with respect to T in the first argument and t_2-Lipschitz continuous with respect to Q in the second argument and T is λ_T-Lipschitz continuous and Q is λ_Q-Lipschitz continuous, we have

$$\begin{aligned}\big\|N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big\| &= \big\|N(w_n, v_n) - N(w_{n-1}, v_n) \\ &\quad + N(w_{n-1}, v_n) - N(w_{n-1}, v_{n-1})\big\|\end{aligned}$$

$$\begin{aligned}
&\le \left\| N(w_n, v_n) - N(w_{n-1}, v_n) \right\| \\
&\quad + \left\| N(w_{n-1}, v_n) - N(w_{n-1}, v_{n-1}) \right\| \\
&\le t_1 \|w_n - w_{n-1}\| + t_2 \|v_n - v_{n-1}\| \\
&\le t_1 \mathcal{D}\big(T(u_n), T(u_{n-1})\big) + t_2 \mathcal{D}\big(Q(u_n), Q(u_{n-1})\big) \\
&\le t_1 \lambda_T \|u_n - u_{n-1}\| + t_2 \lambda_Q \|u_n - u_{n-1}\| \\
&= (t_1 \lambda_T + t_2 \lambda_Q) \|u_n - u_{n-1}\|.
\end{aligned} \tag{4.13}$$

As η is τ-Lipschitz continuous, we have

$$\left\| \eta(u_n, u_{n-1}) \right\| \le \tau \|u_n - u_{n-1}\|. \tag{4.14}$$

Since N is η-relaxed Lipschitz continuous with respect to T and η-relaxed Lipschitz continuous with respect to Q in first and second arguments with constants τ_1 and τ_2, respectively, we have

$$\begin{aligned}
&\big\langle N(w_n, v_n) - N(w_{n-1}, v_{n-1}), j\big(\eta(u_n, u_{n-1})\big)\big\rangle \\
&\quad = \big\langle N(w_n, v_n) - N(w_{n-1}, v_n), j\big(\eta(u_n, u_{n-1})\big)\big\rangle \\
&\qquad + \big\langle N(w_{n-1}, v_n) - N(w_{n-1}, v_{n-1}), j\big(\eta(u_n, u_{n-1})\big)\big\rangle \\
&\quad \le -\tau_1 \|u_n - u_{n-1}\|^2 - \tau_2 \|u_n - u_{n-1}\|^2 \\
&\quad \le -(\tau_1 + \tau_2) \|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.15}$$

Using (4.12)-(4.15), (4.11) becomes

$$\begin{aligned}
&\left\| H(Au_n, Bu_n) - H(Au_{n-1}, Bu_{n-1}) - \lambda\big[N(w_n, v_n) - N(w_{n-1}, v_{n-1})\big] \right\|^2 \\
&\quad \le r_1^2 \|u_n - u_{n-1}\|^2 - 2\lambda(t_1\lambda_T + t_2\lambda_Q)\|u_n - u_{n-1}\| \\
&\qquad \times \big[r_1\|u_n - u_{n-1}\| + \lambda(t_1\lambda_T + t_2\lambda_Q)\|u_n - u_{n-1}\| + \tau\|u_n - u_{n-1}\|\big] \\
&\qquad + 2\lambda\big(-(\tau_1 + \tau_2)\big)\|u_n - u_{n-1}\|^2 \\
&\quad = r_1^2 \|u_n - u_{n-1}\|^2 - 2\lambda(t_1\lambda_T + t_2\lambda_Q)\|u_n - u_{n-1}\| \\
&\qquad \times \big[\big\{r_1 + \lambda(t_1\lambda_T + t_2\lambda_Q) + \tau\big\}\|u_n - u_{n-1}\|\big] - 2\lambda(\tau_1 + \tau_2)\|u_n - u_{n-1}\|^2 \\
&\quad = r_1^2 \|u_n - u_{n-1}\|^2 - 2\lambda(t_1\lambda_T + t_2\lambda_Q)\big[r_1 + \lambda(t_1\lambda_T + t_2\lambda_Q) + \tau\big] \\
&\qquad \times \|u_n - u_{n-1}\|^2 - 2\lambda(\tau_1 + \tau_2)\|u_n - u_{n-1}\|^2 \\
&\quad = \big[r_1^2 - 2\lambda(t_1\lambda_T + t_2\lambda_Q)\big[r_1 + \lambda(t_1\lambda_T + t_2\lambda_Q) + \tau\big] - 2\lambda(\tau_1 + \tau_2)\big]\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.16}$$

Using r_2-Lipschitz continuity of $H(\cdot,\cdot)$ with respect to B and (4.16), (4.10) becomes

$$\|u_{n+1} - u_n\| \le \theta \|u_n - u_{n-1}\|, \tag{4.17}$$

where

$$\theta = \frac{\tau}{\mu\alpha^2 - \gamma\beta^2}\sqrt{\theta_1} + \frac{\tau r_2}{\mu\alpha^2 - \gamma\beta^2}$$

and

$$\theta_1 = \left[r_1^2 - 2\lambda(t_1\lambda_T + t_2\lambda_Q)\left[r_1 + \lambda(t_1\lambda_T + t_2\lambda_Q) + \tau\right] - 2\lambda(\tau_1 + \tau_2)\right]. \tag{4.18}$$

From (4.7), it follows that $\theta < 1$, so $\{u_n\}$ is a Cauchy sequence in X, thus there exists a $u \in X$ such that $u_n \to u$ as $n \to \infty$. Also from (4.8) and (4.9), it follows that $\{w_n\}$ and $\{v_n\}$ are also Cauchy sequences in X, thus there exist w and v in X such that $w_n \to w$, $v_n \to v$ as $n \to \infty$. By the continuity of $R_{\lambda,M}^{H(\cdot,\cdot)-\eta}$, H, A, B, η, N, T and Q and from (4.4) of Algorithm 4.1, it follows that

$$\begin{aligned} u_{n+1} &= R_{\lambda,M}^{H(\cdot,\cdot)-\eta}\left[H\left(A(u_n), B(u_n)\right) - \lambda N(w_n, v_n)\right] && (4.19)\\ \to \quad u &= R_{\lambda,M}^{H(\cdot,\cdot)-\eta}\left[H\left(A(u), B(u)\right) - \lambda N(w, v)\right] \quad (n \to \infty). && (4.20)\end{aligned}$$

Now, we prove that $w \in T(u)$. In fact, since $w_n \in T(u_n)$, we have

$$\begin{aligned} d\left(w, T(u)\right) &\leq \|w - w_n\| + d\left(w_n, T(u)\right)\\ &\leq \|w - w_n\| + \mathcal{D}\left(T(u_n), T(u)\right)\\ &\leq \|w - w_n\| + \lambda_T\|u_n - u\| \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

which means that $d(w, T(u)) = 0$. Since $T(u) \in CB(X)$, it follows that $w \in T(u)$. Similarly, we can show that $v \in Q(u)$. By Lemma 4.1, we conclude that (u, w, v) is a solution of generalized set-valued variational-like inclusion problem (4.1). This completes the proof. □

From Theorem 4.1, we can obtain the following theorem which is similar to the Theorem 4.3 of Ahmad *et al.* [12].

Theorem 4.2 *Let X be a Hilbert space. Let A, B, H and T be the same as in Theorem* 4.1 *and $M : X \to 2^X$ be the set-valued, $H(\cdot,\cdot)$-cocoercive mapping with respect to A and B. Assume that*

(i) *T is $\mathcal{D}$-Lipschitz continuous with constant λ_T*;
(ii) *$H(A, B)$ is cocoercive with respect to A with constant $\mu > 0$ and relaxed cocoercive with respect to B with constant $\gamma > 0$*;
(iii) *A is α-expansive*;
(iv) *B is β-Lipschitz continuous*;
(v) *$H(A, B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B*;
(vi) $(r_1 + r_2) < [(\mu\alpha^2 - \gamma\beta^2) - \lambda]$; $\mu > \gamma$, $\alpha > \beta$.

Then the problem (4.2) *admits a solution (u, w) with $u \in X$ and $w \in T(u)$ and the iterative sequences $\{u_n\}$ and $\{w_n\}$ converge strongly to u and w, respectively.*

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