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New proofs of Schur-concavity for a class of symmetric functions

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Abstract

By properties of the Schur-convex function, Schur-concavity for a class of symmetric functions is simply proved uniform.

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1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{x = (x_1, ..., x_n) : x_i \in \mathbb{R}, i = 1, ..., n\},$$
$$\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) : x_i > 0, i = 1, ..., n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$ respectively.

For convenience, we introduce some definitions as follows.

Definition 1. [1,2] Let $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{y} = (y_1, ..., y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2,..., n$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, ϕ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y})$. ϕ is said to be decreasing if and only if $-\phi$ is increasing.

Definition 2. [1,2] Let $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{y} = (y_1, ..., y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2,..., n - 1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, ϕ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$. ϕ is said to be a Schur-concave function on Ω if and only if $-\phi$ is Schur-convex function on Ω.

Definition 3. [1,2] Let $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{y} = (y_1, ..., y_n) \in \mathbb{R}^n$.

(*i*) $\Omega \subseteq \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \le \alpha \le 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function ϕ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha x + (1-\alpha)y) \le \alpha\varphi(x) + (1-\alpha)\varphi(y)$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0,1]$. ϕ is said to be a concave function on Ω if and only if $-\phi$ is convex function on Ω.

Recall that the following so-called Schur's condition is very useful for determining whether or not a given function is Schur-convex or Schur-concave.

Theorem A. [[1], p. 5] *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. ϕ: $\Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then ϕ is the Schur-convex (Schur-concave) function, if and only if ϕ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \ge 0(\le 0) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

In recent years, by using Theorem A, many researchers have studied the Schur-convexity of some of symmetric functions.

Chu et al. [3] defined the following symmetric functions

$$F_n(x,k) = \prod_{1\le i_1<\ldots<i_k\le n} \frac{\sum_{j=1}^{k} x_{i_j}}{\sum_{j=1}^{k}(1+x_{i_j})}, k = 1, \ldots, n, \tag{2}$$

and established the following results by using Theorem A.

Theorem B. *For $k = 1,\ldots, n$, $F_n(\boldsymbol{x}, k)$ is an Schur-concave function on $\mathbb{R}^n_+$.*

Jiang [4] are discussed the following symmetric functions

$$H_k^*(x) = \prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} x_{i_j}^{1/k}, k = 1, \ldots, n, \tag{3}$$

and established the following results by using Theorem A.

Theorem C. *For $k = 1, \ldots, n$, $H_k^*(\boldsymbol{x})$ is an Schur-concave function on $\mathbb{R}^n_+$.*

Xia and Chu [5] investigated the following symmetric functions

$$\phi_n(\boldsymbol{x},k) = \prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} \frac{x_{i_j}}{1+x_{i_j}}, k = 1, \ldots, n, \tag{4}$$

and established the following results by using Theorem A.

Theorem D. *For $k = 1,\ldots, n$, $F_n(\boldsymbol{x}, k)$ is an Schur-concave function on $\mathbb{R}^n_+$.*

In this note, by properties of the Schur-convex function, we simply prove Theorems B, C and D uniform.

2. New proofs three theorems

To prove the above three theorems, we need the following lemmas.

Lemma 1. [[1], p. 67], [2]*If ϕ is symmetric and convex (concave) on symmetric convex set Ω, then ϕ is Schur-convex (Schur-concave) on Ω.*

Lemma 2. [[1], p. 73],[2]*Let* $\Omega \subset \mathbb{R}^n$, $\phi: \Omega \to \mathbb{R}_+$. *Then* $\ln\phi$ *is Schur-convex (Schur-concave) if and only if* ϕ *is Schur-convex (Schur-concave).*

Lemma 3. [[1], p. 446], [2]*Let* $\Omega \subset \mathbb{R}^n$ *be open convex set,* $\phi : \Omega \to \mathbb{R}$. *For* $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, *defined one variable function* $g(t) = \phi(t\boldsymbol{x} + (1 - t)\boldsymbol{y})$ *on interval* (0, 1). *Then* ϕ *is convex (concave) on* Ω *if and only if g is convex (concave) on* (0, 1) *for all* $\boldsymbol{x},\boldsymbol{y} \lfloor \Omega$.

Lemma 4. *Let* $\boldsymbol{x} = (x_1,\dots, x_m)$ *and* $\boldsymbol{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$. *Then the following functions are concave on* (0,1).

(i) $f(t) = \ln \sum_{j=1}^{m} (tx_j + (1-t)y_j) - \ln \sum_{j=1}^{m} (1 + tx_j + (1-t)y_j)$,

(ii) $g(t) = \ln \sum_{j=1}^{m} (tx_j + (1-t)y_j)^{1/m}$,

(iii) $h(t) = \frac{1}{m} \ln \psi(t)$, *where*

$$\psi(t) = \sum_{j=1}^{m} \frac{tx_j + (1-t)y_j}{1 + tx_j + (1-t)y_j}.$$

Proof. (i) Directly calculating yields

$$f'(t) = \sum_{j=1}^{m} (x_j - y_j) \left[\frac{1}{tx_j + (1-t)y_j} - \frac{1}{1 + tx_j + (1-t)y_j} \right]$$

and

$$\begin{aligned} f''(t) &= -\sum_{j=1}^{m} (x_j - y_j)^2 \left[\frac{1}{(tx_j + (1-t)y_j)^2} - \frac{1}{(1 + tx_j + (1-t)y_j)^2} \right] \\ &= -\sum_{j=1}^{m} (x_j - y_j)^2 \frac{1 + 2tx_j + 2(1-t)y_j}{(tx_j + (1-t)y_j)^2 (1 + tx_j + (1-t)y_j)^2}. \end{aligned}$$

Since $f''(t) \le 0$, $f(t)$ is concave on (0,1).

(ii) Directly calculating yields

$$g'(t) = \frac{\frac{1}{m} \sum_{j=1}^{m} (x_j - y_j)^{\frac{1}{m}-1}}{\sum_{j=1}^{m} (tx_j + (1-t)y_j)^{1/m}}$$

and

$$g''(t) = -\frac{\left[\frac{1}{m} \sum_{j=1}^{m} (x_j - y_j)^{\frac{1}{m}-1} \right]^2}{\sum_{j=1}^{m} (tx_j + (1-t)y_j)^{2/m}}.$$

Since $g''(t) \le 0$, $f(t)$ is concave on (0,1)

(iii) By computing,

$$h'(t) = \frac{1}{m} \frac{\psi'(t)}{\psi(t)},$$

$$h''(t) = \frac{1}{m} \frac{\psi''(t)\psi(t) - (\psi'(t))^2}{\psi^2(t)},$$

where

$$\psi'(t) = \sum_{j=1}^{m} \frac{x_j - y_j}{(1 + tx_j + (1-t)y_j)^2}$$

and

$$\psi''(t) = -\sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(1 + tx_j + (1-t)y_j)^3}.$$

Thus,

$$\begin{aligned}\psi''(t)\psi(t) - (\psi'(t))^2 = &-\sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(1 + tx_j + (1-t)y_j)^3} \sum_{j=1}^{m} \frac{tx_j + (1-t)y_j}{1 + tx_j + (1-t)y_j} \\ &- \left[\sum_{j=1}^{m} \frac{x_j - y_j}{(1 + tx_j + (1-t)y_j)^2}\right]^2 \le 0,\end{aligned}$$

and then $h''(t) \le 0$, so $f(t)$ is concave on (0,1).

The proof of Lemma 4 is completed.

Proof of Theorem A: For any $1 \le i_1 < \cdots < i_k \le n$, by Lemma 3 and Lemma 4(i), it follows that $\ln \sum_{j=1}^{k} x_{i_j} - \ln \sum_{j=1}^{k} (1 + x_{i_j})$ is concave on $\mathbb{R}_+^n$, and then $\ln F_n(\boldsymbol{x}, k) = \prod_{1 \le i_1 < \cdots < i_k \le n} \left(\ln \sum_{j=1}^{k} x_{i_j} - \ln \sum_{j=1}^{k} (1 + x_{i_j})\right)$ is concave on $\mathbb{R}_+^n$. Furthermore, it is clear that $\ln F_n(\boldsymbol{x}, k)$ is symmetric on $\mathbb{R}_+^n$, by Lemma 1, it follows that $\ln F_n(\boldsymbol{x}, k)$ is concave on $\mathbb{R}_+^n$, and then from Lemma 2 we conclude that $F_n(\boldsymbol{x}, k)$ is also concave on $\mathbb{R}_+^n$.

The proof of Theorem A is completed.

Similar to the proof of Theorem A, by Lemma 4 (ii) and Lemma 4 *(iii)*, we can prove Theorems B and C, respectively. Omitted detailed process.

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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