Liu and Jiang *Journal of Inequalities and Applications* 2012, **2012**:128
http://www.journalofinequalitiesandapplications.com/content/2012/1/128

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

# Jensen's inequality for monetary utility functions

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**Abstract**

In this paper, we prove that Jensen's inequality holds true for all monetary utility functions with respect to certain convex or concave functions by studying the properties of monetary utility functions, convex functions and concave functions.

**Keywords:** monetary utility functions, Jensen's inequality, convex functions, concave functions

## 1 Introduction and preliminaries

### 1.1 Introduction

Monetary utility functions have recently attracted much attention in the mathematical finance community, see e.g. [1-4]. According to [2], a monetary utility function $U$ can be identified with a convex risk measure $\rho$ by the formula $U(\xi) = -\rho(\xi)$; convex risk measure, introduced in [5,6], is a popular notion in particular since the Basel II accord. Some times we want to not only measure the utility of an uncertain random variable but also estimate the utility of its function (see e.g. 2.2). Jensen's inequality will be a useful tool solving this problem.

It is well-known that Jensen's inequality holds true for classical expectation, which, in terms of operator, can be seen as a particular type of monetary utility functions. But with respect to some convex or concave functions, it does not hold true for all monetary utility functions, as stated in our Example 2.1. This suggests a natural question: with respect to which kind of convex or concave functions does it hold true? In this paper, we study this question and give a sufficient and reasonable condition under which Jensen's inequality holds for all monetary utility functions.

### 1.2 Notations and assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Denote by $\mathbb{L}^\infty$ the collection of all real-valued essentially bounded $\mathcal{F}$-measurable random variables in $(\Omega, \mathcal{F}, \mathbb{P})$. The Definition 1.1 and Remark 1.1 are cited from [2].

**Definition 1.1**. A function $U : \mathbb{L}^\infty \mapsto \mathbb{R}$ is called a monetary utility function if it is concave, non-decreasing with respect to the order of $\mathbb{L}^\infty$, satisfies the normalization condition $U(0) = 0$ and has the cash-invariance property ($U(X + b) = U(X) + b$ for every $X \in \mathbb{L}^\infty$ and $b \in \mathbb{R}$).

*Remark* **1.1**. The normalization $U(0) = 0$ does not restrict the generality as it may be obtained by adding a constant to $U$.

## 2 Jensen's inequality for monetary utility functions

In this section, we will show our main results and give two examples. For proving the results, we need Proposition 2.1 on monetary utility functions.

**Proposition 2.1**. *Let* $U : \mathbb{L}^\infty \mapsto \mathbb{R}$*be a monetary utility function. Then for any* $k \in \mathbb{R}$, $X \in \mathbb{L}^\infty$, *we have*

$$\text{(i) } U(kX) \geq kU(X), \quad \text{if } 0 \leq k \leq 1; \tag{2.1}$$

$$\text{(ii) } U(kX) \leq kU(X), \quad \text{if } k \leq 0 \text{ or } k \geq 1. \tag{2.2}$$

*Proof.*

(i) If $0 \leq k \leq 1$, for the concavity of $U$, we have

$$U(kX + (1-k)Y) \geq kU(X) + (1-k)U(Y).$$

Take $Y = 0$ and consider $U(0) = 0$, then we have

$$U(kX) \geq kU(X) + (1-k)U(0) = kU(X).$$

(ii) If $k \geq 1$, then $0 < \frac{1}{k} \leq 1$. By (2.1) we have

$$U\left(\left(\frac{1}{k}\right)kX\right) \geq \frac{1}{k}U(kX).$$

It follows that

$$U(kX) \leq kU(X). \tag{2.3}$$

If $-1 \leq k \leq 0$, then $0 \leq -k \leq 1$. By (2.1) we have

$$U((-k)X) \geq (-k)U(X).$$

Since

$$0 = U(0) = U\left(\frac{1}{2}kX + \frac{1}{2}(-kX)\right) \geq \frac{1}{2}U(kX) + \frac{1}{2}U(-kX),$$

therefore

$$U(-kX) \leq -U(kX), \tag{2.4}$$

$$-kU(X) \leq U(-kX) \leq -U(kX).$$

Hence

$$U(kX) \leq kU(X).$$

If $k \leq -1$, then $-k \geq 1$. By (2.3) we have

$$U(kX) = U((-k)(-X)) \leq (-k)U(-X).$$

Combining the above inequality with (2.4) which is actually true for all k in R, we have

$$U(kX) \leq (-k)U(-X) \leq (-k)(-U(X)) = kU(X).$$

The proof of Proposition 2.1 is complete.

□

Now let us introduce the main results of this paper, i.e. Jensen's inequality for monetary utility functions.

**Theorem 2.1**. *Let $\phi$ be a convex function on $\mathbb{R}$. Suppose that for any $\varphi'_-(x) \le 1$and $\varphi'_-(x) \le 1$. Then for any $X \in \mathbb{L}^\infty$and any monetary utility function $U : \mathbb{L}^\infty \mapsto \mathbb{R}$, we have $\phi(U(X)) \le U(\phi(X))$.*

*Proof*. As stated in Definition 1.1, for every $X \in \mathbb{L}^\infty$, $U(X)$ is finite, i.e. $U(X) \in \mathbb{R}$, so we have

$$\varphi'_+(U(X)) \ge 0 \quad \text{and} \quad \varphi'_-(U(X)) \le 1.$$

For any

$$\alpha \in [\varphi'_-(U(X)),\ \varphi'_+(U(X))] \cap [0, 1],$$

based on the subgradient inequality in [7], we have

$$\varphi(x) \ge \alpha(x - U(X)) + \varphi(U(X)),\ \forall x \in \mathbb{R}.$$

For the arbitrariness of $x$, we have

$$\varphi(X) \ge \alpha(X - U(X)) + \varphi(U(X)).$$

Consider the monotonicity and cash-invariance property of $U$ and (2.1) in Proposition 2.1, then we have

$$\begin{aligned} U(\varphi(X)) &\ge U(\alpha(X - U(X)) + \varphi(U(X))) \\ &= U(\alpha X) - \alpha U(X) + \varphi(U(X)) \\ &\ge \alpha U(X) - \alpha U(X) + \varphi(U(X)). \end{aligned}$$

Hence

$$\varphi(U(X)) \le U(\varphi(X)).$$

□

It is also possible to obtain the Jensen inequality for monetary utility functions with respect to certain concave functions (Theorem 2.2) and prove it by the subgradient inequality in [7] and (2.2) in Proposition 2.1. As the proof is very similar with the proof of Theorem 2.1, we omit it and just give the result.

**Theorem 2.2**. *Let $\psi$ be a concave function on$\mathbb{R}$. Suppose that for any $\psi'_-(x) \ge 1$or $\psi'_-(x) \ge 1$. Then for any $X \in \mathbb{L}^\infty$and any monetary utility function $U : \mathbb{L}^\infty \mapsto \mathbb{R}$, we have $\psi(U(X)) \ge U(\psi(X))$.*

Then let us illustrate the reasonableness of the conditions on convex and concave functions in Theorems 2.1 and 2.2 through an example. Actually, Jensen's inequality usually is not true for all monetary utility functions even when the related convex or concave function is a linear function.

*Example* **2.1**. Let $\phi(x) = kx + a$ ($k <0$ or $k >1$) and $\psi(x) = hx + b$ ($0 < h <1$), obviously $\phi$ (respectively $\psi$) is a convex (respectively concave) function on $\mathbb{R}$ that do not satisfy the condition in Theorem 2.1 (respectively Theorem 2.2). We consider a particular type of monetary utility function $U$, the entropic utility function in [4], which is defined as

$$U(X) := -\ln \mathbb{E}[\exp(-X)]. \tag{2.5}$$

We choose a $X \in \mathbb{L}^\infty$ such that $\mathbb{P}\{X = 0\} = \mathbb{P}\{X = 1\} = 0.5$. It is easy to check that $U(kX) < kU(X)$ and $U(hX) > hU(X)$. Then we have

$$\varphi(U(X)) = kU(X) + a > U(kX) + a = U(kX + a) = U(\varphi(X)),$$
$$\psi(U(X)) = hU(X) + b < U(hX) + b = U(hX + b) = U(\psi(X)).$$

Thus Jensen's inequality does not holds. So the conditions in Theorems 2.1 and 2.2 are reasonable.

At the end of this paper, let us discuss an application of Jensen's inequality for monetary utility functions.

*Example* **2.2**. We still consider the entropic utility function. Sometimes, we want to estimate the entropic utility of $X^+$ or $-X^-$ using the entropic utility of the future outcome $X$. For this kind of problem, Jensen's inequality will be a useful tool. Let $\phi(x) = x^+$, $\psi(x) = -x^-$, then $\phi$ is a convex function satisfying the condition in Theorem 2.1 and $\psi$ is a concave function satisfying the condition in Theorem 2.2 By Theorems 2.1 and 2.2 we have

$$-\ln \mathbb{E}[\exp(-X^+)] \geq (-\ln \mathbb{E}[\exp(-X)])^+,$$
$$-\ln \mathbb{E}[\exp(-(-X^-))] \leq -(-\ln \mathbb{E}[\exp(-X)])^-.$$

**Acknowledgements**
The authors would like to thank the referees for their detailed comments and valuable suggestions. This research was supported by the National Natural Science Foundation of China (No. 10971220), the FANEDD (No. 200919), and the National Undergraduate Innovation Experiment Project of China (No. 091029030).

**Authors' contributions**
JL study the properties of the functions, carried out the proof of the results and drafted the manuscript. LJ conceived of the study, designed the research method, and helped to draft the manuscript. All authors read and approved the final manuscript.

**Competing interests**
The authors declare that they have no competing interests.

Received: 5 July 2011 Accepted: 7 June 2012 Published: 7 June 2012

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doi:10.1186/1029-242X-2012-128
**Cite this article as:** Liu and Jiang: **Jensen's inequality for monetary utility functions.** *Journal of Inequalities and Applications* 2012 **2012**:128.