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Convergence of a proximal point algorithm for maximal monotone operators in Hilbert spaces

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Abstract

In this article, we consider the proximal point algorithm for the problem of approximating zeros of maximal monotone mappings. Strong convergence theorems for zero points of maximal monotone mappings are established in the framework of Hilbert spaces.

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1. Introduction

The theory of maximal monotone operators has emerged as an effective and powerful tool for studying many real world problems arising in various branches of social, physical, engineering, pure and applied sciences in unified and general framework. Recently, much attention has been payed to develop efficient and implementable numerical methods including the projection method and its variant forms, auxiliary problem principle, proximal-point algorithm and descent framework for solving variational inequalities and related optimization problems (see [1-32] and the references therein). The proximal point algorithm, can be traced back to Martinet [33] in the context of convex minimization and Rockafellar [34] in the general setting of maximal monotone operators, has been extended and generalized in different directions by using novel and innovative techniques and ideas.

In this article, we investigate the problem of approximating a zero of the maximal monotone mapping based on a proximal point algorithm in the framework of Hilbert spaces. Strong convergence of the iterative algorithm is obtained.

2. Preliminaries

Throughout this article, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let T be a set-valued mapping.

(a) The set $D(T)$ defined by

$$D(T) = \{u \in H : T(u) \neq \emptyset\}$$

is called the effective domain of T.

(b) The set $R(T)$ defined by

$$R(T) = \bigcup_{u \in H} T(u)$$

is called the range of T.

(c) The set $G(T)$ defined by

$$G(T) = \{(u, v) \in H \times H : u \in D(T), v \in R(T)\}$$

is said to be the graph of T.

Recall the following definitions.

(c) T is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (u, x), (v, y) \in G(T).$$

(d) T is said to be maximal monotone if it is not properly contained in any other monotone operator.

For a maximal monotone $T : D(T) \to 2^H$, we can defined the resolvent of T by

$$J_t = (I + tT)^{-1}, \quad t > 0. \tag{2.1}$$

It is well known that J_t: $H \to D(T)$ is nonexpansive, and $F(J_t) = T^{-1}(0)$, where $F(J_t)$ denotes the set of fixed points of J_t. The Yosida approximation T_t is defined by

$$T_t = \frac{1}{t}(I - J_t), \quad t > 0.$$

It is well known that $T_t x \in T J_t x$, $\forall x \in H$ and $\|T_t x\| \leq |Tx|$, where

$$|Tx| = \inf\{\|y\| : y \in Tx\},$$

for all $x \in D(T)$.

Let C be a nonempty, closed and convex subset of H. Next, we always assume that T: $C \to 2^H$ is a maximal monotone mapping with $T^{-1}(0) \neq \emptyset$, where $T^{-1}(0)$ denotes the set of zeros of T.

The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of zero points for maximal monotone mappings. A classical method to solve the following set-valued equation

$$0 \in Tx, \tag{2.2}$$

is the proximal point algorithm. To be more precise, start with any point $x_0 \in H$, and update x_{n+1} iteratively conforming to the following recursion

$$x_n \in x_{n+1} + \beta_n T x_{n+1}, \quad n \geq 0, \tag{2.3}$$

where $\{\beta_n\} \subset [\beta, \infty)$, $(\beta > 0)$ is a sequence of real numbers. However, as pointed in [15], the ideal form of the method is often impractical since, in many cases, to solve the problem (2.3) exactly is either impossible or the same difficult as the original problem (2.2). Therefore, one of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of T.

In 1976, Rockafellar [35] gave an inexact variant of the method

$$x_0 \in H, \quad x_n + e_{n+1} \in x_{n+1} + \lambda_n T x_{n+1}, \quad n \geq 0, \tag{2.4}$$

where $\{e_n\}$ is regarded as an error sequence. This is an inexact proximal point algorithm. It was shown that, if

$$\sum_{n=0}^{\infty} \|e_n\| < \infty, \tag{A}$$

then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a zero of T provided that $T^{-1}(0) \neq \emptyset$. In [16], Güller obtained an example to show that Rockafellar's proximal point algorithm (1.4) does not converge strongly, in general.

Recently, many authors studied the problems of modifying Rockafellar's proximal point algorithm so that strong convergence is guaranteed. Cho et al. [13] proved the following result.

Theorem CKZ. *Let H be a real Hilbert space, Ω a nonempty closed convex subset of H, and T: $\Omega \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_Ω be the metric projection of H onto Ω. Suppose that, for any given $x_n \in H$, $\beta_n > 0$ and $e_n \in H$, there exists $\bar{x}_n \in \Omega$ conforming to the SVME (2.4), where $\{\beta_n\} \subset (0, +\infty)$ with $\beta_n \to \infty$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} \|e_n\|^2 < \infty. \tag{B}$$

Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ *such that*

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

for any fixed $u \in \Omega$, define the sequence $\{x_n\}$ iteratively as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_\Omega(\bar{x}_n - e_n), \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to a fixed point z of T, where $z = \lim_{t\to\infty} J_t u$.

In this article, motivated by Theorem CKZ, we continue to consider the problem of approximating a zero of the maximal monotone mapping T. Strong convergence theorems are established under mild restrictions imposed on the error sequence $\{e_n\}$ comparing with the restriction (B). The results which include Cho et al. [13] as a special case also improve the corresponding results announced by many others.

In order to prove our main result, we need the following lemmas.

Lemma 2.1. (Bruck [[35], Lemma 1]). *Let H be a Hilbert space and C a nonempty, closed and convex subset H. For all $u \in C$, $\lim_{t\to\infty} J_t u$ exists and it is the point of T^{-1}* (0) *nearest u.*

Lemma 2.2 (Eckstein [[15], Lemma 2]). *For any given $x_n \in C$, $\lambda_n > 0$, and $e_n \in H$, there exists $\bar{x}_n \in C$ conforming to the following set-valued mapping equation (in short, SVME):*

$$x_n + e_n \in \bar{x}_n + \lambda_n T \bar{x}_n. \tag{2.5}$$

Furthermore, for any $p \in T^{-1}(0)$, *we have*

$$\langle x_n - \bar{x}_n, x_n - \bar{x}_n + e_n \rangle \le \langle x_n - p, x_n - \bar{x}_n + e_n \rangle$$

and

$$\left\| \bar{x}_n - e_n - p \right\|^2 \le \left\| x_n - p \right\|^2 - \|x_n - \bar{x}_n\|^2 + \|e_n\|^2.$$

Lemma 2.3 (Liu [36]). *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* (0,1) *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

Theorem 3.1. *Let H be a real Hilbert space, C a nonempty, closed and convex subset of H and T*: $C \to 2^H$ *a maximal monotone operator with* $T^{-1}(0) \neq \emptyset$. *Let* P_C *be a metric projection from H onto C. For any* $x_n \in H$ *and* $\lambda_n > 0$, *find* $\bar{x}_n \in C$ *and* $e_n \in H$ *conforming to the SVME* (2.5), *where* $\{\lambda_n\} \subset (0, \infty)$ *with* $\lambda_n \to \infty$ *as* $n \to \infty$ and

$$\|e_n\| \le \eta_n \|x_n - \bar{x}_n\| \tag{C}$$

with $\sup_{n\ge 0} \eta_n = \eta < 1$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* [0, 1] *satisfying* $\alpha_n + \beta_n < 1$ *and the following control conditions:*

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n) P_C(\bar{x}_n - e_n). \quad n \ge 0, \tag{3.1}$$

where $u \in C$ *is a fixed element. Then the sequence* $\{x_n\}$ *generated by* (3.1) *strongly converges to a zero point z of T, where* $z = \lim_{t\to\infty} J_t u$, *if and only if* $e_n \to 0$ *as* $n \to \infty$.

Proof. First, show that the necessity. Assume that $x_n \to z$ as $n \to \infty$, where $z \in T^{-1}(0)$. It follows from (2.5) that

$$\begin{aligned} \|\bar{x}_n - z\| &\le \|x_n - z\| + \|e_n\| \\ &\le \|x_n - z\| + \eta_n \|x_n - \bar{x}_n\| \\ &\le (1 + \eta_n) \|x_n - z\| + \eta_n \|\bar{x}_n - z\| . \end{aligned}$$

This implies that

$$\|\bar{x}_n - z\| \le \frac{1 + \eta_n}{1 - \eta_n} \|x_n - z\| .$$

It follows that $\bar{x}_n \to z$ as $n \to \infty$. Note that

$$\|e_n\| \le \eta_n \|x_n - \bar{x}_n\| \le \eta_n(\|x_n - z\| + \|z - \bar{x}_n\|).$$

This shows that $e_n \to 0$ as $n \to \infty$.

Next, we show the sufficiency. The proof is divided into three steps.

Step 1. Show that $\{x_n\}$ is bounded.

From the assumption (C), we see that

$$\|e_n\| \leq \|x_n - \bar{x}_n\|.$$

For any $p \in T^{-1}(0)$, it follows from Lemma 2.2 that

$$\begin{aligned}\left\|P_C(\bar{x}_n - e_n) - p\right\|^2 &\leq \left\|\bar{x}_n - e_n - p\right\|^2 \\ &\leq \left\|x_n - p\right\|^2 - \|x_n - \bar{x}_n\|^2 + \|e_n\|^2 \\ &\leq \left\|x_n - p\right\|^2.\end{aligned}$$

That is,

$$\left\|P_C(\bar{x}_n - e_n) - p\right\| \leq \left\|x_n - p\right\|. \tag{3.2}$$

It follows from (3.2) that

$$\begin{aligned}\left\|x_{n+1} - p\right\| &= \left\|\alpha_n(u-p) + (1-\alpha_n)[P_C(\bar{x}_n - e_n) - p]\right\| \\ &\leq \alpha_n\left\|u - p\right\| + (1-\alpha_n)\left\|P_C(\bar{x}_n - e_n) - p\right\| \\ &\leq \alpha_n\left\|u - p\right\| + (1-\alpha_n)\left\|x_n - p\right\|.\end{aligned} \tag{3.3}$$

Putting

$$M = \max\{\left\|x_0 - p\right\|, \left\|u - p\right\|\},$$

we show that $\|x_n\| \leq M$ for all $n \geq 0$. It is easy to see that the result holds for $n = 0$. Assume that the result holds for some $n \geq 0$. That is, $\|x_n - p\| \leq M$. Next, we prove that $\|x_{n+1} - p\| \leq M$. Indeed, we see from (3.3) that

$$\left\|x_{n+1} - p\right\| \leq M.$$

This shows that the sequence $\{x_n\}$ is bounded.

Step 2. Show that $\limsup_{n\to\infty} \langle u - z, x_{n+1} - z\rangle \leq 0$, where $z = \lim_{t\to\infty} J_t u$.

From Lemma 2.1, we see that $\lim_{t\to\infty} J_t u$ exists, which is the point of $T^{-1}(0)$ nearest to u. Since T is maximal monotone, $T_t u \in TJ_t u$ and $T_{\lambda n}x_n \in TJ_{\lambda n}x_n$, we see

$$\begin{aligned}&\left\langle u - J_t u, J_{\lambda_n}x_n - J_t u\right\rangle \\ &= -t\left\langle T_t u, J_t u - J_{\lambda_n}x_n\right\rangle \\ &= -t\left\langle T_t u - T_{\lambda_n}x_n, J_t u - J_{\lambda_n}x_n\right\rangle - t\left\langle T_{\lambda_n}x_n, J_t u - J_{\lambda_n}x_n\right\rangle \\ &= -\frac{t}{\lambda_n}\left\langle x_n - J_{\lambda_n}x_n, J_t u - J_{\lambda_n}x_n\right\rangle.\end{aligned}$$

Since $\lambda_n \to \infty$ as $n \to \infty$, for any $t > 0$, we have

$$\limsup_{n\to\infty}\left\langle u - J_t u, J_{\lambda_n}x_n - J_t u\right\rangle \leq 0. \tag{3.4}$$

On the other hand, by the nonexpansivity of $J_{\lambda n}$, we obtain

$$\left\|J_{\lambda_n}(x_n + e_n) - J_{\lambda_n}x_n\right\| \leq \left\|(x_n + e_n) - x_n\right\| = \|e_n\|.$$

From the assumption $e_n \to 0$ as $n \to \infty$ and (3.4), we arrive at

$$\limsup_{n\to\infty} \langle u - J_t u, J_{\lambda_n}(x_n + e_n) - J_t u\rangle \le 0. \tag{3.5}$$

From (2.5), we see that

$$\left\|P_C(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\right\| \le \left\|(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\right\| \le \|e_n\|.$$

That is,

$$\lim_{n\to\infty} \left\|P_C(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\right\| = 0. \tag{2.6}$$

Combining (3.5) with (3.6), we arrive at

$$\limsup_{n\to\infty} \langle u - J_t u, P_C(\bar{x}_n - e_n) - J_t u\rangle \le 0. \tag{3.7}$$

On the other hand, we see from the algorithm (3.1) that

$$x_{n+1} - P_C(\bar{x}_n - e_n) = \alpha_n[u - P_C(\bar{x}_n - e_n)] + \beta_n[x_n - P_C(\bar{x}_n - e_n)].$$

It follows from the condition $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ that

$$x_{n+1} - P_C(\bar{x}_n - e_n) \to 0 \quad \text{as } n \to \infty,$$

which combines with (3.7) yields that

$$\limsup_{n\to\infty} \langle u - J_t u, x_{n+1} - J_t u\rangle \le 0, \quad \forall t \ge 0. \tag{3.8}$$

From $z = \lim_{t\to\infty} J_t u$ and (3.8), we arrive at

$$\limsup_{n\to\infty} \langle u - z, x_{n+1} - z\rangle \le 0. \tag{3.9}$$

Step 3. Show that $x_n \to z$ as $n \to \infty$.

It follows from (3.2) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \langle \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)P_C(\bar{x}_n - e_n) - z, x_{n+1} - z\rangle \\
&\le \alpha_n \langle u - z, x_{n+1} - z\rangle + \beta_n \langle x_n - z, x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n - \beta_n)\langle P_C(\bar{x}_n - e_n) - z, x_{n+1} - z\rangle \\
&\le \alpha_n \langle u - z, x_{n+1} - z\rangle + \beta_n \|x_n - z\| \|x_{n+1} - z\| \\
&\quad + (1 - \alpha_n - \beta_n)\left\|P_C(\bar{x}_n - e_n) - z\right\| \|x_{n+1} - z\| \\
&\le \alpha_n \langle u - z, x_{n+1} - z\rangle + \beta_n \|x_n - z\| \|x_{n+1} - z\| \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - z\| \|x_{n+1} - z\| \\
&= \alpha_n \langle u - z, x_{n+1} - z\rangle + (1 - \alpha_n)\|x_n - z\| \|x_{n+1} - z\| \\
&\le \alpha_n \langle u - z, x_{n+1} - z\rangle + \frac{1 - \alpha_n}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2).
\end{aligned}$$

This implies that

$$\|x_{n+1} - z\|^2 \le (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n \langle u - z, x_{n+1} - z\rangle. \tag{3.10}$$

Applying Lemma 2.3 to (3.10), we obtain that $x_n \to z$ as $n \to \infty$. This completes the proof.

As a corollary of Theorem 3.1, we have the following.

Corollary 3.2. *Let H be a real Hilbert space, C a nonempty, closed and convex subset of H and T: $C \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_C be a metric projection from H onto C. For any $x_n \in H$ and $\lambda_n > 0$, find $\bar{x}_n \in C$ and $e_n \in H$ conforming to the SVME* (2.5), *where $\{\lambda_n\} \subset (0, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and*

$$\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$$

with $\sup_{n\geq 0} \eta_n = \eta < 1$. Let $\{\alpha_n\}$ be a real sequence in (0,1) *satisfying the following control conditions:*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{x_n\}$ be a sequence generated by the following manner:

$$x_0 \in H, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)P_C(\bar{x}_n - e_n). \quad n \geq 0,$$

where $u \in C$ is a fixed element. Then the sequence $\{x_n\}$ strongly converges to a zero point z of T, where $z = \lim_{t\to\infty} J_t u$, if and only if $e_n \to 0$ as $n \to \infty$.

Remark 3.3. Corollary 3.2 improves Theorem CKZ by relaxing the restriction imposed on the sequence $\{e_n\}$. In [34], Rockafellar obtained a weak convergence by assuming that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, see [34] for more details.

Next, as applications of Theorem 3.1, we consider the problem of finding a minimizer of a convex function.

Let H be a Hilbert space, and f: $H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y \rangle, \quad z \in H\}, \quad \forall x \in H.$$

Theorem 3.4. *Let H be a real Hilbert space and f: $H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function. Let $\{\lambda_n\}$ be a sequence in $(0, +\infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\{e_n\}$ a sequence in H with $e_n \to \infty$ as $n \to \infty$. Assume that*

$$\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$$

with $sup_{n\geq 0}\ \eta_n = \eta < 1$. Let $\bar{x}_n$ be the solution of SVME (2.5) *with T replacing by ∂f. That is,*

$$x_n + e_n \in \bar{x}_n + \lambda_n \partial f(\bar{x}_n), \quad \forall n \geq 0.$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in [0, 1] *satisfying $\alpha_n + \beta_n < 1$ and the following control conditions:*

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in H, \\ \bar{x}_n = argmin_{x\in H}\{f(x) + \dfrac{1}{2\lambda_n}\|x - x_n - e_n\|^2\}, \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)(\bar{x}_n - e_n). \quad n \geq 0, \end{cases}$$

where $u \in H$ is a fixed element. If $\partial f(0) \neq \emptyset$, the sequence $\{x_n\}$ converges strongly to a minimizer of f nearest to u.

Proof. Since f: $H \to (-\infty, +\infty]$ is a proper convex lower semi-continuous function, we have that the subdifferential ∂f of f is maximal monotone by Theorem 1 of [34]. Notice that

$$\bar{x}_n = \arg\min_{x \in H} \left\{ f(x) + \frac{1}{2\beta_n} \|x - x_n - e_n\|^2 \right\}$$

is equivalent to the following

$$0 \in \partial f(\bar{x}_n) + \frac{1}{\lambda_n}(\bar{x}_n - x_n - e_n).$$

It follows that

$$x_n + e_n \in \bar{x}_n + \lambda_n \partial f(\bar{x}_n), \quad \forall n \geq 0.$$

By Theorem 3.1, we can obtain the desired conclusion immediately.

As a corollary of Theorem 3.4, we have the following.

Corollary 3.5. *Let H be a real Hilbert space and f: $H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function. Let $\{\lambda_n\}$ be a sequence in $(0, +\infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\{e_n\}$ a sequence in H with $e_n \to \infty$ as $n \to \infty$. Assume that*

$$\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$$

with $\sup_{n \geq 0} \eta_n = \eta < 1$. Let $\bar{x}_n$ be the solution of SVME (2.5) with T replacing by ∂f. That is,

$$x_n + e_n \in \bar{x}_n + \lambda_n \partial f(\bar{x}_n), \quad \forall n \geq 0.$$

Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ ssatisfying the following control conditions:

$$\lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in H, \\ \bar{x}_n = \arg\min_{x \in H} \{f(x) + \frac{1}{2\lambda_n} \|x - x_n - e_n\|^2\}, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(\bar{x}_n - e_n). \quad n \geq 0, \end{cases}$$

where $u \in H$ is a fixed element. If $\partial f(0) \neq \emptyset$, the sequence $\{x_n\}$ converges strongly to a minimizer of f nearest to u.

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Authors' contributions

All authors contributed equally to this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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