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Strong convergence of composite general iterative methods for one-parameter nonexpansive semigroup and equilibrium problems

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Abstract

In this paper, we introduce both explicit and implicit schemes for finding a common element in the common fixed point set of a one-parameter nonexpansive semigroup $\{T(s)|0 \leq s < \infty\}$ and in the solution set of an equilibrium problems which is a solution of a certain optimization problem related to a strongly positive bounded linear operator. Strong convergence theorems are established in the framework of Hilbert spaces. As an application, we consider the optimization problem of a k-strict pseudocontraction mapping. The results presented improve and extend the corresponding results of many others.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$. Recall, a mapping T with domain $D(T)$ and range $R(T)$ in H is called nonexpansive iff for all $x, y \in D(T)$,

$$||Tx - Ty|| \leq ||x - y||.$$

Let C be a closed convex subset of a Hilbert space H, a family $\Im = \{T(s)|\ 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C iff it satisfies the following conditions:

(a) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$ and $T(0) = I$;

(b) $||T(s)x - T(s)y|| \leq ||x - y||$ for all $x, y \in C$ and $s \geq 0$.

(c) the mapping $T(\cdot)x$ is continuous, for each $x \in C$.

We denote by $F(\Im)$ the set of common fixed points of $\{T(t): t \geq 0\}$. That is, $F(\Im) = \cap_{0 \leq s < \infty} F(T(s))$. It is clear that $T(s)T(t) = T(s + t) = T(t)T(s)$ for $s, t \geq 0$.

Recall that f is called to be weakly contractive [1] iff for all $x, y \in D(T)$,

$$||f(x) - f(y)|| \leq ||x - y|| - \varphi(||x - y||),$$

for some ϕ: $[0, +\infty) \to [0, +\infty)$ is a continuous and strictly increasing function such that ϕ is positive on $(0, +\infty)$ and $\phi(0) = 0$. If $\phi(t) = (1 - k)t$, then f is called to be contractive with the contractive coefficient k. If $\phi(t) \equiv 0$, then f is said to be nonexpansive.

Let C be a nonempty closed convex subset of H and F: $C \times C \to R$ be a bifunction, where R is the set of real numbers. The equilibrium problem (for short, EP) is to find $x \in C$ such that for all $y \in C$,

$$F(x, y) \geq 0, \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping T: $C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $x \in EP(F)$ if and only if $x \in C$ is a solution of the variational inequality $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP; see, for example [2-5] and references therein. Some solution methods have been proposed to solve the EP; see, for example, [6-12] and references therein.

To study the equilibrium problems, we assume that the bifunction F: $C \times C \to R$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, y α $F(x, y)$ is convex and lower semi-continuous.

Recently, Takahashi and Takahashi [10] introduced the following iterative method

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n \ \ n \geq 1 \end{cases}$$

for approximating a common element in the fixed point set of a single nonexpansive mapping and in the solution set of the equilibrium problem. To be more precise, they proved the following Theorem.

Theorem TT Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4) and let T be a nonexpansive mapping of C into H such that $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n & n \geq 1 \end{cases}$$

where $\{\alpha_n\} \in 0[1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(T) \cap EP(F)$, where $z = P_{(T)\cap EP(F)} f(z)$.

Subsequently, many authors studied the problem of finding a common element in the fixed point set of nonexpansive mappings, in the solution set of variational inequalities and in the solution set of equilibrium problems, for instance see [10-20].

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [21-26] and the references therein. Let A be a strongly positive linear bounded operator (i.e., there is a constant $\bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}||x||^2$, $\forall x \in H$), and T be a nonexpansive mapping on H. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle \tag{1.2}$$

where $F(T)$ is the fixed point set of the mapping T on H and b is a given point in H.

Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b \quad n \geq 0 \tag{1.3}$$

It is proved [23] (see also [24]) that the sequence $\{x_n\}$ generated by (1.3) converges strongly to the unique solution of the minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

In 2007, related to a certain optimization problem, Marino and Xu [27] introduced the following viscosity approximation method for nonexpansive mappings. Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0 \tag{1.4}$$

and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*,\ x - x^*\rangle \geq 0, \quad x \in C,$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax,\ x\rangle - h(x),$$

where h is a potential function for γf (that is, $h'(x) = \gamma f(x)$ *for* $x \in H$).

In 2009, Cho et al. [28] extended the result of Marino and Xu [27] to the class of k-strictly pseudo-contractive mappings and proved the convergent theorem.

Motivated by the ongoing research and the above mentioned results, we introduce both explicit and implicit schemes for finding a common element in the common fixed point set of a one-parameter nonexpansive semigroup $\{T(s)|0 \leq s < \infty\}$, in the solution set of an equilibrium problems, in the solution set of variational inequalities in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results in [9,10,14,15,27-30].

In order to prove our main results, we need the following lemmas.

The following lemma can be found in [7].

Lemma 1.1 Let C be a nonempty closed convex subset of H and let $F: C \times C \to R$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\},$$

for all $z \in H$. Then the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3)$F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Remark 1.1 The mapping T_r is also nonexpansive for all $r > 0$.

Lemma 1.2[27] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq ||A||^{-1}$. Then $||I - \rho A|| \leq 1 - \rho\bar{\gamma}$.

Lemma 1.3[31] Let C be a nonempty bounded closed convex subset of H and let $\Im = \{T(s): 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \geq 0$,

$$\lim_{t\to\infty} \sup_{x\in C} ||\frac{1}{t}\int_0^t T(s)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)|| = 0.$$

Lemma 1.4 Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t): 0 \leq t < \infty\}$be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties:

$(i)x_n \rightharpoonup z$;

(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} ||T(t)x_n - x_n|| = 0$,

where $x_n \rightharpoonup z$ denote that $\{x_n\}$ converges weakly to z, then $z \in F(\Im)$.

Proof. This Lemma is the continuous version of Lemma 2.3 of Tan and Xu[32]. This proof given in [32] is easily extended to the continuous case.

Lemma 1.5[33] Let C be a nonempty closed convex subset of a real Hilbert space H, T be nonexpansive mapping from C into self. Then the mapping $I - T$ is demiclosed at zero, i.e.,

$$x_n \rightharpoonup x, \ ||x_n - Tx_n|| \to 0 \ \textit{implies} \ x = Tx.$$

Lemma 1.6[22] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} = (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0.$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$.

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.7[34] Let $\{\lambda_n\}$ and $\{\beta_n\}$ be two nonnegative real number sequences and $\{\alpha_n\}$ a positive real number sequence satisfying the conditions $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$ or $\sum_{n=0}^{\infty} \beta_n < \infty$. Let the recursive inequality

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n) + \beta_n, \quad n = 0, 1, 2 \cdots,$$

be given, where $\psi(\lambda)$ is a continuous and strict increasing function for all $\lambda \geq 0$ with $\psi(0) = 0$. Then $\{\lambda_n\}$ converges to zero, as $n \to \infty$.

2. Implicit viscosity iterative algorithm

Theorem 2.1 Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ into R satisfying (A1)-(A4). Let f be a weakly contractive mapping with a function ϕ on H, A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ on H, $\Im = \{T(s): s \geq 0\}$ be a nonexpansive semigroup on C, respectively. Assume that $\Omega := F(\Im) \cap EP(F) \neq \emptyset$, $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{t_n\} \subset (0,\infty)$ be real sequences satisfying the conditions

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \lim_{n\to\infty} t_n = \infty.$$

then for any $0 < \gamma \leq \bar{\gamma}$ and any $r > 0$, there exists a unique sequence $\{x_n\} \subset H$ satisfying the following condition

$$\begin{cases} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ z_n = (1 - \beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \beta_n u_n, \\ x_n = (I - \alpha_n A)z_n + \alpha_n \gamma f(x_n), & \forall n \geq 1. \end{cases} \tag{2.1}$$

Furthermore, the sequence $\{x_n\}$ converges strongly to $z^* \in \Omega$ which uniquely solves the following variational inequality

$$\langle(\gamma f - A)z^*, z - z^*\rangle \leq 0, \quad \textit{for any } z \in \Omega. \tag{2.2}$$

Proof. We divide the proof into six steps.

Step 1.

Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, with no loss of generality, that $\alpha_n < ||A||^{-1}$ for all $n \geq 1$. Then, $\alpha_n < \frac{1}{\gamma}$ for all $n \geq 1$.

First, we show that the sequence $\{x_n\}$ generated from (2.1) is well defined. For each $n \geq 1$, define a mapping S_n^f in H as follows

$$S_n^f := (I - \alpha_n A)[(1 - \beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)T_r ds + \beta_n T_r] + \alpha_n \gamma f.$$

Observe from Lemma 1.1 that T_r be a nonexpansive for each $r > 0$, thus we have that for any $x, y \in C$,

$$\begin{aligned} &||S_n^f x - S_n^f y|| \\ \leq\ & ||I - \alpha_n A||[(1 - \beta_n)\frac{1}{t_n}\int_0^{t_n} ||T(s)T_r x - T(s)T_r y||ds + \beta_n||T_r x - T_r y||] \\ &+\alpha_n\gamma||f(x) - f(y)|| \\ \leq\ & (1 - \alpha_n\bar{\gamma})[(1 - \beta_n)||x - y|| + \beta_n||x - y||] + \alpha_n\gamma||f(x) - f(y)|| \\ \leq\ & [1 - \alpha_n(\bar{\gamma} - \gamma)]||x - y|| - \alpha_n\gamma\varphi(||x - y||) \\ \leq\ & ||x - y|| - \psi(||x - y||), \end{aligned}$$

where $\psi(x - y) = \alpha_n\gamma\phi(||x - y||)$. This shows that S_n^f is a weakly contractive mapping with a function ψ on H for each $n \geq 1$. Therefore, by Theorem 5 of [11], S_n^f has a unique fixed point (say) $x_n \in H$. This means that Eq.(2.1) has a unique solution for each $n \geq 1$, namely,

$$\begin{aligned} x_n &= (I - \alpha_n A)[(1 - \beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)T_r x_n ds + \beta_n T_r x_n] + \alpha_n\gamma f(x_n) \\ &= (I - \alpha_n A)[(1 - \beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \beta_n u_n] + \alpha_n\gamma f(x_n), \end{aligned}$$

where note from Lemma 1.1 that u_n can be re-written as $u_n = T_r x_n$.

Next, we show that $\{x_n\}$ is bounded. Indeed, for any $z \in \Omega$, note that $z = T_r z$. It follows from Lemma 1.1 that

$$||u_n - z|| = ||T_r x_n - z|| = ||T_r x_n - T_r z|| \leq ||x_n - z||. \tag{2.3}$$

Notice that

$$\begin{aligned} ||z_n - z|| &\leq (1 - \beta_n)||u_n - z|| + \beta_n||u_n - z|| \\ &= ||u_n - z|| \leq ||x_n - z||. \end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned} &||x_n - z||^2 = \langle x_n - z, x_n - z\rangle \\ &= \langle (I - \alpha_n A)(z_n - z), x_n - z\rangle + \alpha_n\gamma\langle f(x_n) - f(z), x_n - z\rangle \\ &\quad + \alpha_n\langle \gamma f(z) - Az, x_n - z\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})||x_n - z||^2 + \alpha_n\gamma||x_n - z||^2 - \alpha_n\gamma\varphi(||x_n - z||)||x_n - z|| \\ &\quad + \alpha_n\langle \gamma f(z) - Az, x_n - z\rangle \\ &= [1 - \alpha_n(\bar{\gamma} - \gamma)]||x_n - z||^2 + \alpha_n\langle \gamma f(z) - Az, x_n - z\rangle - \alpha_n\gamma\varphi(||x_n - z||)||x_n - z|| \\ &\leq ||x_n - z||^2 + \alpha_n\langle \gamma f(z) - Az, x_n - z\rangle - \alpha_n\gamma\varphi(||x_n - z||)||x_n - z||. \end{aligned} \tag{2.5}$$

Therefore,

$$\varphi(||x_n - z||) \leq \frac{1}{\gamma}||\gamma f(z) - Az||,$$

which implies that $\{\phi(||x_n - z||)\}$ is bounded. We obtain that $\{||x_n - z||\}$ is bounded by the property of ϕ. So $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{z_n\}$, $\{f(u_n)\}$, $\{Az_n\}$ from Eq.(2.3) and Eq.(2.4). We may, without loss of generality, assume that there exists a bounded set $K \subset C$ such that $x_n, u_n, z_n \in K$, for each $n \geq 1$.

Step 2. We claim that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup z^*$ and $z^* \in F(\Im)$.

Indeed, for any $z \in \Omega$, denote $w_n := \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds$, then $||w_n-z|| \leq ||u_n-z|| \leq ||x_n-z||$. From Eq.(2.1), the boundedness of $\{f(x_n)\}$, $\{Az_n\}$, $\{u_n\}$, $\{w_n\}$ and the conditions $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, we see that

$$||z_n - w_n|| = \beta_n||u_n - w_n|| \to 0 \ (n \to \infty) \tag{2.6}$$

and

$$||x_n - z_n|| = \alpha_n||\gamma f(x_n) - Az_n|| \to 0 \ (n \to \infty). \tag{2.7}$$

In view of Eq.(2.6) and Eq.(2.7), we obtain that

$$||x_n - w_n|| \to 0 \ (n \to \infty). \tag{2.8}$$

Let $K_1 = \{\omega \in K : \varphi(||\omega - z||) \leq \frac{1}{\gamma}||\gamma f(z) - Az||\}$, then K_1 is a nonempty bounded closed convex subset of H and $T(s)$-invariant. Since $\{x_n\} \subset K_1$ and K_1 is bounded, there exists $r > 0$ such that $K_1 \subset B_r$, it follows from Lemma 1.3 that

$$\limsup_{s\to\infty} \limsup_{n\to\infty} ||w_n - T(s)w_n|| = 0. \tag{2.9}$$

By virtue of Eq.(2.8) and Eq.(2.9), we arrive at

$$\limsup_{s\to\infty} \limsup_{n\to\infty} ||x_n - T(s)x_n|| = 0. \tag{2.10}$$

On the other hand, since $\{x_n\}$ is bounded, we know that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z^*$. By Lemma 1.4 and Eq.(2.10), we arrive at $z^* \in F(\Im)$.

In Eq.(2.5), interchange z^* and z to obtain

$$\psi(||x_{n_k} - z^*||) \leq \langle \gamma f(z^*) - Az^*, \ x_{n_k} - z^* \rangle,$$

where $\psi(||x_{n_k} - z^*||) = \varphi(||x_{n_k} - z^*||)||x_{n_k} - z^*||$. From $x_{n_k} \rightharpoonup z^*$, we get that

$$\limsup_{k\to\infty} \psi(||x_{n_k} - z^*||) \leq \lim_{n\to\infty} \langle \gamma f(z^*) - Az^*, x_{n_k} - z^* \rangle = 0.$$

namely,

$$\psi(||x_{n_k} - z^*||) \to 0 \ as \ k \to \infty,$$

which implies that $x_{n_k} \to z^*$ as $k \to \infty$ by the property of ψ, and thus $z_{n_k} \to z^*$.

Step 3. We shall show that $\lim_{n\to\infty} ||u_n - x_n|| = 0$ and $z^* \in EP(F)$, where z^* is obtained in Step 2.

Since T_r is firmly nonexpansive, from Lemma 1.1(2), we see for any $z \in \Omega$ that

$$\begin{aligned} \|u_n - z\|^2 &= \|T_r x_n - T_r z\|^2 \\ &\leq \langle T_r x_n - T_r z, \ x_n - z \rangle \\ &= \langle u_n - z, \ x_n - z \rangle \\ &= \frac{1}{2}(\|u_n - z\|^2 + \|x_n - z\|^2 - \|u_n - x_n\|^2), \end{aligned}$$

from which it follows that

$$||u_n - z||^2 \leq ||x_n - z||^2 - ||u_n - x_n||^2. \tag{2.11}$$

On the other hand, it follows from Eq.(2.1) and Eq.(2.11) that

$$\begin{aligned} &\|x_n - z\|^2 \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|z_n - z\|^2 + 2\alpha_n \langle \gamma f(x_n) - Az, \ x_n - z \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|u_n - z\|^2 + 2\alpha_n \gamma \|x_n - z\|^2 + 2\alpha_n ||\gamma f(z) - Az|| ||x_n - z|| \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - z\|^2 - (1 - \alpha_n \bar{\gamma})^2 \|u_n - x_n\|^2 + 2\alpha_n \gamma \|x_n - z\|^2 \\ &\quad + 2\alpha_n ||\gamma f(z) - Az|| ||x_n - z||. \end{aligned} \tag{2.12}$$

Moreover, we have from Eq.(2.12) that

$$\begin{aligned} &(1-\alpha_n\bar{\gamma})^2\|u_n-x_n\|^2 \\ &\leq (1-\alpha_n\bar{\gamma})^2\|x_n-z\|^2-\|x_n-z\|^2+2\alpha_n\gamma\|x_n-z\|^2 \\ &\quad +2\alpha_n||\gamma f(z)-Az||||x_n-z|| \\ &\leq 2\alpha_n\gamma\|x_n-z\|^2+2\alpha_n||\gamma f(z)-Az||||x_n-z|| \\ &\leq 4\alpha_n M, \end{aligned} \tag{2.13}$$

where M is a constant such that $M \geq \max\{\sup_{n\geq 1}\{\gamma||x_n - z||^2\}, \sup_{n\geq 1}\{||\gamma f(z) - Az||\,||x_n - z||\}\}$. From the condition $\alpha_n \to 0(n \to \infty)$, we get from Eq.(2.13) that $\limsup_{n\to\infty} ||u_n - x_n|| = 0$, which implies that as $n \to \infty$, $||u_n - x_n|| \to 0$. Because $||u_n - x_n|| = ||T_r x_n - x_n|| \to 0(n \to \infty)$, we see from Lemma 1.5 and Lemma 1.1 that $z^* \in F(T_r) = EP(F)$. Therefore, $z^* \in \Omega$.

Step 4. We claim that z^* is the unique solution of the variational inequality (2.2).

Firstly, we show the uniqueness of the solution to the variational inequality (2.2) in Ω. In fact, suppose $p, q \in \Omega$ satisfy Eq.(2.2), we see that

$$\langle (A-\gamma f)p, p-q\rangle \leq 0, \tag{2.14}$$

$$\langle (A-\gamma f)q, q-p\rangle \leq 0. \tag{2.15}$$

Adding these two inequalities (2.14) (2.15) yields

$$\begin{aligned} 0 &\geq \langle A(p-q),\ p-q\rangle - \gamma\langle f(p)-f(q),\ p-q\rangle \\ &\geq \bar{\gamma}\left\|p-q\right\|^2-\gamma\left\|p-q\right\|^2+\gamma\varphi(||p-q||)||p-q|| \\ &= (\bar{\gamma}-\gamma)\left\|p-q\right\|^2+\gamma\varphi(||p-q||)||p-q||, \end{aligned}$$

thus

$$\varphi(||p-q||) \leq \frac{\gamma-\bar{\gamma}}{\gamma}||p-q||.$$

From $\frac{\gamma-\bar{\gamma}}{\gamma} \leq 0$, we get that

$$\varphi(||p-q||) \leq 0.$$

By the property of ϕ, we must have $p = q$ and the uniqueness is proved.

Next we show that z^* is a solution in Ω to the variational inequality (2.2).

In fact, since

$$x_n = (I-\alpha_n A)(1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + (I-\alpha_n A)\beta_n u_n + \alpha_n\gamma f(x_n),$$

we derive that

$$\begin{aligned} &Ax_n-\gamma f(x_n) \\ &= -\frac{1}{\alpha_n}(I-\alpha_n A)(1-\beta_n)(I-\frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n \\ &\quad +\frac{1}{\alpha_n}[(I-\alpha_n A)u_n-(I-\alpha_n A)x_n]. \end{aligned}$$

For any $z \in \Omega$, it follows that

$$\begin{aligned}
&\langle A(x_n) - \gamma f(x_n),\ u_n - z\rangle \\
&= -\frac{(1-\beta_n)}{\alpha_n}\langle (I - \alpha_n A)(I - \frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n,\ u_n - z\rangle \\
&+ \frac{1}{\alpha_n}\langle (I - \alpha_n A)(u_n - x_n),\ u_n - z\rangle \\
&= -\frac{1-\beta_n}{\alpha_n}\langle (I - \frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n - (I - \frac{1}{t_n}\int_0^{t_n} T(s)ds)z,\ u_n - z\rangle \\
&+ (1-\beta_n)\langle A(I - \frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n,\ u_n - z\rangle \\
&+ \frac{1}{\alpha_n}\langle u_n - x_n,\ u_n - z\rangle + \langle Ax_n - Au_n,\ u_n - z\rangle.
\end{aligned} \tag{2.16}$$

Now we consider the right side of Eq.(2.16). Observe from Eq.(2.1) that

$$\langle u_n - x_n,\ u_n - z\rangle \le rF(u_n,\ z).$$

Note from $z \in \Omega \subset EP(F)$ that $F(z,\ u_n) \geq 0$, then $F(u_n,\ z) \leq -F(z,\ u_n) \leq 0$, which implies that

$$\frac{1}{\alpha_n}\langle u_n - x_n,\ u_n - z\rangle \le 0.$$

On the other hand, it is easily seen that $I - \frac{1}{t_n}\int_0^{t_n} T(s)ds$ is monotone, that is

$$\langle (I - \frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n - (I - \frac{1}{t_n}\int_0^{t_n} T(s)ds)z,\ u_n - z\rangle \ge 0.$$

Thus, we obtain from Eq.(2.16) that

$$\begin{aligned}
&\langle (A(x_n) - \gamma f(x_n),\ u_n - z\rangle \\
&\le (1-\beta_n)\langle A(I - \frac{1}{t_n}\int_0^{t_n} T(s)ds)u_n,\ u_n - z\rangle + \langle Ax_n - Au_n,\ u_n - z\rangle.
\end{aligned} \tag{2.17}$$

Also, we notice from $\| x_n - u_n \| \to 0$ $(n \to \infty)$ and $x_{n_k} \to z^* \in \Omega$ that

$$\limsup_{k\to\infty} \langle A(I - \frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s)ds)u_{n_k}, u_{n_k} - z\rangle = 0, \tag{2.18}$$

and

$$\limsup_{k\to\infty} \langle A(x_{n_k} - u_{n_k}), u_{n_k} - z\rangle = 0. \tag{2.19}$$

Now replacing n in Eq.(2.17) with n_k and taking *limsup*, we have from Eq.(2.18) and Eq.(2.19) that

$$\langle (A - \gamma f)z^*,\ z^* - z\rangle \le 0. \tag{2.20}$$

for any $z \in \Omega$. This is, $z^* \in \Omega$ is unique solution of Eq.(2.2).

Step 5. We claim that

$$\limsup_{n\to\infty} \langle \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - z^*, \gamma f(z^*) - Az^*\rangle \le 0. \tag{2.21}$$

To show Eq.(2.21), we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds - z^*,\ \gamma f(z^*) - Az^*\rangle\\ &= \lim_{i\to\infty}\langle\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)u_{n_i}ds - z^*,\ \gamma f(z^*) - Az^*\rangle.\end{aligned} \tag{2.22}$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to z.

We may, assume without loss of generality, that $x_{n_i} \rightharpoonup z$, then $u_{n_i} \rightharpoonup z$, note from Step 2 and Step 3 that $z \in \Omega$ and thus $\frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s)u_{n_i}ds \rightharpoonup z$. It follows from Eq.(2.22) that

$$\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds - z^*,\ \gamma f(z^*) - Az^*\rangle = \langle z - z^*,\ \gamma f(z^*) - Az^*\rangle \le 0.$$

So Eq.(2.21) holds, thanks to Eq.(2.20).

Step 6. We claim that $x_n \to z^*$ as $n \to \infty$.

First, from Eq.(2.8) and Eq.(2.21) we conclude that

$$\limsup_{n\to\infty}\langle\gamma f(z^*) - Az^*, x_n - z^*\rangle \le 0. \tag{2.23}$$

Now we compute $||x_n - z^*||^2$ and have the following estimates:

$$\begin{aligned}&||x_n - z^*||^2\\ &= ||(I - \alpha_n A)(z_n - z^*) + \alpha_n(\gamma f(x_n) - Az^*)||^2\\ &\le (1 - \alpha_n\bar{\gamma})^2||z_n - z^*||^2 + 2\alpha_n\langle\gamma f(x_n) - Az^*,\ x_n - z^*\rangle\\ &\le (1 - \alpha_n\bar{\gamma})^2||z_n - z^*||^2 + 2\alpha_n\gamma||x_n - z^*||^2\\ &\quad -2\alpha_n\gamma\varphi(||x_n - z^*||) + 2\alpha_n\langle\gamma f(z^*) - Az^*,\ x_n - z^*\rangle\\ &\le (1 - \alpha_n\bar{\gamma})^2||x_n - z^*||^2 + 2\alpha_n\gamma||x_n - z^*||^2\\ &\quad +2\alpha_n\langle\gamma f(z^*) - Az^*,\ x_n - z^*\rangle - 2\alpha_n\gamma\varphi(||x_n - z^*||)\\ &\le (1 + \alpha_n^2\bar{\gamma}^2)||x_n - z^*||^2 - 2\alpha_n\gamma\varphi(||x_n - z^*||) + 2\alpha_n\langle\gamma f(z^*) - Az^*,\ x_n - z^*\rangle.\end{aligned}$$

It follows that

$$\varphi(||x_n - z^*||) \le \frac{\bar{\gamma}^2}{2\gamma}\alpha_n||x_n - z^*||^2 + \frac{1}{\gamma}\langle\gamma f(z^*) - Az^*,\ x_n - z^*\rangle.$$

By virtue of the boundedness of $\{x_n\}$, Eq.(2.23) and the condition $\alpha_n \to 0 (n \to \infty)$, we can conclude that $\lim_{n\to\infty} \phi(||x_n - z^*||) = 0$. By the property of ϕ, we obtain that $x_n \to z^* \in \Omega$ as $n \to \infty$. This completes the proof of Theorem 2.1.

From Theorem 2.1, we can derive the desired conclusion immediately for a single nonexpansive mapping T.

Corollary 2.1 Let C be a nonempty closed convex subset of a Hilbert space H. Let F a bifunction from $C \times C \to R$ satisfying (A1)-(A4), f be a weakly contractive mapping with a function ϕ on H, A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ on H, and T be a nonexpansive mapping from C into itself, respectively. Assume that $\Omega = F(T) \cap EP(F) \neq \emptyset$, $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ are real sequences such that:

$\lim_{n\to\infty} \alpha_n = 0$, $\beta_n = o(\alpha_n)$. Then for any $0 < \gamma \le \bar{\gamma}$ and any $r > 0$, there exists a unique $\{x_n\} \subset H$ such that

$$\begin{cases} F(u_n, y) + \dfrac{1}{r}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C, \\ z_n = (1 - \beta_n)Tu_n + \beta_n u_n \\ x_n = (I - \alpha_n A)z_n + \alpha_n \gamma f(x_n), \ \forall n \ge 1. \end{cases}$$

Furthermore, the sequence $\{x_n\}$ converges strongly to $z^* \in \Omega$ which solves the variational inequality (2.2).

Remark 2.1 Putting $\beta_n = 0$, $\phi(t) = (1 - k)t$ and $u_n = x_n$ in Theorem 2.1, we can obtain Theorem 3.1 in [30].

Remark 2.2 The parameter γ can be allowed to take the coefficient $\bar{\gamma}$ in Theorem 2.1 and Corollary 2.1. Our results contain the ones in [23] and [27] as special cases.

3. Explicit viscosity iterative algorithm

Theorem 3.1 Let C be a nonempty closed convex subset of a Hilbert space H. Let F a bifunction from $C \times C \to R$ satisfying (A1)-(A4), f be a weakly contractive mapping with a function ϕ on H, A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ on H, and $\Im = \{T(s): s \ge 0\}$ be a nonexpansive semigroup on C, respectively. Assume that $\Omega = F(\Im) \cap EP(F) \ne \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\{t_n\} \subset (0, \infty)$ are real sequences satisfying the following restrictions:

(C_1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(C_2) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(C_3) $\sum_{n=0}^{\infty} \frac{1}{t_n} < \infty$, $\sum_{n=0}^{\infty} |\frac{1}{t_n} - \frac{1}{t_{n-1}}| < \infty$.

For any $0 < \gamma \le \bar{\gamma}$ and any $r > 0$, let a sequence $\{y_n\}$ be iteratively generated from $y_1 \in C$ by:

$$\begin{cases} F(v_n, y) + \frac{1}{r}\langle y - v_n, v_n - y_n\rangle \ge 0, \ \forall y \in C, \\ z_n = (1 - \beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds + \beta_n v_n \\ y_{n+1} = (I - \alpha_n A)z_n + \alpha_n \gamma f(y_n), \quad \forall n \ge 1. \end{cases} \tag{3.1}$$

Then $\{y_n\}$ converges strongly to the unique solution in F to the inequality (2.2).

Proof. Firstly, we show that $\{y_n\}$ is bounded.

Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, with no loss of generality, that $\alpha_n < ||A||^{-1}$ for all $n \ge 1$. Then, $\alpha_n < \frac{1}{\bar{\gamma}}$ for all $n \ge 1$.

For any $z \in \Omega$, note from Lemma 1.1 that v_n can be re-written as $v_n = T_r y_n$ for each $n \ge 1$ and $z = T_r z$. It follows from Lemma 1.1 that

$$||v_n - z|| = ||T_r y_n - z|| = ||T_r y_n - T_r z|| \le ||y_n - z||.$$

Notice that

$$\begin{aligned} ||z_n - z|| &\le (1 - \beta_n)\,||v_n - z|| + \beta_n||v_n - z|| \\ &= ||v_n - z|| \le ||y_n - z||. \end{aligned}$$

From which it follows that

$$\begin{aligned}
&||y_{n+1} - z|| \\
&\leq ||I - \alpha_n A|| \, ||z_n - z|| + \alpha_n \gamma ||f(y_n) - f(z)|| + \alpha_n ||\gamma f(z) - Az|| \\
&\leq (1 - \alpha_n \bar{\gamma}) ||z_n - z|| + \alpha_n \gamma ||y_n - z|| - \alpha_n \gamma \varphi(||y_n - z||) + \alpha_n ||\gamma f(z) - Az|| \\
&\leq (1 - \alpha_n \bar{\gamma}) ||y_n - z|| + \alpha_n \gamma ||y_n - z|| + \alpha_n ||\gamma f(z) - Az|| \\
&= [1 - \alpha_n (\bar{\gamma} - \gamma)] ||y_n - z|| + \alpha_n ||\gamma f(z) - Az||.
\end{aligned}$$

By induction,

$$||y_{n+1} - z|| \leq \max\{||y_0 - z||, \frac{||\gamma f(z) - Az||}{\bar{\gamma} - \gamma}\}, \quad n \geq 0.$$

and $\{y_n\}$ is bounded, which leads to the boundedness of $\{v_n\}$, $\{z_n\}$, $\{f(v_n)\}$, $\{Az_n\}$. We may, without loss of generality, assume that there exists a bounded set $K \subset C$ such that $y_n, v_n, z_n \in K$, for each $n \geq 1$.

Using a similar method of proof as in the Step 2 of the proof of Theorem 2.1, we can conclude that

$$||y_{n+1} - w_n|| \to 0 \text{ as } n \to \infty. \tag{3.2}$$

and

$$||y_{n+1} - z_n|| \to 0 \text{ as } n \to \infty. \tag{3.3}$$

Let $K_1 = \{\omega \in K : ||\omega - z|| \leq \max\{||y_0 - z||, \frac{||\gamma f(z) - Az||}{\bar{\gamma} - \gamma}\}\}$, then K_1 is a nonempty bounded closed convex subset of H and $T(s)$-invariant. Since $\{y_{n+1}\} \subset K_1$ and K_1 is bounded, there exists $r > 0$ such that $K_1 \subset B_r$, it follows from Lemma 1.3 that

$$\limsup_{s\to\infty} \limsup_{n\to\infty} ||\omega_n - T(s) w_n|| = 0. \tag{3.4}$$

By virtue of Eq.(3.2) and Eq.(3.4), we arrive at

$$\limsup_{s\to\infty} \limsup_{n\to\infty} ||y_{n+1} - T(s) y_{n+1}|| = 0. \tag{3.5}$$

Next we shall prove that $y_{n+1} - y_n \to 0$ as $n \to \infty$.

Note that

$$\begin{aligned}
&||y_{n+1} - y_n|| \\
&\leq ||\alpha_n \gamma f(y_n) - ||\alpha_n \gamma f(y_{n-1})|| + \alpha_n \gamma f(y_{n-1}) - \alpha_{n-1} \gamma f(y_{n-1})|| \\
&\quad + ||(I - \alpha_n A) z_n - (I - \alpha_n A) z_{n-1}|| + ||(I - \alpha_n A) z_{n-1} - (I - \alpha_{n-1} A) z_{n-1}|| \\
&\leq \alpha_n \gamma ||y_n - y_{n-1}|| - \alpha_n \gamma \varphi(||y_n - y_{n-1}||) + |\alpha_n - \alpha_{n-1}| \gamma || f(y_{n-1})|| \\
&\quad + (1 - \alpha_n \gamma) ||z_n - z_{n-1}|| + |\alpha_{n-1} - \alpha_n| \, ||Az_{n-1}||.
\end{aligned} \tag{3.6}$$

Also

$$\begin{aligned}
&||z_n - z_{n-1}|| \\
&\leq ||(1 - \beta_n) w_n - (1 - \beta_n) w_{n-1}|| + ||(1 - \beta_n) w_{n-1} - (1 - \beta_{n-1}) w_{n-1}|| \\
&\quad + ||\beta_n v_n - \beta_n v_{n-1}|| + ||\beta_n v_{n-1} - \beta_{n-1} v_{n-1}|| \\
&\leq (1 - \beta_n) ||w_n - w_{n-1}|| + |\beta_{n-1} - \beta_n| —— w_{n-1}|| \\
&\quad + \beta_n ||v_n - v_{n-1}|| + |\beta_n - \beta_{n-1}| \, ||v_{n-1}||,
\end{aligned} \tag{3.7}$$

and

$$||v_n - v_{n-1}|| = ||T_r y_n - T_r y_{n-1}|| \le ||y_n - y_{n-1}||. \tag{3.8}$$

where $w_n := \frac{1}{t_n}\int_0^{t_n} T(s)v_n ds$. Now we compute $||w_n - w_{n-1}||$,

$$\begin{aligned} &||w_n - w_{n-1}|| \\ \le\ & ||\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)v_{n-1} ds|| + ||\frac{1}{t_n}\int_0^{t_n} T(s)v_{n-1} ds \\ & -\frac{1}{t_{n-1}}\int_0^{t_n} T(s)v_{n-1} ds|| + ||\frac{1}{t_{n-1}}\int_0^{t_n} T(s)v_{n-1} ds - \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)v_{n-1} ds|| \\ \le\ & ||v_n - v_{n-1}|| + |\frac{1}{t_n} - \frac{1}{t_{n-1}}|\ ||\int_0^{t_n} T(s)v_{n-1} ds|| + \frac{1}{t_{n-1}}||\int_{t_n}^{t_n - 1} T(s)\, v_{n-1} ds||. \end{aligned} \tag{3.9}$$

Substituting Eq.(3.7)-(3.9) into Eq.(3.6), we arrive at

$$\begin{aligned} &||y_{n+1} - y_n|| \\ \le\ & ||y_n - y_{n-1}|| - \alpha_n \gamma \varphi\left(||y_n - y_{n-1}||\right) + |\alpha_n - \alpha_{n-1}|M \\ & + |\frac{1}{t_n} - \frac{1}{t_{n-1}}|M + \frac{1}{t_{n-1}}M + |\beta_{n-1} - \beta_n|M, \end{aligned}$$

for some positive constant M. Thanks to the conditions (C_1) - (C_3) and Lemma 1.7, we conclude that

$$||y_{n+1} - y_n|| \ \to 0, \quad as\ n \to \infty. \tag{3.10}$$

Now, we show

$$\lim_{n\to\infty} ||v_n - y_n|| \ = \lim_{n\to\infty} ||T_r y_n - y_n|| \ = 0. \tag{3.11}$$

Indeed, using a similar method of proof as in the Step 3 of the proof of Theorem 2.1, we can obtain that

$$\begin{aligned} &\left\|y_{n+1} - z\right\|^2 \\ \le\ & (1-\alpha_n\bar{\gamma})^2\left\|y_n - z\right\|^2 - (1-\alpha_n\bar{\gamma})^2\left\|v_n - y_n\right\|^2 + 2\alpha_n\gamma||y_n - z||\,||y_{n+1} - z|| \\ & + 2\alpha_n||\gamma f(z) - Az||\,||y_{n+1} - z||, \end{aligned}$$

from which it follows that

$$\begin{aligned} &(1-\alpha_n\bar{\gamma})^2\left\|v_n - y_n\right\|^2 \\ \le\ & (1-\alpha_n\bar{\gamma})^2\left\|y_n - z\right\|^2 - \left\|y_{n+1} - z\right\|^2 + 2\alpha_n\gamma||y_n - z||\,||y_{n+1} - z|| \\ & + 2\alpha_n||\gamma f(z) - Az||\,||y_{n+1} - z|| \\ \le\ & \left\|y_n - z\right\|^2 - \left\|y_{n+1} - z\right\|^2 + 4\alpha_n M_1 \\ =\ & \left[||y_n - z|| + ||y_{n+1} - z||\right]\left[||y_n - z|| - ||y_{n+1} - z||\right] + 4\alpha_n M_1 \\ \le\ & M_1||y_n - y_{n+1}|| + 4\alpha_n M_1, \end{aligned}$$

where M_1 is an appropriate constant such that $M_1 = \max\{\sup_{n \ge 1}\{\gamma||y_n - z||\ ||y_{n+1} - z||\}, \sup_{n \ge 1}\{||\gamma f(z) - Az||\ ||y_{n+1} - z||\}, \sup_{n \ge 1}\{||y_n - z|| + ||y_{n+1} - z||\}\}$. From the condition $\alpha_n \to 0 (n \to \infty)$ and Eq.(3.10), we get that $\lim\sup_{n\to\infty} ||v_n - y_n|| = 0$, which implies that the Eq.(3.11) holds.

It follows from Theorem 2.1 that there is a unique solution $z^* \in \Omega$ to the variational inequality (2.2).

Next, we show that $\limsup_{n\to\infty} \langle \gamma f(z^*) - Az^*, y_{n+1} - z^*\rangle \leq 0$

Indeed, we can take a subsequence $\{y_{n_k+1}\}$ of $\{y_{n_k}\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f\left(z^*\right) - Az^*, y_{n+1} - z^*\rangle = \lim_{n\to\infty}\langle \gamma f\left(z^*\right) - Az^*, y_{n_k+1} - z^*\rangle$$

We may assume that $y_{n_k+1} \rightharpoonup z$ since $\{y_{n+1}\}$ is bounded. From Eq.(3.5) and Lemma 1.4, we conclude $z \in F(\mathfrak{I})$. Similarly, from Eq.(3.11) and Lemma 1.1 and Lemma 1.5, we have $z \in EP(F)$. Therefore, $z \in \Omega$.

In view of the variational inequality (2.2), we conclude

$$\begin{aligned}&\limsup_{n\to\infty}\langle \gamma f\left(z^*\right) - Az^*, y_{n+1} - z^*\rangle = \lim_{n\to\infty}\langle \gamma f\left(z^*\right) - Az^*, y_{n_k+1} - z^*\rangle\\ &= \langle \gamma f\left(z^*\right) - Az^*, z - z^*\rangle \leq 0\end{aligned}$$

From (3.4), we see that

$$\limsup_{n\to\infty}\langle \gamma f\left(z^*\right) - Az^*, z_n - z^*\rangle \leq 0. \tag{3.12}$$

Finally, we show that $y_n \to z^*$. As a matter of fact,

$$\begin{aligned}&\left\|y_{n+1} - z^*\right\|^2\\ &= ||(I - \alpha_n A)(z_n - z^*) + \alpha_n(\gamma f(y_n) - Az^*)||^2\\ &= ||(I - \alpha_n A)(z_n - z^*)||^2 + \alpha_n^2\left\|\gamma f(y_n) - Az^*\right\|^2\\ &\quad +2\alpha_n\langle (I - \alpha_n A)(z_n - z^*), \gamma f(y_n) - Az^*\rangle\\ &\leq (1 - \alpha_n\bar{\gamma})^2\left\|z_n - z^*\right\|^2 + \alpha_n^2\left\|\gamma f(y_n) - Az^*\right\|^2 + 2\alpha_n\langle z_n - z^*, \gamma f\left(y_n\right) - Az^*\rangle\\ &\quad -2\alpha_n^2\langle A(z_n - z^*), \gamma f(y_n) - Az^*\rangle \qquad (3.13)\\ &\leq (1 - \alpha_n\bar{\gamma})^2\left\|y_n - z^*\right\|^2 + \alpha_n^2\left\|\gamma f\left(y_n\right) - Az^*\right\|^2 + 2\alpha_n\gamma\langle z_n - z^*, f\left(y_n\right) - f(z^*)\rangle\\ &\quad +2\alpha_n\langle z_n - z^*, \gamma f(z^*) - Az^*\rangle - 2\alpha_n^2\rangle A(z_n - z^*), \gamma f(y_n) - Az^*\rangle\\ &\leq [(1 - \alpha_n\bar{\gamma})^2 + 2\alpha_n\gamma]\left\|y_n - z^*\right\|^2 + \alpha_n[2\langle z_n - z^*, \gamma f(z^*) - Az^*\rangle\\ &\quad +\alpha_n\left\|\gamma f(y_n) - Az^*\right\|^2 + 2\alpha_n||A(z_n - z^*)||\,||\gamma f(y_n) - Az^*||]\\ &= [1 - 2\alpha_n(\bar{\gamma} - \gamma)]\left\|y_n - z^*\right\|^2 + \alpha_n A_n,\end{aligned}$$

where

$$\begin{aligned}A_n &= 2\langle z_n - z^*, \gamma f(z^*) - Az^*\rangle + \alpha_n\left[\left\|\gamma f\left(y_n\right) - Az^*\right\|^2\right.\\ &\quad\left. +2||A(z_n - z^*)||\,||\gamma f\left(y_n\right) - Az^*|| + \bar{\gamma}^2\left\|y_n - z^*\right\|^2\right].\end{aligned}$$

Since $\{y_n\}$ is bounded, there must exist a constant $M_2 > 0$ such that

$$\left\|\gamma f\left(y_n\right) - Az^*\right\|^2 + 2||A(z_n - z^*)||\,||\gamma f\left(y_n\right) - Az^*|| + \bar{\gamma}^2\left\|y_n - z^*\right\|^2 \leq M_2.$$

It then follows from Eq.(3.13) that

$$\left\|y_{n+1} - z^*\right\|^2 \leq [1 - 2\alpha_n\left(\bar{\gamma} - \gamma\right)]\left\|y_n - z^*\right\|^2 + \alpha_n B_n, \tag{3.14}$$

where $B_n = 2\langle z_n - z^*, \gamma f(z^*) - Az^*\rangle + \alpha_n M_2$. From the conditions (C_1) - (C_2), Eq.(3.12) and Lemma 1.6, we obtain from Eq.(3.14) that $y_n \to z^*$ in norm. This completes the proof of Theorem 3.1.

From Theorem 3.1, we can derive the desired conclusion immediately for a nonexpansive mapping T.

Corollary 3.1 Let H be a Hilbert space, C be a nonempty closed convex subset of H, F a bifunction from $C \times C \to R$ satisfying (A1)-(A4). Let f be a weakly contractive mapping with a function ϕ on H, and A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ on H, and T be a nonexpansive mapping from C into itself. Assume that $\Omega = F(T) \cap EP(F) \neq \emptyset$, for any $0 < \gamma \leq \bar{\gamma}$ and any $r > 0$, let $y_1 \in C$, and $\{y_n\}$ be a sequence generated in

$$\begin{cases} F\left(v_n, y\right) + \frac{1}{r}\langle y - v_n, v_n - y_n\rangle \geq 0, & \forall y \in C, \\ z_n = (1-\beta_n)\,Tv_n + \beta_n v_n \\ y_{n+1} = (I - \alpha_n A)\,z_n + \alpha_n \gamma f\left(y_n\right), & \forall n \geq 1. \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ are real sequences satisfying the conditions (C_1) - (C_2) in Theorem 3.1. Then the sequence $\{y_n\}$ converges strongly to $z^* \in \Omega$ which uniquely solves the variational inequality (2.2).

Remark 3.1 Putting $A = I$, $\beta_n = 0$ and $\phi(t) = (1 - k)t$ in Corollary 3.1, we can easily conclude Theorem TT [10].

Remark 3.2 Putting $\beta_n = 0$, $\phi(t) = (1 - k)t$ and $u_n = x_n$ in Theorem 3.1, we can obtain Theorem 3.2 in [30].

4. Application

In this section, we shall consider another class of important nonlinear operator: k-strict pseudocontractions.

Recall that a mapping $S: C \to C$ is said to be a k-strict pseudocontraction if there exists a constant $k \in (0, 1)$ such that

$$\left\|Sx - Sy\right\|^2 \leq \left\|x - y\right\|^2 + k\left\|(I - S)\,x - (I - S)\,y\right\|^2$$

for all $x, y \in C$. Note that the class of k-strict pseudocontractions strictly includes the class of nonexpansive mappings.

Corollary 4.1 Let C be a nonempty closed convex subset of a Hilbert space H, F be a bifunction from $C \times C \to R$ satisfying (A1)-(A4). Let f be a weakly contractive mapping with a function ϕ on H, A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ on H, $T: C \to H$ be a k-strictly pseudo-contractive mapping for some $0 \leq k < 1$, respectively. Assume that $\Omega = F(T) \cap EP(F) \neq \emptyset$, for any $0 < \gamma \leq \bar{\gamma}$ and any $r > 0$, let $y_1 \in C$, and $\{y_n\}$ be a sequence generated in

$$\begin{cases} F\left(v_n, y\right) + \frac{1}{r}\langle y - v_n, v_n - y_n\rangle \geq 0, & \forall y \in C, \\ z_n = (1-\beta_n)\,P_C S y_n + \beta_n y_n \\ y_{n+1} = (I - \alpha_n A)\,z_n + \alpha_n \gamma f\left(y_n\right), & \forall n \geq 1. \end{cases}$$

where $S: C \to H$ is a mapping defined by $Sx = kx + (1 - k)Tx$ and P_C is the metric projection of H onto C, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ are real sequences satisfying the conditions

(C_1) - (C_2) in Theorem 3.1. Then the sequence $\{y_n\}$ converges strongly to $z^* \in \Omega$ which uniquely solves the variational inequality (2.2).

Proof. From Lemma 2.3 in [35], we see that $S: C \to H$ is a nonexpansive mapping and $F(T) = F(S)$. It follows from Lemma 2.2 [32] that, $P_C S: C \to C$ is a nonexpansive mapping and $F(P_C S) = F(S) = F(T)$. Hence the result follows from Corollary 3.1.

Remark 4.1 Putting $v_n = y_n$ and $\phi(t) = (1 - k)t$ in Corollary 4.1, we can obtain Theorem J in [29], Further, putting $\beta_n = 0$, we can obtain Theorem CKQ in [28].

5. Numerical examples

Now, we give some real numerical examples in which the conditions satisfy the ones of Theorems 3.1 and 2.1 and some numerical experiment results to explain the main results Theorems 3.1 and 2.1 as follows:

Example 5.1. Let $H = R$ and $C = 0[1]$. For each $x \in C$, we define $f(x) = \frac{1}{4}x^2$, $A(x) = 2x$, $T(x) = \frac{1}{2}x^2$. Let $\alpha_n = \beta_n = \frac{1}{n} \in [0, 1]$, $n \in N$. For each $(x, y) \in H \times H$, we define $F(x, y) = x^2 + y$. Then $\{y_n\}$ is the sequence generated by

$$y_{n+1} = \frac{1}{2}(1-\frac{2}{n})(1-\frac{1}{n})y_n^2 + (1-\frac{2}{n})\frac{1}{n}y_n + \frac{1}{8n}y_n^2 \tag{5.1}$$

and $y_n \to y^* = 0$ as $n \to \infty$, where $y^* = 0 \in F(T) \cap EP(F)$.

Proof. It is obvious that the bifunction $F(x, y)$ satisfies the conditions (A1)-(A4) and $f(x) = \frac{1}{4}x^2$ is a weakly contractive mapping with a function $\varphi(t) = \frac{1}{2}t$ on R, $T(x) = \frac{1}{2}x^2$ is a nonexpansive mapping on C and $F(T) = \{0\}$, $A(x) = 2x$ is a strongly positive linear bounded operator with coefficient $\bar{\gamma} = 1$ on R, and the bifunction $F(x, y) = x^2 + y$ satisfies conditions (A1)-(A4) and $EP(F) = \{y: y \geq 0\}$. $\alpha_n = \beta_n = \frac{1}{n} \in [0, 1]$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ and $F(T) \cap EP(F) = \{0\}$.

Hence, the conditions satisfy the ones of Theorem 3.1. Substituting all of the given conditions to the scheme (3.1), we have (5.1). Following the proof of Theorem 3.1, we easily obtain $\{y_n\}$ converges strongly to $y^* = 0 \in F(T) \cap EP(F)$.

The proof is completed.

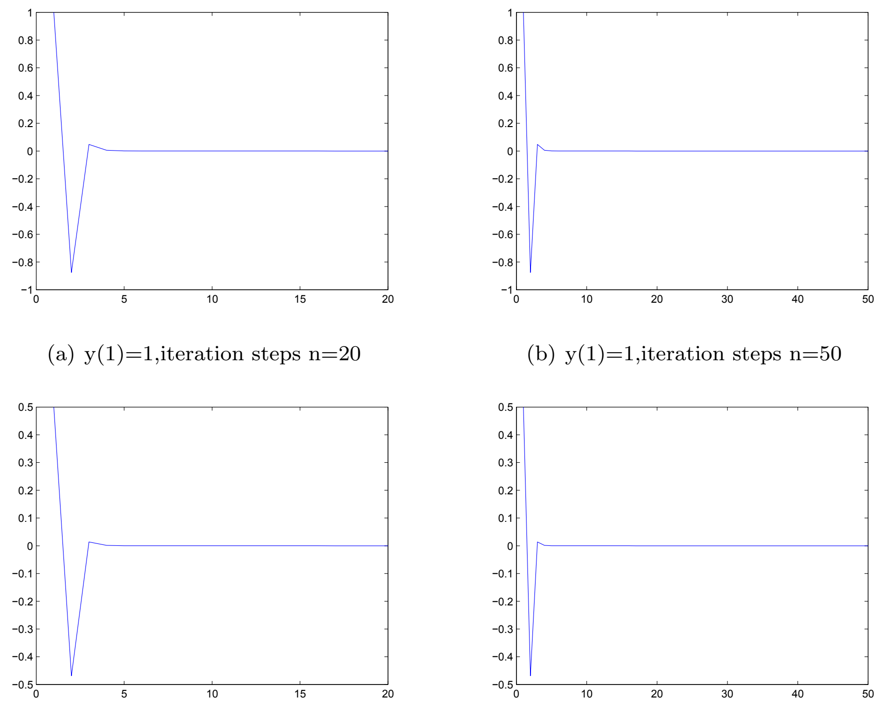
Example 5.2. Let $C = 0[2]$, $H, f, A, T, \alpha_n, \beta_n, F$ be as in Example 5.1., $F(T) \cap EP(F) = \{0, 2\}$. Then there exists a unique sequence $\{x_n\} \subset H$ satisfying the following equation

$$x_n = \frac{1}{2}(1-\frac{2}{n})(1-\frac{1}{n})x_n^2 + (1-\frac{2}{n})\frac{1}{n}x_n + \frac{1}{8n}x_n^2 \tag{5.2}$$

Furthermore, $x_n \to x^* = 2$ as $n \to \infty$, where $x^* = 2 \in F(T) \cap EP(F)$.

Proof. As in the proof of Example 5.1, the conditions satisfy the ones of Theorem 2.1. Substituting all of the given conditions to the scheme (2.1), we have (5.2), and if $x_n \neq 0$ for each n, (5.2) is equal to the following equation

$$x_n = \frac{1-(1-\frac{2}{n})\frac{1}{n}}{\frac{1}{2}(1-\frac{2}{n})(1-\frac{1}{n}) + \frac{1}{8n}}$$



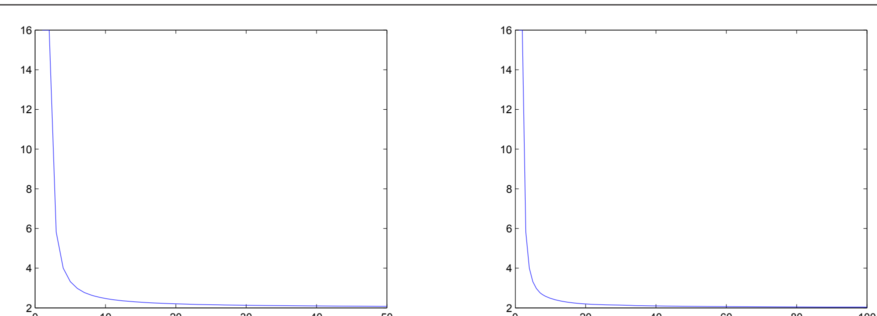
(a) y(1)=1,iteration steps n=20 (b) y(1)=1,iteration steps n=50

(c) y(1)=0.5,iteration steps n=20 (d) y(1)=0.5,iteration steps n=50

Figure 1 The case of iteration sequence (5.1).

Following the proof of Theorem 2.1, we easily obtain $\{x_n\}$ converges strongly to $x^* = 2 \in F(T) \cap EP(F)$. The proof is completed.

Next, we give the numerical experiment results using software Matlab 7.0 and get Figures 1 and 2, which show that the iteration processes of the sequence $\{y_n\}$ as initial point $y(1) = 1$, $y(1) = 0.5$ and the sequence $\{x_n\}$, respectively. From the figures, we can see that $\{y_n\}$ converges to 0 and $\{x_n\}$ converges to 2, and the more the iteration steps are, the more fast the sequence $\{y_n\}$ and $\{x_n\}$ converges to 0 and 2, respectively.



(a) iteration steps n=50 (b) iteration steps n=100

Figure 2 The case of iteration sequence (5.2).

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Authors' contributions
XX and SL carried out the proof of convergence of the theorems. LL, HS and LZ carried out the check of the manuscript. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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