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Sharp bounds by the power mean for the generalized Heronian mean

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Abstract

In this article, we answer the question: For $p, \omega \in \mathbb{R}$ with $\omega > 0$ and $p(\omega - 2) \neq 0$, what are the greatest value $r_1 = r_1(p, \omega)$ and the least value $r_2 = r_2(p, \omega)$ such that the double inequality $M_{r_1}(a, b) < H_{p,\omega}(a, b) < M_{r_2}(a, b)$ holds for all $a, b > 0$ with $a \neq b$? Here $H_{p,\omega}(a, b)$ and $M_r(a, b)$ denote the generalized Heronian mean and rth power mean of two positive numbers a and b, respectively.

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1 Introduction

In the recent past, the bivariate means have been the subject of intensive research. In particular, many remarkable inequalities can be found in the literature [1-26].

The power mean $M_r(a, b)$ of order r of two positive numbers a and b is defined by

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.1}$$

It is well-known that $M_r(a, b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Let $A(a, b) = (a + b)/2$, $G(a,b) = \sqrt{ab}$, $H(a, b) = 2ab/(a + b)$, $I(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$ $(b \neq a)$, $I(a, b) = a$ $(b = a)$, and $L(a, b) = (b-a)/(\log b - \log a)$ $(b \neq a)$, $L(a, b) = a$ $(b = a)$ be the arithmetic, geometric, harmonic, identric, and logarithmic means of two positive numbers a and b, respectively. Then

$$\begin{aligned} &\min\{a,b\} \le H(a,b) = M_{-1}(a,b) \le G(a,b) = M_0(a,b) \\ &\le L(a,b) \le I(a,b) \le A(a,b) = M_1(a,b) \le \max\{a,b\} \end{aligned} \tag{1.2}$$

for all $a, b > 0$, and each inequality becomes equality if and only if $a = b$.

The classical Heronian mean $He(a, b)$ of two positive numbers a and b is defined by ([27], see also [28])

$$He(a,b) = \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) = \frac{a+\sqrt{ab}+b}{3}. \tag{1.3}$$

In [27], Alzer and Janous established the following sharp double inequality (see also [[28], p. 350]):

$$M_{\log 2/\log 3}(a,b) < He(a,b) < M_{2/3}(a,b)$$

for all a, b >0 with $a \neq b$.

Mao [29] proved that

$$M_{1/3}(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) < M_{1/2}(a,b)$$

for all a, b >0 with $a \neq b$, and $M_{1/3}(a, b)$ is the best possible lower power mean bound for the sum $\frac{1}{3}A(a,b) + \frac{2}{3}G(a,b)$.

For any $\alpha \in (0, 1)$, Janous [30] found the greatest value p and the least value q such that $M_p(a, b) < \alpha A(a, b) + (1 - \alpha)G(a, b) < M_q(a, b)$ for all a, b >0 with $a \neq b$.

The following sharp bounds for L, I, $(LI)^{1/2}$ and $(L + I)/2$ in terms of power mean are given in [10,21-25,31,32]:

$$\begin{aligned} M_0(a,b) &< L(a,b) < M_{1/3}(a,b), \\ M_{2/3}(a,b) &< I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) &< \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b), \\ M_{\log 2/(1+\log 2)}(a,b) &< \frac{1}{2}[L(a,b) + I(a,b)] < M_{1/2}(a,b) \end{aligned}$$

for all a, b >0 with $a \neq b$.

In [6,7] the authors established the following sharp inequalities:

$$\begin{aligned} &M_{-1/3}(a,b) < \frac{2}{3}G(a,b) + \frac{1}{3}H(a,b) < M_0(a,b), \\ &M_{-2/3}(a,b) < \frac{1}{3}G(a,b) + \frac{2}{3}H(a,b) < M_0(a,b), \\ &M_0(a,b) < A^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{(1+2\alpha)/3}(a,b), \\ &M_0(a,b) < G^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{(1-\alpha)/3}(a,b) \end{aligned}$$

for all for all a, b >0 with $a \neq b$ and $\alpha \in (0, 1)$.

For $\omega \geq 0$ and $p \in \mathbb{R}$ the generalized Heronian mean $H_{p,\omega}(a, b)$ of two positive numbers a and b was introduced in [33] as follows:

$$H_{p,\omega}(a,b) = \begin{cases} \left[\frac{a^p+\omega(ab)^{p/2}+b^p}{\omega+2}\right]^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.4}$$

It is not difficult to verify that $H_{p,\omega}(a, b)$ is continuous with respect to $p \in \mathbb{R}$ for fixed a, b >0 and $\omega \geq 0$, strictly increasing with respect to $p \in \mathbb{R}$ for fixed a, b >0 with $a \neq b$ and $\omega \geq 0$, strictly decreasing with respect to $\omega \geq 0$ for fixed a, b >0 with $a \neq b$ and p >0 and strictly increasing with respect to $\omega \geq 0$ for fixed a, b >0 with $a \neq b$ and p <0.

From (1.1) and (1.3) together with (1.4) we clearly see that $H_{p,0}(a, b) = M_p(a, b)$, $H_{p,2}(a,b) = M_{\frac{p}{2}}(a,b)$, $H_{0,\omega}(a, b) = M_0(a, b)$ and $H_{1,1}(a, b) = H_e(a, b)$ for all a, b >0 and $\omega \geq 0$.

The purpose of this article is to answer the question: For $p, \omega \in \mathbb{R}$ with ω >0 and $p(\omega - 2) \neq 0$, what are the greatest value $r_1 = r_1(p, \omega)$ and the least value $r_2 = r_2(p, \omega)$

such that the double inequality $M_{r_1}(a,b) < H_{p,\omega}(a,b) < M_{r_2}(a,b)$ holds for all a, b >0 with $a \neq b$?

2 Main result

In order to establish our main results we need the following Lemma 2.1.

Lemma 2.1. (see [30]). $(\omega + 2)^2 > 2^{\omega+2}$ for $\omega \in (0, 2)$, and $(\omega + 2)^2 < 2^{\omega+2}$ for $\omega \in (2, +\infty)$.

Theorem 2.1. For all a, b >0 with $a \neq b$ we have

$$M_{\frac{2}{\omega+2}p}(a,b) < H_{p,\omega}(a,b) < M_{\frac{\log 2}{\log(\omega+2)}p}(a,b)$$

for $(p, \omega) \in \{(p, \omega): p > 0, \omega > 2\} \cup \{(p, \omega): p < 0, 0 < \omega < 2\}$ and

$$M_{\frac{2}{\omega+2}p}(a,b) > H_{p,\omega}(a,b) > M_{\frac{\log 2}{\log(\omega+2)}p}(a,b)$$

for $(p, \omega) \in \{(p, \omega): p > 0, 0 < \omega < 2\} \cup \{(p, \omega): p < 0, \omega > 2\}$, and the parameters $\frac{2}{\omega+2}p$ and $\frac{\log 2}{\log(\omega+2)}p$ are the best possible in either case.

Proof. Without loss of generality, we can assume that $a > b$ and put $t = \frac{a}{b} > 1$.

Firstly, we compare the value of $M_{\frac{2}{\omega+2}p}(a,b)$ with that of $H_{p,\omega}(a, b)$. From (1.1) and (1.4) we have

$$\begin{aligned} &\log[M_{\frac{2}{\omega+2}p}(a,b)] - \log[H_{p,\omega}(a,b)] \\ &= \frac{\omega+2}{2p}\log\frac{1+t^{\frac{2}{\omega+2}p}}{2} - \frac{1}{p}\log\frac{1+\omega t^{\frac{p}{2}}+t^p}{\omega+2}. \end{aligned} \tag{2.1}$$

Let

$$f(t) = \frac{\omega+2}{2p}\log\frac{1+t^{\frac{2p}{2+\omega}}}{2} - \frac{1}{p}\log\frac{1+\omega t^{\frac{p}{2}}+t^p}{\omega+2}. \tag{2.2}$$

Then simple computations lead to

$$f(1) = 0, \tag{2.3}$$

$$\begin{aligned} f'(t) &= \frac{t^{\frac{2p}{\omega+2}}g(t)}{2t(1+t^{\frac{2p}{\omega+2}})(1+\omega t^{\frac{p}{2}}+t^p)}, \\ g(t) &= -2t^{\frac{\omega p}{\omega+2}} + \omega t^{\frac{p}{2}} - \omega t^{\frac{\omega-2}{2(\omega+2)}p} + 2, \end{aligned} \tag{2.4}$$

$$g(1) = 0, \tag{2.5}$$

$$g'(t) = \omega p t^{\frac{(\omega-2)p}{2(\omega+2)}-1}h(t), \tag{2.6}$$

$$h(t) = -\frac{2}{\omega+2}t^{\frac{p}{2}} + \frac{1}{2}t^{\frac{2p}{\omega+2}} - \frac{\omega-2}{2(\omega+2)}, \quad h(1) = 0, \tag{2.7}$$

$$h'(t) = \frac{p}{\omega+2}t^{\frac{2p}{\omega+2}-1}\left[1 - t^{\frac{(\omega-2)p}{2(\omega+2)}}\right]. \tag{2.8}$$

We divide the comparison into two cases.

Case 1. If $(p, \omega) \in \{(p, \omega): p > 0, \omega > 2\} \cup \{(p, \omega): p < 0, 0 < \omega < 2\}$, then from (2.8) we clearly see that

$$h'(t) < 0 \tag{2.9}$$

for $t > 1$.

Therefore, $M_{\frac{2}{\omega+2}p}(a,b) < H_{p,\omega}(a,b)$ follows from (2.1)-(2.7) and (2.9).

Case 2. If $(p, \omega) \in \{(p, \omega): p > 0, 0 < \omega < 2\} \cup \{(p, \omega): p < 0, \omega > 2\}$, then (2.8) leads to

$$h'(t) > 0 \tag{2.10}$$

for $t > 1$.

Therefore, $M_{\frac{2}{\omega+2}p}(a,b) > H_{p,\omega}(a,b)$ follows from (2.1)-(2.7) and (2.10).

Secondly, we compare the value of $M_{\frac{\log 2}{\log(\omega+2)}p}(a,b)$ with that of $H_{p,\omega}(a, b)$. From (1.1) and (1.4) we have

$$\begin{aligned}&\log[M_{\frac{\log 2}{\log(\omega+2)}p}(a,b)] - \log[H_{p,\omega}(a,b)] \\ &= \frac{\log(\omega+2)}{p\log 2}\log\frac{1+t^{\frac{\log 2}{\log(\omega+2)}p}}{2} - \frac{1}{p}\log\frac{1+\omega t^{\frac{p}{2}}+t^p}{\omega+2}.\end{aligned} \tag{2.11}$$

Let

$$F(t) = \frac{\log(\omega+2)}{p\log 2}\log\frac{1+t^{\frac{\log 2}{\log(\omega+2)}p}}{2} - \frac{1}{p}\log\frac{1+\omega t^{\frac{p}{2}}+t^p}{\omega+2}. \tag{2.12}$$

Then simple computations lead to

$$F(1) = \lim_{t\to+\infty}F(t) = 0, \tag{2.13}$$

$$F'(t) = \frac{t^{\frac{\log 2}{\log(\omega+2)}p}G(t)}{t(1+t^{\frac{\log 2}{\log(\omega+2)}p})(1+\omega t^{\frac{p}{2}}+t^p)}, \tag{2.14}$$

$$G(t) = -t^{(1-\frac{\log 2}{\log(\omega+2)})p} + \frac{\omega}{2}t^{\frac{p}{2}} - \frac{\omega}{2}t^{\frac{1}{2}(1-\frac{2\log 2}{\log(\omega+2)})p} + 1, \tag{2.15}$$

$$G(1) = 0, \tag{2.16}$$

$$G'(t) = pt^{\frac{1}{2}(1-\frac{2\log 2}{\log(\omega+2)})p-1}H(t), \tag{2.17}$$

$$H(t) = (\frac{\log 2}{\log(\omega + 2)} - 1)t^{\frac{p}{2}} + \frac{\omega}{4}t^{\frac{\log 2}{\log(\omega+2)}p} - \frac{\omega}{4}(1 - \frac{2\log 2}{\log(\omega + 2)}), \tag{2.18}$$

$$H(1) = \frac{(\omega + 2)\log 2}{2\log(\omega + 2)} - 1, \tag{2.19}$$

$$H'(t) = \frac{\log 2 - \log(\omega + 2)}{2\log(\omega + 2)}p[t^{\frac{1}{2}(1-\frac{2\log 2}{\log(\omega+2)})p} - \frac{\omega\log 2}{2(\log(\omega + 2) - \log 2)}] \times t^{\frac{\log 2}{\log(\omega+2)}p-1}. \tag{2.20}$$

We divide the comparison into four cases.

Case A. If $p > 0$ and $\omega > 2$, then from (2.15) and (2.18)-(2.20) together with Lemma 2.1 we clearly see that

$$\lim_{t\to+\infty} G(t) = -\infty, \tag{2.21}$$

$$\lim_{t\to+\infty} H(t) = -\infty, \tag{2.22}$$

$$H(1) > 0, \tag{2.23}$$

and there exists $a_1 > 1$ such that

$$H'(t) > 0 \tag{2.24}$$

for $t \in [1, a_1)$ and

$$H'(t) < 0 \tag{2.25}$$

for $t \in (a_1, +\infty)$.

From (2.24) and (2.25) we know that $H(t)$ is strictly increasing in $[1, a_1]$ and strictly decreasing in $[a_1, +\infty)$. Then (2.22) and (2.23) together with the monotonicity of $H(t)$ imply that there exists $a_2 > 1$ such that $H(t) > 0$ for $t \in [1, a_2)$ and $H(t) < 0$ for $t \in (a_2, +\infty)$. It follows from (2.17) that $G(t)$ is strictly increasing in $[1, a_2]$ and strictly decreasing in $[a_2, +\infty)$.

From (2.16) and (2.21) together with the monotonicity of $G(t)$ we know that there exists $a_3 > 1$ such that $G(t) > 0$ for $t \in (1, a_3)$ and $G(t) < 0$ for $t \in (a_3, +\infty)$. Then (2.14) leads to that $F(t)$ is strictly increasing in $[1, a_3]$ and strictly decreasing in $[a_3, +\infty)$.

Therefore, $M_{\frac{\log 2}{\log(\omega+2)}}(a, b) > H_{p,\omega}(a, b)$ follows from (2.11)-(2.13) and the monotonicity of $F(t)$.

Case B. If $p > 0$ and $0 < \omega < 2$, then (2.15) and (2.18)-(2.20) together with Lemma 2.1 lead to

$$\lim_{t\to+\infty} G(t) = +\infty, \tag{2.26}$$

$$\lim_{t\to+\infty} H(t) = +\infty, \tag{2.27}$$

$$H(1) < 0, \tag{2.28}$$

and there exists $b_1 > 1$ such that

$$H'(t) < 0 \tag{2.29}$$

for $t \in [1, b_1)$ and

$$H'(t) > 0 \tag{2.30}$$

for $t \in (b_1, +\infty)$.

From (2.27)-(2.30) we clearly see that there exists $b_2 > 1$ such that $H(t) < 0$ for $t \in [1, b_2)$ and $H(t) > 0$ for $t \in (b_2, +\infty)$. Then (2.17) implies that $G(t)$ is strictly decreasing in $[1, b_2]$ and strictly increasing in $[b_2, +\infty)$. It follows from (2.16) and (2.26) together with the monotonicity of $G(t)$ that there exists $b_3 > 1$ such that $G(t) < 0$ for $t \in (1, b_3)$ and $G(t) > 0$ for $t \in (b_3, +\infty)$. Then (2.14) leads to that $F(t)$ is strictly decreasing in $[1, b_3]$ and strictly increasing in $[b_3, +\infty)$.

Therefore, $M_{\frac{\log 2}{\log(\omega+2)}}(a,b) < H_{p,\omega}(a,b)$ follows from (2.11)-(2.13) and the monotonicity of $F(t)$.

Case C. If $p < 0$ and $\omega > 2$, then it follows from (2.15) and (2.18)-(2.20) together with Lemma 2.1 that

$$\lim_{t\to+\infty} G(t) = 1, \tag{2.31}$$

$$\lim_{t\to+\infty} H(t) = \frac{\omega}{4}\left(\frac{2\log 2}{\log(\omega+2)} - 1\right) < 0, \tag{2.32}$$

$$H(1) > 0, \tag{2.33}$$

$$H'(t) < 0 \tag{2.34}$$

for $t \in [1, +\infty)$.

From (2.32)-(2.34) we clearly see that there exists $c_1 > 1$ such that $H(t) > 0$ for $t \in [1, c_1)$ and $H(t) < 0$ for $t \in (c_1, +\infty)$. Then (2.17) implies that $G(t)$ is strictly decreasing in $[1, c_1]$ and strictly increasing in $[c_1, +\infty)$.

It follows from (2.16) and (2.31) together with the monotonicity of $G(t)$ that there exists $c_2 > 1$ such that $G(t) < 0$ for $t \in (1, c_2)$ and $G(t) > 0$ for $t \in (c_2, +\infty)$. Then (2.14) leads to that $F(t)$ is strictly decreasing in $[1, c_2]$ and strictly increasing in $[c_2, +\infty)$.

Therefore, $M_{\frac{\log 2}{\log(\omega+2)}}(a,b) < H_{p,\omega}(a,b)$ follows from (2.11)-(2.13) and the monotonicity of $F(t)$.

Case D. If $p < 0$ and $0 < \omega < 2$, then (2.15) and (2.18)-(2.20) together with Lemma 2.1 lead to

$$\lim_{t\to+\infty} G(t) = -\infty, \tag{2.35}$$

$$\lim_{t\to+\infty} H(t) = \frac{\omega}{4}\left(\frac{2\log 2}{\log(\omega+2)} - 1\right) > 0, \tag{2.36}$$

$$H(1) < 0, \tag{2.37}$$

$$H'(t) > 0 \tag{2.38}$$

for $t > 1$.

From (2.17) and (2.36)-(2.38) we clearly see that there exists $d_1 > 1$ such that $G(t)$ is strictly increasing in $[1, d_1]$ and strictly decreasing in $[d_1, +\infty)$. It follows from (2.14), (2.16), (2.35) and the monotonicity of $G(t)$ that there exists $d_2 > 1$ such that $F(t)$ is strictly increasing in $[1, d_2]$ and strictly decreasing in $[d_2, +\infty)$.

Therefore, $M_{\frac{\log 2}{\log(\omega+2)}}(a,b) > H_{p,\omega}(a,b)$ follows from (2.11)-(2.13) and the monotonicity of $F(t)$.

Thirdly, we prove that the parameter $\frac{2}{\omega+2}p$ is the best possible in either case.

For any $p, r \in \mathbb{R}$ with $pr \neq 0$, $\omega \geq 0$ and $x > 0$, one has

$$\begin{aligned} &\log[M_r(1, 1+x)] - \log[H_{p,\omega}(1, 1+x)] \\ &= \frac{1}{r}\log\frac{1+(1+x)^r}{2} - \frac{1}{p}\log\frac{1+\omega(1+x)^{\frac{p}{2}}+(1+x)^p}{\omega+2}. \end{aligned} \tag{2.39}$$

Let $x \to 0$, then the Taylor expansion leads to

$$\begin{aligned} &\frac{1}{r}\log\frac{1+(1+x)^r}{2} - \frac{1}{p}\log\frac{1+\omega(1+x)^{\frac{p}{2}}+(1+x)^p}{\omega+2} \\ &= \frac{(\omega+2)r-2p}{4(\omega+2)}x^2 + o(x^2). \end{aligned} \tag{2.40}$$

If $(p, \omega) \in \{(p, \omega): p > 0, \omega > 2\} \cup \{(p, \omega): p < 0, 0 < \omega < 2\}$, then equations (2.39) and (2.40) imply that for any $r > \frac{2}{\omega+2}p$ there exists $\delta_1 = \delta_1(r, p, \omega) > 0$ such that $M_r(1, 1 + x) > H_{p,\omega}(1, 1 + x)$ for $x \in (0, \delta_1)$.

If (p, ω) $\{(p, \omega): p > 0, 0 < \omega < 2\} \cup \{(p, \omega): p < 0, \omega > 2\}$, then from (2.39) and (2.40) we know that for any $r > \frac{2}{\omega+2}p$ there exists $\delta_2 = \delta_2(r, p, \omega) > 0$ such that $M_r(1, 1 + x) < H_{p,\omega}(1, 1 + x)$ for $x \in (0, \delta_2)$.

Finally, we prove that the parameter $\frac{\log 2}{\log(\omega+2)}p$ is the optimal parameter in either case.

For any $p, r \in \mathbb{R}$ with $pr > 0$, $\omega \geq 0$ and $x > 0$ we have

$$\begin{aligned} &\lim_{x\to+\infty}[\log M_r(1,x) - \log H_{p,\omega}(1,x)] \\ &= \frac{1}{p}\log(\omega+2) - \frac{1}{r}\log 2. \end{aligned} \tag{2.41}$$

If $(p, \omega) \in \{(p, \omega): p > 0, \omega > 2\} \cup \{(p, \omega): p < 0, 0 < \omega < 2\}$, then equation (2.41) implies that for any $r < \frac{\log 2}{\log(\omega+2)}p$ there exists $X_1 = X_1(r, p, \omega) > 1$ such that $M_r(1, x) < H_{p,\omega}(1, x)$ for $x \in (X_1, +\infty)$.

If (p, ω) $\omega \in$ $\{(p, \omega): p > 0, 0 < \omega < 2\} \cup \{(p, \omega): p < 0, \omega > 2\}$, then equation (2.41) leads to that for any $r > \frac{\log 2}{\log(\omega+2)}p$ there exists $X_2 = X_2(r, p, \omega) > 1$ such that $M_r(1, x) > H_{p,\omega}(1, x)$ for $x \in (X_2, +\infty)$.

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Authors' contributions
Y-ML provided the main idea in this article. B-YL carried out the proof of the inequalities in Theorem 2.1. Y-MC carried out the proof of the optimality in Theorem 2.1. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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