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Perturbed projection and iterative algorithms for a system of general regularized nonconvex variational inequalities

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Abstract

The purpose of this paper is to introduce a new system of general nonlinear regularized nonconvex variational inequalities and verify the equivalence between the proposed system and fixed point problems. By using the equivalent formulation, the existence and uniqueness theorems for solutions of the system are established. Applying two nearly uniformly Lipschitzian mappings S_1 and S_2 and using the equivalent alternative formulation, we suggest and analyze a new perturbed *p*-step projection iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q} = (\mathcal{S}_1, \mathcal{S}_2)$ which is the unique solution of the system of general nonlinear regularized nonconvex variational inequalities. We also discuss the convergence analysis of the proposed iterative algorithm under some suitable conditions.
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1 Introduction

The theory of variational inequalities introduced by Stampacchia [1] in the early 1960s have enjoyed vigorous growth for the last 30 years. Variational inequality theory describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics, and engineering sciences. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative (see [2-7]). One of the most interesting and important problems in variational inequality theory is the development of an efficient numerical method. There is a substantial number of numerical methods including projection method and its variant forms, Wiener-Holf (normal) equations, auxiliary principle, and descent framework for solving variational inequalities and complementarity problems. For the applications, physical formulations, numerical methods and other aspects of variational inequalities (see [1-52] and the references therein). Projection method and its variant forms represent important tool for finding the approximate solution of various types of variational and quasi-variational inequalities, the origin of which can be traced back to Lions and Stampacchia [21]. The projection type methods were developed in 1970s and 1980s. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed point problem using the concept of projection. This alternative formulation enables us to suggest some iterative methods for computing the approximate solution (see [36,42,43]).

It is worth mentioning that most of the results regarding the existence and iterative approximation of solutions to variational inequality problems have been investigated and considered so far to the case where the underlying set is a convex set. Recently, the concept of convex set has been generalized in many directions, which has potential and important applications in various fields. It is well known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases, for more details (see for example [11,12,17,45,46,30,31]). In recent years, Bounkhel et al. [17], Cho et al. [40], Moudafi [24], Noor [25,26] and Pang et al. [30] have considered variational inequalities and equilibrium problems in the context of uniformly prox-regular sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique.

On the other hand, related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems. Motivated and inspired by the problems, Noor and Huang [27] considered the problem of finding the common element of the set of the solutions of variational inequalities and the set of the fixed points of the nonexpansive mappings. It is well known that every nonexpansive mapping is a Lipschitzian mapping. Lipschitzian mappings have been generalized by various authors. Sahu [50] introduced and investigated nearly uniformly Lipschitzian mappings as a generalization of Lipschitzian mappings.

Motivated and inspired by the recent results in this area, in this paper, we introduce and consider a new system of general nonlinear regularized nonconvex variational inequalities involving four different nonlinear operators. We first establish the equivalence between the system of general nonlinear regularized nonconvex variational inequalities and fixed point problems and, by the equivalent formulation, we discuss the existence and uniqueness of solution of the proposed system. By using two nearly uniformly Lipschitzian mappings S_1 and S_2 and the equivalent alternative formulation, we suggest and analyze a new perturbed p-step iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q} = (S_1, S_2)$ which is the unique solution of the system of general nonlinear regularized nonconvex variational inequalities. We also discuss the convergence analysis of the proposed iterative algorithm under some suitable conditions.

2 Preliminaries

Throughout this paper, let $\mathcal{H}$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $||\cdot||$ and K be a nonempty convex subset of $\mathcal{H}$. We denote by $d_K(\cdot)$ or $d(\cdot, K)$ the usual distance function to the subset K, i.e., $d_K(u) = \inf_{v\in K} ||u - v||$ Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis [31,44-46].

Definition 2.1. Let $u \in \mathcal{H}$is a point not lying in K. A point $v \in K$ is called a *closest point* or a *projection* of u onto K if, $d_K(u) = ||u - v||$. The set of all such closest points is denoted by $P_K(u)$, i.e.,

$$P_K(u) := \{v \in K : d_K(u) = ||u - v||\}.$$

Definition 2.2. The *proximal normal cone* of K at a point $u \in K$ is given by

$$N_K^P(u) := \{\xi \in \mathcal{H} : u \in P_K(u + \alpha\xi), \quad \text{for some} \quad \alpha > 0\}.$$

Clarke et al. [45], in Proposition 1.1.5, give a characterization of $N_K^P(u)$ as the following:

Lemma 2.3. *Let K be a nonempty closed subset in $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that $\langle \xi, v - u\rangle \leq \alpha\, ||v - u||^2$ for all $v \in K$.*

The above inequality is called the *proximal normal inequality*. The special case in which K is closed and convex is an important one. In Proposition 1.1.10 of [45], the authors give the following characterization of the proximal normal cone the closed and convex subset $K \subset \mathcal{H}$:

Lemma 2.4. *Let K be a nonempty, closed and convex subset in $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if $\langle \xi, v - u\rangle \leq 0$ for all $v \in K$.*

Definition 2.5. Let X be a real Banach space and $f: X \to \mathbb{R}$ be Lipschitz with constant τ near a given point $x \in X$; that is, for some $\varepsilon > 0$, we have $|f(y) - f(z)| \leq \tau || y - z||$ for all $y, z \in B(x; \varepsilon)$, where $B(x; \varepsilon)$ denotes the open ball of radius $\varepsilon > 0$ and centered at x. The *generalized directional derivative* of f at x in the direction v, denoted as $f^\circ(x; v)$, is defined as follows:

$$f^\circ(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t is a positive scalar.

The generalized directional derivative defined earlier can be used to develop a notion of tangency that does not require K to be smooth or convex.

Definition 2.6. The *tangent cone* $T_K(x)$ to K at a point x in K is defined as follows:

$$T_K(x) := \{v \in \mathcal{H} : d_K^\circ(x; v) = 0\}.$$

Having defined a tangent cone, the likely candidate for the normal cone is the one obtained from $T_K(x)$ by polarity. Accordingly, we define the *normal cone* of K at x by polarity with $T_K(x)$ as follows:

$$N_K(x) := \{\xi : \langle \xi, v\rangle \leq 0, \quad \forall v \in T_K(x)\}.$$

Definition 2.7. The *Clarke normal cone*, denoted by $N_K^C(x)$, is given by $N_K^C(x) = \overline{co}[N_K^P(x)]$, where $\overline{co}[S]$ means the closure of the convex hull of S.

It is clear that one always has $N_K^P(x) \subseteq N_K^C(x)$. The converse is not true in general. Note that $N_K^C(x)$ is always closed and convex cone, whereas $N_K^P(x)$ is always convex, but may not be closed (see [31,44,45]).

In 1995, Clarke et al. [46], introduced and studied a new class of nonconvex sets, called proximally smooth sets; subsequently Poliquin et al. [31] investigated the aforementioned sets, under the name of uniformly prox-regular sets. These have been successfully used in many nonconvex applications in areas, such as optimizations, economic models, dynamical systems, differential inclusions, etc. For such applications, see [14-16,18]. This class seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumptions on K. We take the following characterization proved in [46] as a definition of this class. We point out that the original definition was given in terms of the differentiability of the distance function (see [46]).

Definition 2.8. For any $r \in (0, +\infty]$, a subset K_r of $\mathcal{H}$ is called *normalized uniformly prox-regular* (*or uniformly r-prox-regular* [46]) if every nonzero proximal normal to K_r can be realized by an r-ball.

This means that, for all $\bar{x} \in K_r$ and $0 \neq \xi \in N^P_{K_r}(\bar{x})$ with $||\xi|| = 1$,

$$\langle \xi, x - \bar{x} \rangle \leq \frac{1}{2r}||x - \bar{x}||^2, \ \forall x \in K_r.$$

Obviously, the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets (see [19,46]).

Lemma 2.9. [46]*A closed set $K \subseteq \mathcal{H}$is convex if and only if it is proximally smooth of radius r for all r* >0.

If $r = +\infty$, then, in view of Definition 2.8 and Lemma 2.9, the uniform r-prox-regularity of K_r is equivalent to the convexity of K_r, which makes this class of great importance. For the case of that $r = +\infty$, we set $K_r = K$.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. The proof of this results can be found in [31,46].

Proposition 2.10. *Let r >0 and K_r be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set $U(r) = \{u \in \mathcal{H} \ : \ 0 < d_{K_r}(u) < r\}$. Then the following statements hold:*

(1) For all $x \in U(r)$, one has $P_{K_r}(x) \neq \emptyset$;
(2) For all $r' \in (0, r)$, $P_{K_r}$is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $U(r') = \{u \in \mathcal{H} \ : \ 0 < d_{K_r}(u) < r'\}$;
(3) The proximal normal cone is closed as a set-valued mapping.

As a direct consequent of part (c) of Proposition 2.10, we have $N^C_{K_r}(x) = N^P_{K_r}(x)$. Therefore, we define $N_{K_r}(x) := N^C_{K_r}(x) = N^P_{K_r}(x)$ for such a class of sets.

In order to make clear the concept of r-prox-regular sets, we state the following concrete example: The union of two disjoint intervals $[a, b]$ and $[c, d]$ is r-prox-regular with $r = \frac{c-b}{2}$. The finite union of disjoint intervals is also r-prox-regular and r depends on the distances between the intervals.

Definition 2.11. Let $T : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $g : \mathcal{H} \to \mathcal{H}$ be two single-valued operators. Then the operator T is said to be:

(1) *monotone* in the first variable if, for all $x, y \in \mathcal{H}$,

$$\langle T(x, \ u) - T(y, \ v), x - y \rangle \geq 0, \quad \forall u, v \in \mathcal{H};$$

(2) *r-strongly monotone* in the first variable if there exists a constant r >0 such that, for all $x, y \in \mathcal{H}$,

$$\langle T(x, \ u) - T(y, \ v), x - y \rangle \geq r||x - y||^2, \ \forall u, v \in \mathcal{H};$$

(3) *κ-strongly monotone* with respect to g in the first variable if there exists a constant $\kappa > 0$ such that, for all $x, y \in \mathcal{H}$,

$$\langle T(x,\ u) - T(y,\ v), g(x) - g(y)\rangle \geq \kappa ||x - y||^2,\ \forall u, v \in \mathcal{H};$$

(4) *(θ, v)-relaxed cocoercive* in the first variable if there exist constants θ, $v > 0$ such that, for all $x, y \in \mathcal{H}$,

$$\langle T(x,\ u) - T(y,\ v), x - y\rangle \geq -\theta ||T(x,\ u) - T(y,\ v)||^2 +\ v||x - y||^2,\ \forall u, v \in \mathcal{H};$$

(5) *μ-Lipschitz continuous* in the first variable if there exists a constant $\mu > 0$ such that, for all $x, y \in \mathcal{H}$,

$$||T(x,\ u) - T(y,\ v)|| \ \leq \mu || x - y ||, \quad \forall u,\ v \in \mathcal{H}.$$

Definition 2.12. A nonlinear single-valued operator $g : \mathcal{H} \to \mathcal{H}$ is said to be γ-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$||g(x) - g(y)|| \ \leq \gamma ||x - y||, \quad \forall x,\ y \in \mathcal{H}.$$

In the next definitions, several generalizations of the nonexpansive mappings which have been introduced by various authors in recent years are stated.

Definition 2.13. A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be:

(1) *nonexpansive* if

$$||Tx - Ty|| \ \leq \ ||x - y||, \quad \forall x, y \in \mathcal{H};$$

(2) *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$||Tx - Ty|| \ \leq L|| x - y||, \quad \forall x, y \in \mathcal{H};$$

(3) *generalized Lipschitzian* if there exists a constant $L > 0$ such that

$$||Tx - Ty|| \ \leq L\left(|| x - y|| + 1\right), \quad \forall x, y \in \mathcal{H};$$

(4) *generalized (L, M)-Lipschitzian* [50] if there exist two constants L, $M > 0$ such that

$$||Tx - Ty|| \ \leq L\left(||x - y|| + M\right), \quad \forall x, y \in \mathcal{H};$$

(5) *asymptotically nonexpansive* [48] if there exists a sequence $\{k_n\} \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,

$$||T^n x - T^n y|| \ \leq k_n ||x - y||, \quad \forall x, y \in \mathcal{H};$$

(6) *pointwise asymptotically nonexpansive* [49] if, for each $n \geq 1$,

$$||T^n x - T^n y|| \ \leq \alpha_n(x) || x - y||, \quad \forall x, y \in \mathcal{H},$$

where $\alpha_n \to 1$ pointwise on X;

(7) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that, for each $n \in \mathbb{N}$,

$$||T^n x - T^n y|| \ \leq L||x - y||, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.14. [50] A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$is said to be:

(1) *nearly Lipschitzian* with respect to the sequence $\{a_n\}$ if, for each n$\in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$||T^n x - T^n y|| \le k_n \left(||x - y|| + a_n\right), \quad \forall x, y \in \mathcal{H}, \tag{2.1}$$

where $\{a_n\}$ is a fix sequence in $[0, \infty)$ with $a_n \to 0$ as $n \to \infty$.

For an arbitrary, but fixed $n \in \mathbb{N}$, the infimum of constants k_n in (2.1) is called *nearly Lipschitz constant*, which is denoted by $\eta(T^n)$. Notice that

$$\eta(T^n) = \sup\left\{\frac{||T^n x - T^n y||}{||x - y|| + a_n} : x, y \in \mathcal{H}, x \neq y\right\}.$$

A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be:

(2) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$, that is,

$$||T^n x - T^n y|| \le ||x - y|| + a_n, \quad \forall x, y \in \mathcal{H};$$

(3) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$, in other words, $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$;

(4) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$, in other words, $k_n = L$ for all n$\in \mathbb{N}$.

Remark 2.15. It should be pointed that

(1) Every nonexpansive mapping is a asymptotically non-expansive mapping and every asymptotically non-expansive mapping is a pointwise asymptotically nonexpansive mapping. In addition, the class of Lipschitzian mappings properly includes the class of pointwise asymptotically nonexpansive mappings.

(2) It is obvious that every Lipschitzian mapping is a generalized Lipschitzian mapping. Furthermore, every mapping with a bounded range is a generalized Lipschitzian mapping. It is easy to see that the class of generalized (L, M)-Lipschitzian mappings is more general than the class of generalized Lipschitzian mappings.

(3) Clearly, the class of nearly uniformly L-Lipschitzian mappings properly includes the class of generalized (L, M)-Lipschitzian mappings and that of uniformly L-Lipschitzian mappings. Note that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

Now, we present some new examples to investigate relations between these mappings.

Example 2.16. Let $\mathcal{H} = \mathbb{R}$ and define a mapping $T : \mathcal{H} \to \mathcal{H}$ as follow:

$$T(x) = \begin{cases} \frac{1}{\gamma}, & x \in [0,\ \gamma], \\ 0, & x \in (-\infty,\ 0) \cup (\gamma,\ \infty), \end{cases}$$

where $\gamma > 1$ is a constant real number. Evidently, the mapping T is discontinuous at the points $x = 0, \gamma$. Since every Lipschitzian mapping is continuous it follows that T is not Lipschitzian. For each $n \in \mathbb{N}$, take $a_n = \frac{1}{\gamma^n}$. Then

$$|Tx - Ty| \le |x - y| + \frac{1}{\gamma} = |x - y| + a_1, \quad \forall x,\ y \in \mathbb{R}.$$

Since $T^n z = \frac{1}{\gamma}$, for all $z \in \mathbb{R}$ and $n \ge 2$, it follows that, for all $x, y \in \mathbb{R}$ and $n \ge 2$,

$$|T^n x - T^n y| \le |x - y| + \frac{1}{\gamma^n} = |x - y| + a_n.$$

Hence T is a nearly nonexpansive mapping with respect to the sequence $\{a_n\} = \{\frac{1}{\gamma^n}\}$.

The following example shows that the nearly uniformly L-Lipschitzian mappings are not necessarily continuous.

Example 2.17. Let $\mathcal{H} = [0, b]$, where $b \in (0, 1]$ is an arbitrary constant real number, and let the self-mapping T of $\mathcal{H}$ be defined as below:

$$T(x) = \begin{cases} \gamma x, & x \in [0,\ b), \\ 0, & x = b, \end{cases}$$

where $\gamma \in (0, 1)$ is also an arbitrary constant real number. It is plain that the mapping T, is discontinuous in the point b. Hence T is not a Lipschitzian mapping. Take, for each $n \in \mathbb{N}$, $a_n = \gamma^{n-1}$. Then, for all $n \in \mathbb{N}$ and $x, y \in [0, b)$, we have

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - \gamma^n y| = \gamma^n |x - y| \le \gamma^n |x - y| + \gamma^n \\ &\le \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

If $x \in [0, b)$ and $y = b$, then, for each $n \in \mathbb{N}$, we have $T^n x = \gamma^n x$ and $T^n y = 0$. Since $0 < |x - y| \le b \le 1$, it follows that, for all $n \in \mathbb{N}$,

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - 0| = \gamma^n x \le \gamma^n b \le \gamma^n < \gamma^n |x - y| + \gamma^n \\ &\le \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

Hence T is a nearly uniformly γ-Lipschitzian mapping with respect to the sequence $\{a_n\} = \{\gamma^{n-1}\}$.

Obviously, every nearly nonexpansive mapping is a nearly uniformly Lipschitzian mapping. In the following example, we show that the class of nearly uniformly Lipschitzian mappings properly includes the class of nearly nonexpansive mappings.

Example 2.18. Let $\mathcal{H} = \mathbb{R}$ and the self-mapping T of $\mathcal{H}$ be defined as follow:

$$T(x) = \begin{cases} \frac{1}{2}, & x \in [0, 1) \cup \{2\}, \\ 2, & x = 1, \\ 0, & x \in (-\infty,\ 0) \cup (1, 2) \cup (2,\ +\infty). \end{cases}$$

Evidently, the mapping T is discontinuous in the points $x = 0, 1, 2$. Hence, T is not a Lipschitzian mapping. Take, for each $n \in \mathbb{N}$, $a_n = \frac{1}{2^n}$. Then T is not a nearly nonexpansive mapping with respect to the sequence $\{\frac{1}{2^n}\}$ because, taking $x = 1$ and $y = \frac{1}{2}$, we have $Tx = 2$, $Ty = \frac{1}{2}$and

$$|Tx - Ty| > |x - y| + \frac{1}{2} = |x - y| + a_1.$$

However,

$$|Tx - Ty| \le 4(|x - y| + \frac{1}{2}) = 4(|x - y| + a_1), \quad \forall x, y \in \mathbb{R}$$

and for all $n \geq 2$,

$$|T^n x - T^n y| \leq 4(|x - y| + \frac{1}{2^n}) = 4(|x - y| + a_n), \quad \forall x,\ y \in \mathbb{R},$$

since $T^n z = \frac{1}{2}$ for all $z \in \mathbb{R}$ and $n \geq 2$. Hence, for each $L \geq 4$, T is a nearly uniformly L-Lipschitzian mapping with respect to the sequence $\{\frac{1}{2^n}\}$.

It is clear that every uniformly L-Lipschitzian mapping is a nearly uniformly L-Lipschitzian mapping. In the next example, we show that the class nearly uniformly L-Lipschitzian mappings properly includes the class of uniformly L-Lipschitzian mappings.

Example 2.19. Let $\mathcal{H} = \mathbb{R}$ and the self-mapping T of $\mathcal{H}$ be defined the same as in Example 2.18. Then T is not a uniformly 4-Lipschitzian mapping. If $x = 1$ and $y \in (1, \frac{3}{2})$, then we have $|T \times - Ty| > 4|x - y|$ because of $0 < |x - y| < \frac{1}{2}$. But, in view of Example 2.18, T is a nearly uniformly 4-Lipschitzian mapping.

The following example shows that the class of generalized Lipschitzian mappings properly includes the class of Lipschitzian mappings and that of mappings with bounded range.

Example 2.20. [35] Let $\mathcal{H} = \mathbb{R}$ and $T : \mathcal{H} \to \mathcal{H}$ be a mapping defined by

$$T(x) = \begin{cases} x - 1, & x \in (-\infty,\ -1), \\ x - \sqrt{1 - (x+1)^2}, & x \in [-1, 0), \\ x + \sqrt{1 - (x-1)^2}, & x \in [0, 1], \\ x + 1, & x \in (1,\ \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

3 System of general regularized nonconvex variational inequalities

In this section, we introduce a new system of general nonlinear regularized nonconvex variational inequalities and establish the existence and uniqueness theorem for a solution of the mentioned system.

Let K_r be an uniformly r-prox-regular subset of $\mathcal{H}$ and let $T_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$and $g_i : \mathcal{H} \to \mathcal{H} (i = 1, 2)$ be four nonlinear single-valued operators. For any given constants ρ, η >0, we consider the problem of finding $(x^*, y^*) \in K_r \times K_r$ such that

$$\begin{cases} \langle \rho T_1(y^*,\ x^*) + x^* - g_1(y^*),\ g_1(x) - x^* \rangle + \frac{1}{2r}||g_1(x) - x^*||^2 \geq 0,\ \forall x \in \mathcal{H}, \\ \langle \eta T_2(x^*,\ y^*) + y^* - g_2(x^*),\ g_2(x) - y^* \rangle + \frac{1}{2r}||g_2(x) - y^*||^2 \geq 0,\ \forall x \in \mathcal{H}. \end{cases} \tag{3.1}$$

The problem (3.1) is called a system of *general nonlinear regularized nonconvex variational inequalities* involving four different nonlinear operators (SGNRNVID).

Some special cases of the system (3.1) can be found in [1,26,28,32-34,47] and the references therein.

Lemma 3.1. *Let T_i, $g_i (i = 1, 2)$, ρ, η be the same as in the system* (3.1) *and suppose further that $g_i(\mathcal{H}) = K_r$for each $i = 1, 2$. Then the system* (3.1) *is equivalent to that of finding $(x^*, y^*) \in K_r \times K_r$ such that*

$$\begin{cases} 0 \in \rho T_1(y^*,\ x^*) + x^* - g_1(y^*) + N^P_{K_r}(x^*), \\ 0 \in \eta T_2(x^*,\ y^*) + y^* - g_2(x^*) + N^P_{K_r}(y^*), \end{cases} \tag{3.2}$$

where $N^P_{K_r}(s)$*denotes the P-normal cone of* K_r *at s in the sense of nonconvex analysis.*

Proof. Let $(x^*, y^*) \lfloor K_r \times K_r$ be a solution of the system (3.1). If $\rho T_1(y^*, x^*) + x^* -g_1(y^*) = 0$, because the vector zero always belongs to any normal cone, then $0 \in \rho T_1(y^*, x^*) + x^* - g_1(y^*) + N^P_{K_r}(x^*)$. If $\rho T_1(y^*, x^*) + x^* - g_1(y^*) \neq 0$, then, for all $x \in \mathcal{H}$, we have

$$\langle -(\rho T_1(y^*, x^*) + x^* - g_1(y^*)), g_1(x) - x^* \rangle \leq \frac{1}{2r} ||g_1(x) - x^*||^2.$$

Since $g_1(\mathcal{H}) = K_r$, by Lemma 2.3, it follows that

$$-(\rho T_1(y^*, x^*) + x^* - g_1(y^*)) \in N^P_{K_r}(x^*),$$

and so

$$0 \in \rho T_1(y^*, x^*) + x^* - g_1(y^*) + N^P_{K_r}(x^*),$$

Similarly, one can establish that

$$0 \in \eta T_2(x^*, y^*) + y^* - g_2(x^*) + N^P_{K_r}(y^*).$$

Conversely, if $(x^*, y^*) \in K_r \times K_r$ is a solution of the system (3.2), then it follows from Definition 2.8 that (x^*, y^*) is a solution of the system (3.1). This completes the proof.

The problem (3.2) is called the *general nonlinear nonconvex variational inclusions system* associated with the system of general nonlinear regularized nonconvex variational inequalities (3.1).

Now, we prove the existence and uniqueness theorem for a solution of the system of general nonlinear regularized nonconvex variational inequalities (3.1). For this end, we need the following lemma in which by using the projection operator technique, we verify the equivalence between the system of general nonlinear regularized nonconvex variational inequalities (3.1) and a fixed point problem.

Lemma 3.2. *Let* T_i, $g_i (i = 1, 2)$, ρ *and* η *be the same as in the system* (3.1) *and suppose further that* $g_i(\mathcal{H}) = K_r$*for each* $i = 1, 2$. *Then* $(x^*, y^*) \in K_r \times K_r$ *is a solution of the system* (3.1) *if and only if*

$$\begin{cases} x^* = P_{K_r}(g_1(y^*) - \rho T_1(y^*, x^*)), \\ y^* = P_{K_r}(g_2(x^*) - \eta T_2(x^*, y^*)) \end{cases} \tag{3.3}$$

provided that $\rho < \frac{r'}{1+||T_1(y^*,x^*)||}$*and* $\eta < \frac{r'}{1+||T_2(x^*,y^*)||}$, *where* $r' \in (0, r)$ *and* P_{K_r}*is the projection of* $\mathcal{H}$*onto* Kr.

Proof. Let $(xi^*, y^*) \lfloor K_r \times K_r$ be a solution of the system (3.1). Since $g_1(y^*)$, $g_2(x^*) \lfloor K_r$, $\rho < \frac{r'}{1+||T_1(y^*,x^*)||}$ *and* $\eta < \frac{r'}{1+||T_2(x^*,y^*)||}$, it is easy to check that two the points $g_1(y^*) - \rho T_1(y^*, x^*)$ and $g_2(x^*) - \eta T_2(x^*, y^*)$ belong to $U(r')$. Therefore, the r-prox regularity of K_r implies that two the sets $P_{K_r}(g_1(y^*) - \rho T_1(y^*, x^*))$ and $P_{K_r}(g_2(x^*) - \eta T_2(x^*, y^*))$ are nonempty and singleton, that is, the equations (3.3) are well defined. By using Lemma 3.1, we have

$$\begin{cases} 0 \in \rho T_1(y^*,\ x^*) + x^* - g_1(y^*) + N^P_{K_r}(x^*), \\ 0 \in \eta T_2(x^*,\ y^*) + y^* - g_2(x^*) + N^P_{K_r}(y^*), \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} g_1(y^*) - \rho T_1(y^*,\ x^*) \in x^* + N^P_{K_r}(x^*) = (I + N^P_{K_r})(x^*), \\ g_2(x^*) - \eta T_2(x^*,\ y^*) \in y^* + N^P_{K_r}(y^*) = (I + N^P_{K_r})(y^*), \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} x^* = P_{K_r}(g_1(y^*) - \rho T_1(y^*,\ x^*)), \\ y^* = P_{K_r}(g_2(x^*) - \eta T_2(x^*,\ y^*)), \end{cases}$$

where I is identity operator and we have used the well-known fact that $P_{K_r} = (I + N^P_{K_r})^{-1}$. This completes the proof.

Theorem 3.3. *Let T_i, $g_i (i = 1, 2)$, ρ and η be the same as in the system* (3.1) *such that $g_i(\mathcal{H}) = K_r$ for each $i = 1, 2$. Suppose that for each $i = 1, 2$, T_i is π_i-strongly monotone with respect to g_i and σ_i -Lipschitz continuous in the first variable and g_i is δ_i-Lipschitz continuous. If the constants ρ and η satisfy the following conditions:*

$$\rho < \frac{r'}{1 + ||T_1(y, x)||}, \quad \eta < \frac{r'}{1 + ||T_2(x, y)||}, \quad \forall x, y \in \mathcal{H}, \tag{3.4}$$

$$\begin{cases} |\rho - \frac{\pi_1}{\sigma_1^2}| < \frac{\sqrt{r^2\pi_1^2 - \sigma_1^2(r^2\delta_1^2 - (r - r')^2)}}{r\sigma_1^2}, \\ |\eta - \frac{\pi_2}{\sigma_2^2}| < \frac{\sqrt{r^2\pi_2^2 - \sigma_2^2(r^2\delta_2^2 - (r - r')^2)}}{r\sigma_2^2}, \\ r\pi_i > \sigma_i\sqrt{r^2\delta_i^2 - (r - r')^2}, \\ r\delta_i > r - r',\ (i = 1, 2), \end{cases} \tag{3.5}$$

where $r' \in (0, r)$, then the system (3.1) *admits a unique solution.*

Proof. Define the mappings $\psi, \phi : \mathcal{H} \times \mathcal{H} \to K_r$ by

$$\begin{aligned} &\psi(x,\ y) = P_{K_r}(g_1(y) - \rho T_1(y,\ x)), \\ &\phi(x,\ y) = P_{K_r}(g_2(x) - \eta T_2(x,\ y)) \end{aligned} \tag{3.6}$$

for all $(x, y) \in \mathcal{H} \times \mathcal{H}$, respectively. Since $g_1(y), g_2(x) \in K_r$ for $x, y \in \mathcal{H}$, all easily check that the mappings ψ and ϕ are well defined. Define $||\cdot||_*$ on $\mathcal{H} \times \mathcal{H}$ by

$$||(x,\ y)||_* = ||x|| + ||y||, \quad \forall (x,\ y) \in \mathcal{H} \times \mathcal{H}.$$

It is obvious that $(\mathcal{H} \times \mathcal{H},\ ||\cdot||_*)$ is a Hilbert space. In addition, define a mapping $F : K_r \times K_r \to K_r \times K_r$ as follows:

$$F(x,\ y) = (\psi(x,\ y),\ \phi(x,\ y)), \quad \forall (x,\ y) \in K_r \times K_r. \tag{3.7}$$

Now, we verify that F is a contraction mapping. Indeed, let $(x, y), (\hat{x}, \hat{y}) \in K_r \times K_r$ be given. Since two the points $g_1(y) - \rho T_1(y, x)$ and $g_1(\hat{y}) - \rho T_1(\hat{y}, \hat{x})$ belong to $U(r')$, by using Proposition 2.10, we have

$$\begin{aligned} ||\psi(x,\ y) - \psi(\hat{x}, \hat{y})|| &= ||P_{K_r}(g_1(y) - \rho T_1(y,\ x)) - P_{K_r}(g_1(\hat{y}) - \rho T_1(\hat{y}, \hat{x}))|| \\ &\le \frac{r}{r - r'}||g_1(y) - g_1(\hat{y}) - \rho(T_1(y,\ x) - T_1(\hat{y}, \hat{x}))||. \end{aligned} \tag{3.8}$$

Since T_1 is π_1-strongly monotone with respect to g_1 and σ_1-Lipschitz continuous in the first variable and g_1 is δ_1-Lipschitz continuous, we conclude that

$$\begin{aligned}
&\|g_1(y) - g_1(\hat{y}) - \rho(T_1(y,\ x) - T_1(\hat{y}, \hat{x}))\|^2 \\
&\quad = \|g_1(y) - g_1(\hat{y})\|^2 - 2\rho\langle T_1(y,\ x) - T_1(\hat{y}, \hat{x}),\ g_1(y) - g_1(\hat{y})\rangle + \rho^2\|T_1(y,\ x) - T_1(\hat{y}, \hat{x})\|^2 \\
&\quad \le (\delta_1^2 - 2\rho\pi_1 + \rho^2\sigma_1^2)\|y - \hat{y}\|^2.
\end{aligned} \tag{3.9}$$

Substituting (3.9) in (3.8), it follows that

$$\|\psi(x,\ y) - \psi(\hat{x}, \hat{y})\| \ \le \theta\|y - \hat{y}\|, \tag{3.10}$$

where

$$\theta = \frac{r}{r - r'}\sqrt{\delta_1^2 - 2\rho\pi_1 + \rho^2\sigma_1^2}.$$

Since T_2 is π_2-strongly monotone with respect to g_2 and σ_2-Lipschitz continuous in the first variable and g_2 is δ_2-Lipschitz continuous, by the similar way given in the proofs of (3.8)-(3.10), we can prove that

$$\|\phi(x,\ y) - \phi(\hat{x}, \hat{y})\| \ \le \omega\|x - \hat{x}\|, \tag{3.11}$$

where

$$\omega = \frac{r}{r - r'}\sqrt{\delta_2^2 - 2\eta\pi_2 + \eta^2\sigma_2^2}.$$

It follows from (3.7), (3.10) and (3.11) that

$$\begin{aligned}
\|F(x,\ y) - F(\hat{x}, \hat{y})\|_* &= \|\psi(x,\ y) - \psi(\hat{x}, \hat{y})\| + \|\phi(x,\ y) - \phi(\hat{x}, \hat{y})\| \\
&\le \vartheta\|(x,\ y) - (\hat{x}, \hat{y})\|_*,
\end{aligned} \tag{3.12}$$

where $\vartheta = \max\{\theta, \omega\}$. By the condition (3.5), we note that $0 \le \vartheta < 1$ and so (3.12) guarantees that F is a contraction mapping. According to Banach's fixed point theorem, there exists a unique point $(x^*, y^*) \in K_r \times K_r$ such that $F(x^*, y^*) = (x^*, y^*)$. From (3.6) and (3.7), we conclude that $x^* = P_{K_r}(g_1(y^*) - \rho T_1(y^*, x^*))$ and $y^* = P_{K_r}(g_2(x^*) - \eta T_2(x^*, y^*))$. Now, Lemma 3.2 guarantees that $(x^*, y^*) \in K_r \times K_r$ is a solution of the system (3.1). This completes the proof.

4 Perturbed projection and iterative algorithms

In this section, by applying two nearly uniformly Lipschitzian mappings S_1 and S_2 and using the equivalent alternative formulation (3.3), we suggest and analyze a new perturbed p-step projection iterative algorithm with mixed errors for finding an element of the set of the fixed points of $\mathcal{Q} = (S_1, S_2)$ which is the unique solution of the system of general nonlinear regularized nonconvex variational inequalities (3.1).

Let $S_1 : K_r \to K_r$ be a nearly uniformly L_1-Lipschitzian mapping with the sequence $\{a_n\}_{n=1}^{\infty}$ and $S_2 : K_r \to K_r$ be a nearly uniformly L_2-Lipschitzian mapping with the sequence $\{b_n\}_{n=1}^{\infty}$. We define the self-mapping $\mathcal{Q}$ of $K_r \times K_r$ as follows:

$$\mathcal{Q}(x, y) = (S_1x, S_2y), \quad \forall x, y \in K_r. \tag{4.1}$$

Then $\mathcal{Q} = (S_1, S_2) : K_r \times K_r \to K_r \times K_r$ is a nearly uniformly $\max\{L_1, L_2\}$-Lipschitzian mapping with the sequence $\{a_n + b_n\}_{n=1}$ with respect to the norm $\|\cdot\|_*$ in $\mathcal{H} \times \mathcal{H}$. Since, for any (x, y), (x', y') $K_r \times K_r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|\mathcal{Q}^n(x,\ y) - \mathcal{Q}^n(x',\ y')\|_* \\
&\quad = \|(S_1^n x,\ S_2^n y) - (S_1^n x',\ S_2^n y')\|_* = \ \|(S_1^n x - S_1^n x',\ S_2^n y - S_2^n y')\|_* \\
&\quad = \|S_1^n x - S_1^n x'\| + \|S_2^n y - S_2^n y'\| \ \le L_1(\|x - x'\| + a_n) + L_2(\|y - y'\| + b_n) \\
&\quad \le \max\{L_1,\ L_2\}(\|x - x'\| + \|y - y'\| + a_n + b_n) = \max\{L_1,\ L_2\}(\|(x,\ y) - (x',\ y')\|_* + a_n + b_n).
\end{aligned}$$

We denote the sets of all the fixed points of $S_i (i = 1, 2)$ and $\mathcal{Q}$ by $\mathrm{Fix}(S_i)$ and $Fix(\mathcal{Q})$, respectively, and the set of all the solutions of the system (3.1) by SGNRNVID(K_r, T_i, g_i, $i = 1, 2$). In view of (4.1), for any $(x, y) \in K_r \times K_r$, $(x, y) \in Fix(\mathcal{Q})$ if and only if $x \in \mathrm{Fix}(S_1)$ and $y \in \mathrm{Fix}(S_2)$, that is, $\mathrm{Fix}(\mathcal{Q}) = \mathrm{Fix}(S_1, S_2) = \mathrm{Fix}(S_1) \times \mathrm{Fix}(S_2)$.

We now characterize the given problem. Let the operators T_i, $g_i (i = 1,2)$ and the constants ρ, η be the same as in the system (3.1) and, further, suppose that $g_i(\mathcal{H}) = K_r$ for each $i = 1, 2$. If $(x^*, y^*) \in \mathrm{Fix}(\mathcal{Q}) \cap$ SGNRNVID (K_r, T_i, g_i, $i = 1, 2$), $\rho < \frac{r'}{1+\|T_1(y^*,x^*)\|}$ and $\eta < \frac{r'}{1+\|T_2(y^*,x^*)\|}$, where $r' \in (0, r)$, then by using Lemma 3.2, it is easy to see that for each $n \in \mathbb{N}$,

$$\begin{cases} x^* = S_1^n x^* = P_{K_r}(g_1(y^*) - \rho T_1(y^*,\ x^*)) = S_1^n P_{K_r}(g_1(y^*) - \rho T_1(y^*,\ x^*)), \\ y^* = S_2^n y^* = P_{K_r}(g_2(x^*) - \eta T_2(x^*,\ y^*)) = S_2^n P_{K_r}(g_2(x^*) - \eta T_2(x^*,\ y^*)). \end{cases} \tag{4.2}$$

The fixed point formulation (4.2) enables us to suggest the following iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $Q = (S_1, S_2)$ which is the unique solution of the system of the general nonlinear regularized nonconvex variational inequalities (3.1).

Algorithm 4.1. Let T_i, $g_i (i = 1, 2)$, ρ and η be the same as in the system (3.1) such that $g_i(\mathcal{H}) = K_r$ for each $i = 1, 2$. Furthermore, let the constants ρ and η satisfy the condition (3.4). For an arbitrary chosen initial point $(x_1, y_1) \in \mathcal{H} \times \mathcal{H}$, compute the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $\mathcal{H} \times \mathcal{H}$ in the following way:

$$\begin{cases} x_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})x_n + \alpha_{n,1}(S_1^n P_{K_r}(\Phi(v_{n,1}, \nu_{n,1})) + e_{n,1}) + \beta_{n,1} j_{n,1} + r_{n,1}, \\ y_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})y_n + \alpha_{n,1}(S_2^n P_{K_r}(\Psi(v_{n,1}, \nu_{n,1})) + l_{n,1}) + \beta_{n,1} s_{n,1} + k_{n,1}, \\ v_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})x_n + \alpha_{n,i+1}(S_1^n P_{K_r}(\Phi(v_{n,i+1}, \nu_{n,i+1})) + e_{n,i+1}) + \beta_{n,i+1} j_{n,i+1} + r_{n,i+1}, \\ \nu_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})y_n + \alpha_{n,i+1}(S_2^n P_{K_r}(\Psi(v_{n,i+1}, \nu_{n,i+1})) + l_{n,i+1}) + \beta_{n,i+1} s_{n,i+1} + k_{n,i+1}, \\ \cdots \\ v_{n,p-1} = (1 - \alpha_{n,p} - \beta_{n,p})x_n + \alpha_{n,p}(S_1^n P_{K_r}(\Phi(x_n, y_n)) + e_{n,p}) + \beta_{n,p} j_{n,p} + r_{n,p}, \\ \nu_{n,p-1} = (1 - \alpha_{n,p} - -\beta_{n,p})y_n + \alpha_{n,p}(S_2^n P_{K_r}(\Psi(x_n, y_n)) + l_{n,p}) + \beta_{n,p} s_{n,p} + k_{n,p}, \\ i = 1, 2, \ldots, p-2, \end{cases} \tag{4.3}$$

where

$$\begin{cases} \Phi(v_{n,i},\ \mathcal{V}_{n,i}) = g_1(\mathcal{V}_{n,i}) - \rho T_1(\mathcal{V}_{n,i},\ v_{n,i}), \\ \Psi(v_{n,i},\ \mathcal{V}_{n,i}) = g_2(v_{n,i}) - \eta T_2(v_{n,i},\ \mathcal{V}_{n,i}), \\ \Phi(x_n,\ y_n) = g_1(y_n) - \rho T_1(y_n,\ x_n), \\ \Psi(x_n,\ y_n) = g_2(x_n) - \eta T_2(x_n,\ y_n), \\ i = 1, 2,\ \ldots, p-2, \end{cases}$$

S_1, S_2: $K_r \to K_r$, are two nearly uniformly Lipschitzian mappings, $\{\alpha_{n,i}\}_{n=1}^{\infty}$, $\{\beta_{n,i}\}_{n=1}^{\infty} (i = 1, 2, \ldots, p)$ are $2p$ sequences in interval [0, 1] such that $\alpha_{n,i} + \beta_{n,i} \leq 1$, $\alpha_{n,i} + \beta_{n,i} \leq 1$, $\sum_{n=1}^{\infty} \beta_{n,i} < \infty$ and $\{e_{n,i}\}_{n=1}^{\infty}$, $\{l_{n,i}\}_{n=1}^{\infty}$, $\{j_{n,i}\}_{n=1}^{\infty}$, $\{s_{n,i}\}_{n=1}^{\infty}$, $\{k_{n,i}\}_{n=1}^{\infty} (i = 1,\ 2,\ \ldots\ , p)$, $\{k_{n,i}\}_{n=1}^{\infty} (i = 1,\ 2,\ \ldots\ , p)$ are $6p$ sequences in $\mathcal{H}$ to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions: $\{j_{n,i}\}_{n=1}^{\infty}$, $\{s_{n,i}\}_{n=1}^{\infty} (i = 1,\ 2,\ \ldots\ , p)$ are $2p$ bounded sequences in $\mathcal{H}$ and $\{e_{n,i}\}_{n=1}^{\infty}$, $\{l_{n,i}\}_{n=1}^{\infty}$, $\{r_{n,i}\}_{n=1}^{\infty}$, $\{k_{n,i}\}_{n=1}^{\infty} (i = 1,\ 2,\ \ldots\ , p)$ are $4p$ sequencs in $\mathcal{H}$ such that

$$\begin{cases} e_{n,i} = e'_{n,i} + e''_{n,i}, \quad l_{n,i} = l'_{n,i} + l''_{n,i}, \quad n \in \mathbb{N}, \quad i = 1, 2,\ \ldots, p, \\ \lim_{n\to\infty} \|(e'_{n,i},\ l'_{n,i})\|_* = 0,\ i = 1, 2,\ \ldots, p, \\ \sum_{n=1}^{\infty} \|(e''_{n,i},\ l''_{n,i})\|_* < \infty,\ \sum_{n=1}^{\infty} \|(r_{n,i},\ k_{n,i})\|_* < \infty, \quad i = 1, 2,\ \ldots, p. \end{cases} \tag{4.4}$$

If $S_i \equiv I$ for each $i = 1, 2$, then Algorithm 4.1 reduces to the following iterative algorithm for solving the system (3.1).

Algorithm 4.2. Assume that T_i, $g_i (i = 1, 2)$, ρ and η are the same as in Algorithm 4.1. Moreover, let the constants ρ and η satisfy the condition (3.4). For an arbitrary chosen initial point $(x_1, y_1) \in \mathcal{H} \times \mathcal{H}$, compute the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $\mathcal{H} \times \mathcal{H}$ in the following way:

$$\begin{cases} x_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})x_n + \alpha_{n,1}(P_{K_r}(\Phi(v_{n,1}, \nu_{n,1})) + \mathrm{e}_{n,1}) + \beta_{n,1}j_{n,1} + r_{n,1}, \\ y_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})y_n + \alpha_{n,1}(P_{K_r}(\Psi(v_{n,1}, \nu_{n,1})) + l_{n,1}) + \beta_{n,1}s_{n,1} + k_{n,1}, \\ v_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})x_n + \alpha_{n,i+1}(P_{K_r}(\Phi(v_{n,i+1}, \nu_{n,\mathrm{i}+1})) + e_{n,\mathrm{i}+1}) + \beta_{n,i+1}j_{n,i+1} + r_{n,i+1}, \\ \mathcal{V}_{n,\mathrm{i}} = (1-\alpha_{n,\mathrm{i}+1} - -\beta_{n,\mathrm{i}+1})y_n + \alpha_{n,\mathrm{i}+1}(\mathrm{P}_{\mathrm{K_r}}(\Psi(v_{n,\mathrm{i}+1}, \nu_{n,\mathrm{i}+1})) + l_{n,\mathrm{i}+1}) + \beta_{n,\mathrm{i}+1}s_{n,\mathrm{i}+1} + k_{n,\mathrm{i}+1}, \\ \cdots \\ v_{n,p-1} = (1 - \alpha_{n,p} - \beta_{n,p})x_n + \alpha_{n,p}(P_{K_r}(\Phi(x_n, y_n)) + e_{n,p}) + \beta_{n,p}j_{n,p} + r_{n,p}, \\ \nu_{n,p-1} = (1-\alpha_{n,p} - -\beta_{n,p})y_n + \alpha_{n,p}(P_{K_r}(\Psi(x_n, y_n)) + l_{n,p}) + \beta_{n,p}s_{n,p} + k_{n,p}, \\ i = 1, 2, \ldots, p-2, \end{cases}$$

where $\Phi(v_{n,i}, v_{n,i})$, $\Psi(v_{n,i}, v_{n,i})(i = 1, 2, ..., p - 2)$, $\Phi(x_n, y_n)$, $\Psi(x_n, y_n)$, $\{\alpha_{n,i}\}_{n=1}^{\infty}$, $\{\beta_{n,i}\}_{n=1}^{\infty}$, $\{l_{n,i}\}_{n=1}^{\infty}$, $\{l_{n,i}\}_{n=1}^{\infty}$, $\{j_{n,i}\}_{n=1}^{\infty}$, $\{s_{n,i}\}_{n=1}^{\infty}$, $\{r_{n,i}\}_{n=1}^{\infty}$, $\{k_{n,i}\}_{n=1}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1.

Algorithm 4.3. Let T_i, $g_i (i = 1, 2)$, ρ and η be the same as in Algorithm 4.1. Furthermore, suppose that the constants ρ and η satisfy the condition (3.4). For an arbitrary chosen initial point $(x_1, y_1) \in \mathcal{H} \times \mathcal{H}$, compute the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $\mathcal{H} \times \mathcal{H}$, by the following iterative processes:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S_1^n P_{K_r}(g_1(y_n) - \rho T_1(y_n, x_n)), \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n S_2^n P_{K_r}(g_2(x_n) - \eta T_2(x_n, y_n)), \end{cases}$$

where S_1, S_2 are the same as in Algorithm 4.1 and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0, 1] satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $S_i \equiv I$ for each $i = 1, 2$, then Algorithm 4.3 reduces to the following iterative algorithm for solving the system (3.1).

Algorithm 4.4. Let T_i, $g_i (i = 1, 2)$, ρ and η be the same as in Algorithm 4.1. Furthermore, assume that the constants ρ and η satisfy the condition (3.4). For an arbitrary chosen initial point $(x_1, y_1) \in \mathcal{H} \times \mathcal{H}$, compute the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $\mathcal{H} \times \mathcal{H}$ in the following way:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_{K_r}(g_1(y_n) - \rho T_1(y_n, x_n)), \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n P_{K_r}(g_2(x_n) - \eta T_2(x_n, y_n)), \end{cases}$$

where the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is the same as in Algorithm 4.3.

Remark 4.5. Algorithms 2.1-2.4 in [20], Algorithms 3.1-3.7 in [28], Algorithms 2.1-2.3 in [32], Algorithms 2.1 and 2.2 in [33] and Algorithms 2.1-2.4 in [34] are special cases of Algorithms 4.1-4.4. In brief, for a suitable and appropriate choice of the operators S_i, T_i, $g_i (i = 1, 2)$ and the constants ρ and η, one can obtain a number of new and previously known iterative schemes for solving the system (3.1) and related problems. This clearly shows that Algorithms 4.1-4.4 are quite general and unifying.

Remark 4.6. It should be pointed that

(1) If $e_{n,i} = l_{n,i} = r_{n,i} = k_{n,i} = 0$ for all $n \in \mathbb{N}$ and $i = 1, 2, ..., p$, then Algorithms 4.1 and 4.2 change into the perturbed iterative process with mean errors.

(2) When $e_{n,i} = l_{n,i} = j_{n,i} = s_{n,i} = r_{n,i} = k_{n,i} = 0$ for all $n \in \mathbb{N}$ and $i = 1, 2, ..., p$, then Algorithms 4.1 and 4.2 reduce to the perturbed iterative process without error.

5 Main results

In this section, we establish the strong convergence of the sequences generated by perturbed projection iterative Algorithms 4.1 and 4.2, under some suitable conditions. We need the following lemma for verifying our main results.

Lemma 5.1. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative real sequences satisfying the following condition: There exists a positive integer* n_0 *such that*

$$a_{n+1} \leq (1-t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1, \sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ *and* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to 0} a_n = 0$.

Proof. The proof directly follows from Lemma 2 in Liu [22].

Theorem 5.2. *Let* T_i, $g_i(i = 1, 2)$, ρ *and* η *be the same as in Theorem* 3.3. *Suppose that all the conditions of Theorem* 3.3 *hold and the constants* ρ, η *satisfy the conditions* (3.4) *and* (3.5). *Assume that* $S_1 : K_r \to K_r$ *is a nearly uniformly* L_1*-Lipschitzian mapping with the sequence* $\{a_n\}_{n=1}^{\infty}$, $S_2 : K_r \to K_r$ *is a nearly uniformly* L_2*-Lipschitzian mapping with the sequence* $\{b_n\}_{n=1}^{\infty}$, *and* Q*is a self-mapping of* $K_r \times K_r$, *defined by* (4.1) *such that* $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVID}(K_r, T_i, g_i, i = 1, 2) \neq \emptyset$. *Moreover, for each* $i = 1, 2$, *let* $Li\vartheta < 1$, *where* ϑ *is the same as in* (3.12) *If there exists a constant* $\alpha > 0$ *such that* $\prod_{i=1}^{p} \alpha_{n,i} > \alpha$*for each* $n\in \mathbb{N}$, *then the iterative sequence* $\{(x_n, y_n)\}_{n=1}^{\infty}$*generated by Algorithm 4.1 converges strongly to the only element of* $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVID}(K_r, T_i, g_i, i = 1, 2)$.

Proof. According to Theorem 3.3, the system (3.1) has a unique solution (x^*, y^*) in $K_r \times K_r$. Since, $\rho < \frac{r'}{1+||T_1(y^*,x^*)||}$ and $\eta < \frac{r'}{1+||T_1(y^*,x^*)||}$, it follows from Lemma 3.2 that (x^*, y^*) satisfies the equations (3.3) Since SGNRNVID(K_r, T_i, g_i, $i = 1, 2$) is a singleton set and $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVID}(K_r, T_i, g_i, i = 1, 2) \neq \emptyset$, we conclude that $x^* \in \mathrm{Fix}(S_1)$ and $y^* \in \mathrm{Fix}(S_2)$. Hence, for each $i = 1, 2, ..., p$ and $n\in \mathbb{N}$, we can write

$$\begin{cases} x^* = (1-\alpha_{n,i}-\beta_{n,i})x^* + \alpha_{n,i}S_1^n P_{K_r}(g_1(y^*) - \rho T_1(y^*,\ x^*)) + \beta_{n,i}x^*, \\ y^* = (1-\alpha_{n,i}-\beta_{n,i})y^* + \alpha_{n,i}S_2^n P_{K_r}(g_2(x^*) - \eta T_2(x^*,\ y^*)) + \beta_{n,i}y^*, \end{cases} \tag{5.1}$$

Where the sequences $\{\alpha_{n,i}\}_{n=1}^{\infty}$ and $\{\beta_{n,i}\}_{n=1}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1. Let $\Gamma = \max\{\sup_{n\geq 0}||j_{n,i} - x^*||,\ \sup_{n\geq 0}||s_{n,i} - y^*||,\ i = 1, 2,\ \ldots, p\}$. Since $g_1(y^*)\ g_1(y_n) \in K_r$, $\rho < \frac{r'}{1+||T_1(y^*,x^*)||}$ and $\rho < \frac{r'}{1+||T_1(y^*,x^*)||}$, for all $n\in \mathbb{N}$, we can easily check that the points $g_1(y^*)$ - $\rho T_1(y^*,x^*)$ and $g_1(y_n)$ - $\rho T_1(y_n, x_n)$ ($n\in \mathbb{N}$), belong to $U(r')$. By using (4.3), (5.1), Proposition (2.10) and the assumptions, we have

$$\begin{aligned} &||x_{n+1} - x^*|| \\ &\leq (1-\alpha_{n,1}-\beta_{n,1})||x_n - x^*|| + \alpha_{n,1}||S_1^n P_{K_r}(g_1(v_n, 1) - \rho T_1(v_{n,1}, v_{n,1})) \\ &\quad - S_1^n P_{K_r}(g_1(y^*) - \rho T_1(y^*, x^*))|| + \beta_{n,1}||j_{n,1} - x^*|| + \alpha_{n,1}||e_{n,1}|| + ||r_{n,1}|| \\ &\leq (1-\alpha_{n,1}-\beta_{n,1})||x_n - x^*|| + \alpha_{n,1}L_1\left(||P_{K_r}(g_1(v_{n,1}) - \rho T_1(v_{n,1}, v_{n,1}))\right. \\ &\quad \left. - P_{K_r}(g_1(y^*) - \rho T_1(y^*, x^*))|| +\ a_n\right) + \beta_{n,1}||j_{n,1} - x^*|| + \alpha_{n,1}(||e'_{n,1}|| + ||e''_{n,1}||) + ||r_{n,1}|| \\ &\leq (1-\alpha_{n,1}-\beta_{n,1})||x_n - x^*|| +\ \alpha_{n,1}L_1\left(\frac{r}{r-r'}\ ||g_1(v_n, 1) - g_1(y^*)\right. \\ &\quad \left. - \rho(T_1(v_n, 1, v_{n,1}) - T_1(y^*, x^*))|| + a_n\right)\ + \alpha_{n,1}||e'_{n,1}|| + ||e''_{n,1}|| + ||r_{n,1}|| + \beta_{n,1}\Gamma \\ &\leq (1\text{-}\alpha_{n,1}-\beta_{n,1})||x_n - x^*|| + \alpha_{n,1}L_1\theta||v_n, 1 - y^*|| \\ &\quad + \alpha_{n,1}||e'_{n,1}|| + ||e''_{n,1}|| + ||r_{n,1}|| + \alpha_{n,1}L_1 a_n + \beta_{n,1}\Gamma, \end{aligned} \tag{5.2}$$

where θ is the same as in (3.10). In similar way to the proof of (5.2), we can get

$$\begin{aligned} ||y_{n+1} - y^*|| \leq (1-\alpha_{n,1}-\beta_{n,1})||y_n - y^*|| + \alpha_{n,1}L_2\omega||v_{n,1} - x^*|| \\ + \alpha_{n,1}||l'_{n,1}|| + ||l''_{n,1}|| + ||k_{n,1}|| + \alpha_{n,1}L_2 b_n + \beta_{n,1}\Gamma, \end{aligned} \tag{5.3}$$

where ω is the same as in (3.11). Letting $L = \max\{L_1, L_2\}$ and using (5.2) and (5.3), we obtain

$$\begin{aligned}
&||(x_{n+1}, y_{n+1}) - (x^*, y^*)||_* \\
&\quad \le (1-\alpha_{n,1}-\beta_{n,1})||(x_n, y_n) - (x^*, y^*)||_* + \alpha_{n,1} L\vartheta ||(v_{n,1}, v_{n,1}) - (x^*, y^*)||_* \\
&\qquad + \alpha_{n,1}||(e'_{n,1}, l'_{n,1})||_* + ||(e''_{n,1}, l''_{n,1})||_* + ||(r_{n,1}, k_{n,1})||_* + \alpha_{n,1} L(a_n + b_n) + 2\beta_{n,1}\Gamma,
\end{aligned} \tag{5.4}$$

where ϑ is the same as in (3.12).

As in the proof of the inequalities (5.2)-(5.4), for each $i \in \{1, 2, ..., p - 2\}$, we can prove that

$$\begin{aligned}
&||(v_{n,i}, v_{n,i}) - (x^*, y^*)||_* \\
&\quad \le (1-\alpha_{n,i+1}-\beta_{n,i+1})||(x_n, y_n) - (x^*, y^*)||_* + \alpha_{n,i+1} L\vartheta ||(v_{n,i+1}, v_{n,i+1}) - (x^*, y^*)||_* \\
&\qquad + \alpha_{n,i+1}||(e'_{n,i+1}, l'_{n,i+1})||_* + ||(e''_{n,i+1}, l''_{n,i+1})||_* + ||(r_{n,i+1}, k_{n,i+1})||_* \\
&\qquad + \alpha_{n,i+1} L(a_n + b_n) + 2\beta_{n,i+1}\Gamma
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
&||(v_{n,p-1}, v_{n,p-1}) - (x^*, y^*)||_* \\
&\quad \le (1-\alpha_{n,p}-\beta_{n,p})||(x_n, y_n) - (x^*, y^*)||_* + \alpha_{n,p} L\vartheta ||(x_n, y_n) - (x^*, y^*)||_* \\
&\qquad + \alpha_{n,p}||(e'_{n,p}, l'_{n,p})||_* + ||(e''_{n,p}, l''_{n,p})||_* + ||(r_{n,p}, k_{n,p})||_* + \alpha_{n,p} L(a_n + b_n) + 2\beta_{n,p}\Gamma.
\end{aligned} \tag{5.6}$$

It follows from (5.5) and (5.6) that

$$\begin{aligned}
&||(v_{n,1}, v_{n,1}) - (x^*, y^*)||_* \\
&\quad \le (1-\alpha_{n,2}-\beta_{n,2})||(x_n, y_n) - (x^*, y^*)||_* + \alpha_{n,2} L\vartheta ||(v_{n,2}, v_{n,2}) - (x^*, y^*)||_* \\
&\qquad + \alpha_{n,2}||(e'_{n,2}, l'_{n,2})||_* + ||(e''_{n,2}, l''_{n,2})||_* + ||(r_{n,2}, k_{n,2})||_* + \alpha_{n,2} L(a_n + b_n) + 2\beta_{n,2}\Gamma \\
&\quad \le (1-\alpha_{n,2}-\beta_{n,2})||(x_n, y_n) - (x^*, y^*)||_* + \alpha_{n,2} L\vartheta \big((1-\alpha_{n,3}-\beta_{n,3})\|(x_n, y_n) - (x^*, y^*)\|_* \\
&\qquad + \alpha_{n,3} L\vartheta ||(v_{n,3}, v_{n,3}) - (x^*, y^*)||_* + \alpha_{n,3}||(e'_{n,3}, l'_{n,3})||_* + ||(e''_{n,3}, l''_{n,3})||_* + ||(r_{n,3}, k_{n,3})||_* \\
&\qquad + \alpha_{n,3} L(a_n + b_n) + 2\beta_{n,3}\Gamma\big) + \alpha_{n,2}||(e'_{n,2}, l'_{n,2})||_* + ||(e''_{n,2}, l''_{n,2})||_* \\
&\qquad + ||(r_{n,2}, k_{n,2})||_* + \alpha_{n,2} L(a_n + b_n) + 2\beta_{n,2}\Gamma \\
&\quad = (1-\alpha_{n,2}-\beta_{n,2} + \alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})L\vartheta)||(x_n, y_n) - (x^*, y^*)||_* \\
&\qquad + \alpha_{n,2}\alpha_{n,3} L^2\vartheta^2 ||(v_{n,3}, v_{n,3}) - (x^*, y^*)||_* + \alpha_{n,2}||(e'_{n,2}, l'_{n,2})||_* \\
&\qquad + \alpha_{n,2}\alpha_{n,3} L\vartheta ||(e'_{n,3}, l'_{n,3})||_* + ||(e''_{n,2}, l''_{n,2})||_* + \alpha_{n,2} L\vartheta ||(e''_{n,3}, l''_{n,3})||_* + ||(r_{n,2}, k_{n,2})||_* \\
&\qquad + \alpha_{n,2} L\vartheta ||(r_{n,3}, k_{n,3})||_* + (\alpha_{n,2} L + \alpha_{n,2}\alpha_{n,3} L^2\vartheta)(a_n + b_n) + 2(\beta_{n,2} + \alpha_{n,2}\beta_{n,3} L\vartheta)\Gamma \\
&\quad \le \\
&\quad \cdots \\
&\quad \le \Big(1-\alpha_{n,2}-\beta_{n,2} + \alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})L\vartheta + \alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4}-\beta_{n,4})L^2\vartheta^2 \\
&\qquad + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i}(1-\alpha_{n,p}-\beta_{n,p})L^{p-2}\vartheta^{p-2} + \prod_{i=2}^{p}\alpha_{n,i} L^{p-1}\vartheta^{p-1}\Big) ||(x_n, y_n) - (x^*, y^*)||_* \\
&\qquad + \alpha_{n,2}||(e'_{n,2}, l'_{n,2})||_* + \alpha_{n,2}\alpha_{n,3} L\vartheta ||(e'_{n,3}, l'_{n,3})||_* + \cdots + \prod_{i=2}^{p}\alpha_{n,i} L^{p-2}\vartheta^{p-2}||(e'_{n,p}, l'_{n,p})||_* \\
&\qquad + ||(e''_{n,2}, l''_{n,2})||_* + \alpha_{n,2} L\vartheta ||(e''_{n,3}, l''_{n,3})||_* + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i} L^{p-2}\vartheta^{p-2}||(e''_{n,p}, l''_{n,p})||_* \\
&\qquad + ||(r_{n,2}, k_{n,2})||_* + \alpha_{n,2} L\vartheta ||(r_{n,3}, k_{n,3})||_* + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i} L^{p-2}\vartheta^{p-2}||(r_{n,p}, k_{n,p})||_* \\
&\qquad + \left(\alpha_{n,2} L + \alpha_{n,2}\alpha_{n,3} L^2\vartheta + \alpha_{n,2}\alpha_{n,3}\alpha_{n,4} L^3\vartheta^2 + \cdots + \prod_{i=2}^{p}\alpha_{n,i} L^{p-1}\vartheta^{p-2}\right)(a_n + b_n) \\
&\qquad + 2\left(\beta_{n,2} + \alpha_{n,2}\beta_{n,3} L\vartheta + \alpha_{n,2}\alpha_{n,3}\beta_{n,4} L^2\vartheta^2 + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i}\beta_{n,p} L^{p-2}\vartheta^{p-2}\right)\Gamma.
\end{aligned} \tag{5.7}$$

Applying (5.4) and (5.7), we get

$$
\begin{aligned}
&(x_{n+1}, y_{n+1}) - (x^*, y^*)||_* \\
&\quad \leq (1 - \alpha_{n,1} - \beta_{n,1} + \alpha_{n,1}(1 - \alpha_{n,2} - \beta_{n,2})L\vartheta + \alpha_{n,1}\alpha_{n,2}(1 - \alpha_{n,3} - \beta_{n,3})L^2\vartheta^2 \\
&\qquad + \cdots + \prod_{i=1}^{p-1} \alpha_{n,i}\ (1 - \alpha_{n,p} - \beta_{n,p})L^{p-1}\vartheta^{p-1} + \prod_{i=1}^{p} \alpha_{n,i}L^p\vartheta^p\Bigg) ||(x_n, y_n) - (x^*, y^*)||_* \\
&\qquad + \alpha_{n,1}||(e'_{n,1}, l'_{n,1})||_* + \alpha_{n,1}\alpha_{n,2}L\vartheta||(e'_{n,2}, l'_{n,2})||_* + \cdots + \prod_{i=1}^{p} \alpha_{n,i}L^{p-1}\vartheta^{p-1}||(e'_{n,p}, l'_{n,p})||_* \\
&\qquad + ||(e''_{n,1}, l''_{n,1})||_* + \alpha_{n,1}L\vartheta||(e''_{n,2}, l''_{n,2})||_* + \cdots + \prod_{i=1}^{p-1} \alpha_{n,i}L^{p-1}\vartheta^{p-1}||(e''_{n,p}, l''_{n,p})||_* \\
&\qquad + ||(r_{n,1}, k_{n,1})||_* + \alpha_{n,1}L\vartheta||(r_{n,2}, k_{n,2})||_* + \cdots + \prod_{i=1}^{p-1} \alpha_{n,i}L^{p-1}\vartheta^{p-1}\left\|(r_{n,p}, k_{n,p})\right\|_* \\
&\qquad + \left(\alpha_{n,1}L + \alpha_{n,1}\alpha_{n,2}L^2\vartheta + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}L^3\vartheta^2 + \cdots + \prod_{i=1}^{p} \alpha_{n,i}L^p\vartheta^{p-1}\right)(a_n + b_n) \\
&\qquad + 2\left(\beta_{n,1} + \alpha_{n,1}\beta_{n,2}L\vartheta + \alpha_{n,1}\alpha_{n,2}\beta_{n,3}L^2\vartheta^2 + \cdots + \prod_{i=1}^{p-1} \alpha_{n,i}\beta_{n,p}L^{p-1}\vartheta^{p-1}\right)\Gamma \\
&\quad \leq \left[1 - (1 - L\vartheta)\prod_{i=1}^{p} \alpha_{n,i}L^{p-1}\vartheta^{p-1}\right] ||(x_n, y_n) - (x^*, y^*)||_* + \sum_{i=1}^{p}\prod_{j=1}^{i} \alpha_{n,j}L^{i-1}\vartheta^{i-1}||(e'_{n,i}, l'_{n,i})||_* \\
&\qquad + ||(e''_{n,1}, l''_{n,1})||_* + \sum_{i=2}^{p}\prod_{j=1}^{i-1} \alpha_{n,j}L^{i-1}\vartheta^{i-1}||(e''_{n,i}, l''_{n,i})||_* + ||(r_{n,1}, k_{n,1})||_* \\
&\qquad + \sum_{i=2}^{p}\prod_{j=1}^{i-1} \alpha_{n,j}L^{i-1}\vartheta^{i-1}||(r_{n,i}, k_{n,i})||_* + \sum_{i=1}^{p}\prod_{j=1}^{i} \alpha_{n,j}L^{i}\vartheta^{i-1}(a_n + b_n) \\
&\qquad + 2(\beta_{n,1} + \sum_{i=2}^{p}\prod_{j=1}^{i-1} \alpha_{n,j}\beta_{n,i}L^{i-1}\vartheta^{i-1})\Gamma \\
&\quad \leq \left[1 - (1 - L\vartheta)\prod_{i=1}^{p} \alpha_{n,i}L^{p-1}\vartheta^{p-1}\right] ||(x_n, y_n) - (x^*, y^*)||_* \\
&\qquad + (1 - L\vartheta)\prod_{i=1}^{p} \alpha_{n,i}L^{p-1}\vartheta^{p-1}\frac{\sum_{i=1}^{p}\prod_{j=1}^{i} \alpha_{n,j}L^{i-1}\vartheta^{i-1}||(e''_{n,i}, l''_{n,i})||_* + \sum_{i=1}^{p}\prod_{j=1}^{i} \alpha_{n,j}L^{i}\vartheta^{i-1}(a_n + b_n)}{\alpha(1 - L\vartheta)L^{p-1}\vartheta^{p-1}} \\
&\qquad + ||(e''_{n,1}, l''_{n,1})||_* + \sum_{i=2}^{p}\prod_{j=1}^{i-1} \alpha_{n,j}L^{i-1}\vartheta^{i-1}||(e''_{n,i}, l''_{n,i})||_* + ||(r_{n,1}, k_{n,1})||_* \\
&\qquad + \sum_{i=2}^{p}\prod_{j=1}^{i-1} \alpha_{n,j}L^{i-1}\vartheta^{i-1}||(r_{n,i}, k_{n,i})||_* + 2\left(\beta_{n,1} + \sum_{i=2}^{p}\prod_{j=1}^{i-1} \alpha_{n,j}\beta_{n,i}L^{i-1}\vartheta^{i-1}\right)\Gamma.
\end{aligned}
\tag{5.8}
$$

Since $L\vartheta < 1$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} \beta_{n,i} < \infty$ for each $i \in \{1, 2, ..., p\}$, in view of (4.4), it is obvious that all the conditions of Lemma 5.1 are satisfied. Now, Lemma 5.1 and (5.8) guarantee that $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$ and so the sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ generated by Algorithm 4.1 converges strongly to the unique solution (x^*, y^*) of the system (3.1). This completes the proof.

Corollary 5.3. *Suppose that* T_i, $g_i(i = 1, 2)$, ρ *and* η *are the same as in Theorem* 3.3 *and let all the conditions Theorem* 3.3 *hold. Furthermore, assume that the constants* ρ *and* η *satisfy the conditions* (3.4) *and* (3.5) *and, for each* $i = 1, 2$, $L_i\vartheta < 1$, *where* ϑ *is the same as in* (3.12). *If there exists a constant* $\alpha > 0$ *such that* $\prod_{i=1}^{p} \alpha_{n,i} > \alpha$ *for each* $n \in \mathbb{N}$, *then the iterative sequence* $\{(x_n, y_n)\}_{n=1}^{\infty}$ *generated by Algorithm 4.2 converges strongly to the unique solution of the system* (3.1).

As in the proof of Theorem 5.2, one can prove the convergence of the iterative sequence generated by Algorithms 4.3 and 4.4.

6 Comments on results in the papers [20,28,33,47]

In view of Definition 2.11, we note that the condition relaxed cocoercivity of the operator T is weaker than the condition strongly monotonicity of T. In other words, the class of relaxed cocoercive mappings is more general than the class of strongly monotone mappings. In fact, Chang et al. [20], Verma [33], Huang and Noor [47], Noor and Noor [28] studied the convergence analysis of the proposed iterative algorithms under the condition of strong monotonicity. In the present section, we show that, under the mild condition, that is, the relaxed cocoercivity, the main results in the papers [20,28,33,47] still hold.

Let K be a closed convex subset of $\mathcal{H}$ and let $T : K \times K \to \mathcal{H}$ be a nonlinear operator. Verma [33] and Chang et al. [20] introduced and considered the following system of nonlinear variational inequalities (SNVI): Find $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in K, \rho > 0, \\ \langle \eta T(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \ \forall x \in K, \eta > 0. \end{cases} \tag{6.1}$$

Verma [33] proposed the following two-step iterative algorithm for solving the SNVI (6.1):

Algorithm 6.1. (Algorithm 2.1 [33]) For arbitrary chosen initial points $x^0, y^0 \in K$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$\begin{cases} x^{k+1} = (1 - a^k)x^k + a^k P_K[y^k - \rho T(y^k, x^k)], \\ y^k = P_K[x^k - \eta T(x^k, y^k)], \end{cases}$$

where ρ, η >0 are constants, P_K is the $\mathcal{H}$ projection of H onto K, $0 \leq a^k \leq 1$ and $\sum_{k=0}^{\infty} a^k = \infty$.

He also studied the convergence analysis of the suggested iterative algorithm under some certain conditions as follows:

Theorem 6.2. (Theorem 2.1 [33]) *Let $\mathcal{H}$be a real Hilbert space and let K be a nonempty closed convex subset of $\mathcal{H}$. Let $T : K \times K \to \mathcal{H}$ be (γ, r)-relaxed cocoercive and μ-Lipschitz continuous in the first variable. Suppose that $(x^*, y^*) \in K \times K$ is a solution to the* SNVI (6.1), *the sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm* 6.1,

$$0 \leq a^k \leq 1, \ \sum_{k=0}^{\infty} a^k = \infty.$$

Then the sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^* for

$$0 < \rho < \frac{2(r - \gamma\mu^2)}{\mu^2}, \ 0 < \eta < \frac{2(r - \gamma\mu^2)}{\mu^2}.$$

We note that the condition $0 < \rho < \frac{2(r-\gamma\mu^2)}{\mu^2}$ implies that $r > \gamma\mu^2$. Since $T : K \times K \to \mathcal{H}$ is (γ, r)-relaxed cocoercive and μ-Lipschitz continuous in the first variable, the condition $r > \gamma\mu^2$ guarantees that the operator T is $(r - \gamma\mu^2)$-strongly monotone in the first variable. Hence, one can rewrite the statement of Theorem 6.2 as follows:

Theorem 6.3. *Let $\mathcal{H}$be a real Hilbert space and let K be a nonempty closed convex subset of $\mathcal{H}$. Let $T : K \times K \to \mathcal{H}$be ξ-strongly monotone and μ-Lipschitz continuous in the first variable. Suppose that $(x^*, y^*) \in K \times K$ is a solution to the* SNVI (6.1), *the* *sequences* $\{x^k\}$ *and* $\{y^k\}$ *are generated by Algorithm 6.1 and*

$$0 \le a^k \le 1, \quad \sum_{k=0}^{\infty} a^k = \infty.$$

Then the sequences $\{x^k\}$ *and* $\{y^k\}$, *respectively, converge to* x^* *and* y^* *for* $0 < \rho < \frac{2\xi}{\mu^2}$*and* $0 < \rho < \frac{2\xi}{\mu^2}$.

Remark 6.4. Theorem 2.3 in [33] has been stated with the condition relaxed cocoercivity of the operator T. Similarly, the conditions of Theorem 2.3 imply that the operator T is, in fact, strongly monotone. Hence Theorem 2.3 from [33] has been proved with the condition strongly monotonicity of the operator T, not the mild condition relaxed cocoercivity.

Chang et al. [20] proposed the following two-step iterative method for solving the SNVI (6.1):

Algorithm 6.5. (Algorithm 2.1 [20]) For arbitrary chosen initial points x_0, $y_0 \llcorner K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[y_n - \rho T(y_n,\ x_n)], \\ y_n = (1-\beta_n)x_n + \beta_n P_K[x_n - \eta T(x_n,\ y_n)], \end{cases}$$

where ρ, $\eta > 0$ are two constants, P_K is the projection of $\mathcal{H}$ onto K and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1].

They also studied the convergence analysis of the proposed iterative algorithm under some certain conditions as follows:

Theorem 6.6. (Theorem 3.1 [20]) *Let* $\mathcal{H}$*be a real Hilbert space and let K be a nonempty closed convex subset of* $\mathcal{H}$. *Let* $T(.,.) : K \times K \to \mathcal{H}$ *be two-variable* (γ, r)*-relaxed cocoercive and* μ*-Lipschitz continuous in the first variable. Suppose that* $(x, y) \in K \times K$ *is a solution to the* SNVI (6.1) *and that* $\{x_n\}$ *and* $\{y_n\}$ *are the sequences generated by Algorithm 6.5. If* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* [0, 1] *satisfying the following conditions:*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $\sum_{n=0}^{\infty} (1-\beta_n) < \infty$;
(c) $0 < \rho, \eta < \frac{2(r-\gamma\mu^2)}{\mu^2}$;
(d) $r > \gamma\mu^2$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Similarly, since T is (γ, r)-relaxed cocoercive and μ-Lipschitz continuous in the first variable, the condition (d) implies that the operator T is $(r - \gamma\mu^2)$-strongly monotone in the first variable. Accordingly, we can rewrite the statement of Theorem 6.6 as follows:

Theorem 6.7. *Let* $\mathcal{H}$*be a real Hilbert space and let K be a nonempty closed convex subset of* $\mathcal{H}$. *Let* $T(\cdot,\cdot) : K \times K \to \mathcal{H}$*be two-variable* ξ*-strongly monotone and* μ*-Lipschitz continuous in the first variable. Suppose that* (x^*, y^*) *is a solution to the* SNVI (6.1) *and that* $\{x_n\}$ *and* $\{y_n\}$ *are the sequences generated by Algorithm 6.5. If* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* [0, 1] *satisfying the following conditions:*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $\sum_{n=0}^{\infty} (1 - \beta_n) < \infty$;
(c) $0 < \rho, \eta < \frac{2\xi}{\mu^2}$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Remark 6.8. Theorems 3.2-3.4 in [20] have been stated with the condition relaxed cocoercivity of the operator T. The conditions of the aforesaid Theorems imply that the operator T in these theorems is in fact strongly monotone. Therefore, Theorems 3.2-3.4 in [20] have been stated with the condition strongly monotonicity of the operator T instead of the mild condition relaxed cocoercivity.

For given two different nonlinear operators $T_1, T_2 : K \times K \to \mathcal{H}$, Huang and Noor [47] introduced and considered the problem of finding $(x, y) \in K \times K$ such that

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in K, \rho > 0, \\ \langle \eta T_2(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in K, \eta > 0, \end{cases} \tag{6.2}$$

which is called a system of *nonlinear variational inequalities involving two different nonlinear operators* (SNVID).

They proposed the following two-step iterative algorithm for solving the SNVID (6.2):

Algorithm 6.9. (Algorithm 2.1 [47]) For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n P_K[y_n - \rho T_1(y_n, x_n)], \\ y_{n+1} = P_K[x_{n+1} - \eta T_2(x_{n+1}, y_n)], \end{cases}$$

where $a_n \in [0, 1]$ for all $n \geq 0$, $\rho, \eta > 0$ are two constants and P_K is the projection of $\mathcal{H}$ onto K.

Meanwhile, they studied the convergence analysis of the proposed iterative algorithm under some certain conditions as follows:

Theorem 6.10. (Theorem 3.1 [47]) *Let K be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let (x^*, y^*) be the solution of the* SNVID (6.2). *If $T_1 : K \times K \to \mathcal{H}$ is (γ_1, r_1)-relaxed cocoercive and μ_1-Lipschitz continuous in the first variable, and $T_2 : K \times K \to \mathcal{H}$ is (γ_2, r_2)-relaxed cocoercive and μ_2-Lipschitz continuous in the first variable with conditions*

$$0 < \rho < \min\left\{\frac{2(r_1 - \gamma_1\mu_1^2)}{\mu_1^2}, \frac{2(r_2 - \gamma_2\mu_2^2)}{\mu_2^2}\right\},$$
$$0 < \eta < \min\left\{\frac{2(r_1 - \gamma_1\mu_1^2)}{\mu_1^2}, \frac{2(r_2 - \gamma_2\mu_2^2)}{\mu_2^2}\right\},$$

and $r_1 > \gamma_1\mu_1^2, r_2 > \gamma_2\mu_2^2, \mu_1 > 0, \mu_2 > 0, a_n \in [0, 1], \sum_{n=0}^{\infty} a_n = \infty$, *then, for arbitrarily chosen initial points $x_0, y_0 \in K$,* the sequences $\{x_n\}$ *and* $\{y_n\}$ *obtained from explicit Algorithm 6.9 converge strongly to x^* and y^* respectively.*

In similar way, since, for each $i = 1, 2$, the operator T_i is (γ_i, r_i)-relaxed cocoercive and μ_i-Lipschitz continuous in the first variable, the conditions $r_i > \gamma_i\mu_i^2 (i = 1, 2)$ guarantee that, for each $i = 1, 2$, the operator T_i is $(r_i - \gamma_i\mu_i^2)$-strongly monotone in the first variable. Therefore, one can rewrite the statement of Theorem 6.10 as follows:

Theorem 6.11. *Let K be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$*and let* (x^*, y^*) *be a solution of the* SNVID (6.2). *Suppose that for each* $i = 1, 2$, *the operator* $T_i : K \times K \to \mathcal{H}$ *is* ξ_i*-strongly monotone and* μ_i*-Lipschitz continuous in the first variable. If the constants* $\rho, \eta > 0$ *satisfy the following conditions:*

$$0 < \rho, \eta < \min\left\{\frac{2\xi_1}{\mu_1^2}, \frac{2\xi_2}{\mu_2^2}\right\}, \quad a_n \in [0, 1], \quad \sum_{n=0}^{\infty} a_n = \infty,$$

then, for arbitrarily chosen initial points $x_0, y_0 \in K$, *the sequences* $\{x_n\}$ *and* $\{y_n\}$ *obtained from explicit Algorithm* 6.9 *converge strongly to* x^* *and* y^*, *respectively.*

Remark 6.12. The operator T in Theorems 3.2 and 3.3 from [47] is relaxed cocoercive. But, by using the conditions of the aforesaid theorems we note that the operator T in these theorems is in fact strongly monotone. Accordingly, Theorems 3.2 and 3.3 in [47] have been stated with the strongly monotonicity of the operator T, not the relaxed cocoercivity.

For given different nonlinear operators $T_1, T_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $g, h : \mathcal{H} \to \mathcal{H}$, Noor and Noor [28] introduced and considered the problem of finding $(x, y) \in K \times K$ such that

$$\begin{cases} \langle \rho T_1(y^*,\ x^*) + x^* - g(y^*),\ g(x) - x^* \rangle \geq 0,\ \forall x \in \mathcal{H}\ :\ g(x) \in K,\ \rho > 0, \\ \langle \eta T_2(x^*,\ y^*) + y^* - h(x^*),\ h(x) - y^* \rangle \geq 0,\ \forall x \in \mathcal{H}\ :\ h(x) \in K,\ \eta > 0, \end{cases} \tag{6.3}$$

which is called a system of *general nonlinear variational inequalities involving four different nonlinear operators* (SGNVID).

They proposed the following two-step iterative scheme for solving the SGNVID (6.3):

Algorithm 6.13. (Algorithm 3.1 [28]) For arbitrary chosen initial points $x_0, y_0 \,\llcorner\, K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n P_K[g(y_n) - \rho T_1(y_n,\ x_n)], \\ y_{n+1} = P_K[h(x_{n+1}) - \eta T(x_{n+1},\ y_n)], \end{cases}$$

where $a_n \in [0, 1]$ for all $n \geq 0$, $\rho, \eta > 0$ are two constants and P_K is the projection of $\mathcal{H}$ onto K.

They also studied the convergence analysis of the proposed iterative algorithm under some certain conditions as follows:

Theorem 6.14. (Theorem 4.1 [28]) *Let* (x^*, y^*) *be a solution of the* SGNVID (6.3). Suppose that $T_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is (γ_1, r_1)-relaxed cocoercive and μ_1-Lipschitz continuous in the first variable and $T_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is (γ_2, r_2)-relaxed cocoercive and μ_2-Lipschitz continuous in the first variable. Let g be (γ_3, r_3)-relaxed cocoercive and μ_3-Lipschitz continuous and let h be (γ_4, r_4)-relaxed cocoercive and μ_4-Lipschitz continuous. If

$$\begin{aligned} &\left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2\kappa_1(2 - \kappa_1)}}{\mu_1^2}, \quad r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{\kappa_1(2 - \kappa_1)}, \quad \kappa_1 < 1, \\ &\left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2\kappa_2(2 - \kappa_2)}}{\mu_2^2}, \quad r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{\kappa_1(2 - \kappa_1)}, \quad \kappa_2 < 1, \end{aligned} \tag{6.4}$$

where

$$\kappa_1 = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2) + \mu_3^2}, \quad \kappa_2 = \sqrt{1 - 2(r_4 - \gamma_4\mu_4^2) + \mu_4^2}, \quad a_n \in [0, 1], \quad \sum_{n=0}^{\infty} a_n = \infty, \tag{6.5}$$

then, for arbitrarily chosen initial points x_0, $y_0 \in K$, the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 6.13 converge strongly to x^* and y^*, respectively.

The condition (6.4) implies that, for each $i = 1, 2, r_i > r_i\mu_i^2$. Since, for each $i = 1, 2$, the operator T_i is (γ_i, r_i)-relaxed cocoercive and μ_i-Lipschitz continuous in the first variable, the conditions $r_i > r_i\mu_i^2(i = 1, 2)$ guarantee that, for each $i = 1, 2$, the operator T_i is $(r_i - r_i\mu_i^2)$-strongly monotone in the first variable. Since, for each $i = 1, 2$, $\kappa_i < 1$, it follows from the condition (6.5) that, for each $i = 3, 4$, $r_i > \gamma_i{\mu^2}_i$.

Similarly, since g is (γ_3, r_3)-relaxed cocoercive and μ_3-Lipschitz continuous, and h is (γ_4, r_4)-relaxed cocoercive and μ_4-Lipschitz continuous, the conditions $r_i > \gamma_i{\mu^2}_i(i = 3, 4)$ imply that the operator g is $(r_3 - \gamma_3{\mu^2}_3)$-strongly monotone and the operator h is $(r_4 - \gamma_4{\mu^2}_4)$-strongly monotone. Therefore, one can rewrite the statement of Theorem 6.14 as follows:

Theorem 6.15. *Let* (x^*, y^*) *be a solution of the* SGNVID (6.3). Suppose that for each $i = 1, 2$, the operator $T_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is ξ_i-strongly monotone and μ_i-Lipschitz continuous in the first variable. Let g be ξ_3-strongly monotone and μ_3-Lipschitz continuous and h be ξ_4-strongly monotone and μ_4-Lipschitz continuous. If the constants ρ and η satisfy the following conditions:

$$\begin{cases} |\rho - \frac{\xi_1^2}{\mu_1^2}| < \frac{\sqrt{\xi_1^2 - \mu_1^2\kappa_1(2-\kappa_1)}}{\mu_1^2}, \\ |\eta - \frac{\xi_2^2}{\mu_2^2}| < \frac{\sqrt{\xi_2^2 - \mu_2^2\kappa_2(2-\kappa_1)}}{\mu_2^2}, \\ \xi_i > \mu_i\sqrt{\kappa_i(2-\kappa_i)}, \kappa_i < 1(i = 1, 2), \\ \kappa_i = \sqrt{1 - 2\xi_i + \mu_i^2}, 2\xi_i \leq 1 + \mu_i^2(i = 1, 2), \end{cases}$$

then, for arbitrarily chosen initial points x_0, $y_0 \in K$, the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 6.13 converge strongly to x and y, respectively.

Remark 6.16. The operators $T_i(i = 1, 2)$ in Theorems 4.2 and 4.4 from [28] are relaxed cocoercive. While the conditions of the aforesaid theorems guarantee that the operators $T_i(i = 1, 2)$ in these theorems are in fact strongly monotone. Therefore, Theorems 4.2 and 4.4 in [28] have been stated with the condition strongly monotonicity of the operators $T_i(i = 1, 2)$, not the mild condition relaxed cocoercivity. In addition, Theorem 4.3 in [28] has been stated with the condition relaxed cocoercivity of the operator T. The conditions of Theorem 4.3 imply that the operator T in this theorem is in fact strongly monotone. Therefore, Theorem 4.3 in [28] has been stated with the condition strongly monotonicity of the operator T instead of the mild condition relaxed cocoercivity.

Remark 6.17. In view of the above facts, we note that Theorem 5.2 extends and improves Theorems 3.1-3.4 in [20], Theorems 4.1-4.4 in [28], Theorems 3.1-3.3 in [32] and Theorems 2.1-2.3 in [33] and [34].

7 Conclusion

In this paper, we have introduced and considered a new system of general nonlinear regularized nonconvex variational inequalities involving four different nonlinear operators and established the equivalence between the aforesaid system and a fixed point problem. By this equivalent formulation, we have discussed the existence and uniqueness of solution of the system of general nonlinear regularized nonconvex variational inequalities. This equivalence and two nearly uniformly Lipschitzian mappings $S_i(i = 1,$

2) are used to suggest and analyze a new perturbed p-step iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q} = (S_1, S_2)$ which is the unique solution of the system of general nonlinear regularized nonconvex variational inequalities. In the final section, we have presented some remarks on results presented by Change et al [20], Huang and Noor [47], Noor and Noor [28] and Verma [32-34]. We also have shown that their results are special cases of our results. Several special cases are also discussed. It is expected that the results proved in this paper may simulate further research regarding the numerical methods and their applications in various fields of pure and applied sciences.

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