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# $L^p$ Bounds for the parabolic singular integral operator

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**Abstract**

Let $1 < p < \infty$ and $n \geq 2$. The authors establish the $L^p(\mathbb{R}^{n+1})$ boundedness for a class of parabolic singular integral operators with rough kernels.

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## 1 Introduction

Let $\alpha_1,\dots, \alpha_n$ be fixed real numbers, $\alpha_i \geq 1$. For fixed $x \in \mathbb{R}^n$, the function $F(x, \rho) = \sum_{i=1}^n \frac{x_i{}^2}{\rho^{2\alpha_i}}$ is a decreasing function in $\rho > 0$. We denote the unique solution of the equation $F(x, \rho) = 1$ by $\rho(x)$. Fabes and Rivière [1] showed that $\rho(x)$ is a metric on $\mathbb{R}^n$, and $(\mathbb{R}^n, \rho)$ is called the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^n$.

For $\lambda > 0$, let $A_\lambda = \begin{pmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{pmatrix}$. Suppose that $\Omega(x)$ is a real valued and measurable function defined on $\mathbb{R}^n$. We say is $\Omega(x)$ is homogeneous of degree zero with respect to $A_\lambda$, if for any $\lambda > 0$ and $x \in \mathbb{R}^n$

$$\Omega(A_\lambda x) = \Omega(x). \tag{1.1}$$

Moreover, $\Omega(x)$ satisfies the following condition

$$\int_{S^{n-1}} \Omega(x') J(x') \, d\sigma(x') = 0, \tag{1.2}$$

where $J(x')$ is a function defined on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, which will be defined in Section 2.

In 1966, Fabes and Rivière [1] proved that if $\Omega \in C^1(S^{n-1})$ satisfying (1.1) and (1.2), then the parabolic singular integral operator $T_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where $T_\Omega$ is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{\rho(y)^\alpha} f(x-y) \, dy \quad \text{and} \quad \alpha = \sum_{i=1}^n \alpha_i.$$

In 1976, Nagel et al. [2] improved the above result. They showed $T_\Omega$ is still bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if replacing $\Omega \in C^1(S^{n-1})$ by a weaker condition $\Omega \in L \log^+ L(S^{n-1})$. Recently, Chen et al. [3] improve Theorem A, the result is

**Theorem A.** *If $\Omega \in H^1(S^{n-1})$ satisfies* (1.1) *and* (1.2)*; then the operator $T_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

For a suitable function $\varphi$ on [0, 1), and $\Gamma = \{(y, \varphi(\rho(y)) : y \in \mathbb{R}^n\}$. Define the singular integral operator $T_{\varphi,\Omega}$ in $\mathbb{R}^{n+1}$ along $\Gamma$ by

$$\left(T_{\phi,\Omega}f\right)(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^n} f\left(x - y, x_{n+1} - \phi\left(\rho\left(y\right)\right)\right) \frac{\Omega\left(y\right)}{\rho\left(y\right)^\alpha} dy,$$

where $(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$.

On the other hand, we note that if $\alpha_1 = \ldots = \alpha_n = 1$, then $\rho(x) = |x|$, $\alpha = n$ and $(\mathbb{R}^n, \rho) = (\mathbb{R}^n, |\cdot|)$. In this case, $T_{\varphi,\Omega}$ is just the classical singular integral operator along surfaces of revolution, which was studied by the authors of [4-7].

The purpose of this article is to investigate the $L^p$ boundedness of the parabolic singular integral operator $T_{\varphi,\Omega}$ along $\Gamma$ when $\Omega \in F_\beta(S^{n-1})$. For a $\beta > 0$, $F_\beta(S^{n-1})$ denotes the set of all $\Omega$ which are integrable over $S^{n-1}$ and satisfies

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| \left(\ln \frac{1}{|\theta \cdot \xi|}\right)^{1+\beta} d\theta < \infty. \tag{1.3}$$

Condition (1.3) was introduced by Grafakos and Stefanov [8]. The examples in [8] show that there is the following relationship between $F_\beta(S^{n-1})$ and $H^1(S^{n-1})$:

$$\bigcap_{\beta>0} F_\beta\left(S^{n-1}\right) \nsubseteq H^1\left(S^{n-1}\right) \nsubseteq \bigcup_{\beta>0} F_\beta\left(S^{n-1}\right).$$

We shall state our main results as follows:

**Theorem 1** *Let $m \in \mathbb{N}$. Suppose that $\varphi$ is a polynomial of degree $m$ and $\frac{d^{\alpha_i}\phi(t)}{dt^{\alpha_i}}|_{t=0} = 0$, where $\alpha_i' s$ are the all positive integers which is less than $m$ in $\{\alpha_1,\ldots, \alpha_n\}$. In addition, let $\Omega \in F_\beta(S^{n-1})$ for some $\beta > 0$ and satisfies* (1.1) *and* (1.2)*, then $T_{\varphi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $p \in \left(\frac{2+2\beta}{1+2\beta}, 2+2\beta\right)$.*

**Corollary 1** *Let $m \in \mathbb{N}$. Suppose that $\varphi$ is a polynomial and $\frac{d^{\alpha_i}\phi(t)}{dt^{\alpha_i}}|_{t=0} = 0$, where $\alpha_i' s$ are the all positive integers which is less than $m$ in $\{\alpha_1,\ldots, \alpha_n\}$. In addition, let $\Omega \in \cap_{\beta>0} F_\beta\left(S^{n-1}\right)$ and satisfies* (1.1) *and* (1.2)*, then $T_{\varphi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$.*

## 2 Notations and lemmas

In this section, we give some notations and lemmas which will be used in the proof of Theorem 1. For any $x \in \mathbb{R}^n$, set

$$\begin{aligned}
&x_1 = \rho^{\alpha_1} \cos\varphi_1 \cdots \cos\varphi_{n-2} \cos\varphi_{n-1} \\
&x_2 = \rho^{\alpha_2} \cos\varphi_1 \cdots \cos\varphi_{n-2} \sin\varphi_{n-1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&x_{n-1} = \rho^{\alpha_{n-1}} \cos\varphi_1 \sin\varphi_2 \\
&x_n = \rho^{\alpha_n} \sin\varphi_1.
\end{aligned}$$

Then $dx = \rho^{\alpha-1}J(\phi_1,..., \phi_{n-1})d\rho d\sigma$, where $\alpha = \sum_{i=1}^{n}\alpha_i$, $d\sigma$ is the element of area of $S^{n-1}$ and $\rho^{\alpha-1}J(\phi_1,..., \phi_{n-1})$ is the Jacobian of the above transform. In [1], it was shown there exists a constant $L \geq 1$ such that $1 \leq J(\phi_1,..., \phi_{n-1}) \leq L$ and $J(\phi_1,..., \phi_{n-1}) \in C^\infty((0, 2\pi)^{n-2} \times (0, \pi))$. So, it is easy to see that $J$ is also a $C^\infty$ function in the variable $y' \in S^{n-1}$. For simplicity, we denote still it by $J(y')$.

In order to prove our theorems, we need the following lemmas:

**Lemma 2.1**. ([9]) *Let $d \in \mathbb{N}$. Suppose that $\gamma(t)$: $\mathbb{R}^+ \mapsto \mathbb{R}^d$ satisfies $\gamma'(t) = M\left(\frac{\gamma(t)}{t}\right)$ for a fixed matrix M, and assume $\gamma(t)$ doesn't lie in an affine hyperplane. Then*

$$\int_1^2 e^{i\gamma(t)\cdot\eta}dt \leq C|\eta|^{1/d}.$$

**Lemma 2.2**. ([9]) *Suppose that $\lambda_j'$s and $\alpha_j'$s are fixed real numbers, $\varphi(t)$ is a polynomial and $\Gamma(t) = (\lambda_1 t^{\alpha_1}, \ldots, \lambda_n t^{\alpha_n}, \phi(t))$ is a function from $\mathbb{R}_+$ to $\mathbb{R}^{n+1}$. For suitable f, the maximal function associated to the homogeneous curve $\Gamma$ is defined by*

$$M_\Gamma(f)(x) = \sup_h \frac{1}{h}\int_0^h |f(x - \Gamma(t))|dt, h > 0. \tag{2.1}$$

*Then for $1 < p \leq \infty$, there is a constant $C > 0$, independent of $\lambda_j'$s, the coefficient of $\varphi(t)$ and f, such that*

$$||M_\Gamma(f)||_{L^p} \leq C||f||_{L^p}. \tag{2.2}$$

**Lemma 2.3**. *Let $L : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a linear transformation. Suppose that $\{\sigma_k\}_{k\in\mathbb{Z}}$ is a sequence of uniformly bounded measures on $\mathbb{R}^d$ satisfying*

$$|\widehat{\sigma}_k(\xi)| \leq C\min\left\{|A_{2^k}L\xi|, (\ln(|A_{2^k}L\xi|))^{-1-\beta}\right\} \tag{2.3}$$

*for $\xi \in \mathbb{R}^{n+1}$ and $k \in \mathbb{Z}$. For any $1 < p_0 < \infty$ and $A > 0$*

$$\left\|\left(\sum_{k\in\mathbb{Z}}|\sigma_k * g_k|^2\right)^{1/2}\right\|_{L^{p_0}} \leq A\left\|\left(\sum_{k\in\mathbb{Z}}|g_k|^2\right)^{1/2}\right\|_{L^{p_0}} \tag{2.4}$$

*holds for arbitrary functions $\{g_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^{n+1}$.* Then for $p \in \left(\frac{2+2\beta}{1+2\beta}, 2+2\beta\right)$ *there exists a constant $C_p = C(p, n)$ which is independent of L such that*

$$\left\|\sum_{k\in\mathbb{Z}}\sigma_k * f\right\|_{L^p} \leq C_p\|f\|_{L^p} \tag{2.5}$$

*and*

$$\left\|\left(\sum_{k\in\mathbb{Z}}|\sigma_k * f|^2\right)^{1/2}\right\|_{L^p} \leq C||f||_{L^p} \tag{2.6}$$

*for every $f \in L^p(\mathbb{R}^{n+1})$.*

**Proof**. The main idea of the proof is taken from [7,8], we assume that $L\xi = (\xi_1,..., \xi_n) = \zeta$ for $\xi = (\xi_1,..., \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$. Choose a $\psi \in C_0^\infty(\mathbb{R})$ such that $0\cdot \le \psi \le 1$, $supp(\psi) \subseteq (1/4, 4)$, and

$$\sum_{j\in\mathbb{Z}} \left[\psi\left(2^j t\right)\right]^2 \equiv 1 \tag{2.7}$$

For each $j$, we define $\Phi_j$ in $\mathbb{R}^n$ by

$$\widehat{\Phi_j}(\zeta) = \psi\left(2^j \rho(\zeta)\right)\widehat{f}(\zeta)$$

for $\xi = (\xi_1,..., \xi_{n+1}) \in \mathbb{R}^{n+1}$. If we set

$$Tf = \sum_{k\in\mathbb{Z}} \sigma_k * f, \tag{2.8}$$

and let $\delta$ represent the Dirac delta on $\mathbb{R}$, then by (2.7), for any Schwartz function $f$,

$$Tf = \sum_{j\in\mathbb{Z}} T_j f,$$

where

$$T_j f = \sum_{k\in\mathbb{Z}} \left(\Phi_{j+k}\otimes\delta\right) * \sigma_k * \left(\Phi_{j+k}\otimes\delta\right) * f.$$

By using (2.4) and Littlewood-Paley theory (as in [3]), one obtains that for any $1 < p_0 < \infty$,

$$||T_j f||_{L^{p_0}(\mathbb{R}^{n+1})} \le C_{p_0}||f||_{L^{p_0}(\mathbb{R}^{n+1})}. \tag{2.9}$$

On the other hand, by using Plancherel's theorem and (2.3), If $j > 0$, using the estimate $|\widehat{\sigma_k}(\xi)| \le C|A_{2^k}\zeta|$ we have

$$\begin{aligned}
\left\|T_j(f)\right\|_{L^2(\mathbb{R}^{n+1})} &\le \sum_k \int_{2^{-j-k-1}\le\rho(\zeta)\le 2^{-j-k+1}} |\widehat{f}(\xi)|^2 |A_{2^k}\zeta|^2 d\xi \\
&= \sum_k \int_{2^{-j-k-1}\le\rho(\zeta)\le 2^{-j-k+1}} |\widehat{f}(\xi)|^2 \\
&\quad \left(2^{2k\alpha_1}\rho(\zeta)^{2\alpha_1}(\zeta'_1)^2 + \ldots + 2^{2k\alpha_n}\rho(\zeta)^{2\alpha_n}(\zeta'_n)^2\right) d\xi \\
&\le 2^{-2j\min\{\alpha_j\}} \sum_k \int_{2^{-j-k-1}\le\rho(\zeta)\le 2^{-j-k+1}} |\widehat{f}(\xi)|^2 \\
&\quad \left((\zeta'_1)^2 + \ldots + (\zeta'_n)^2\right) d\xi \\
&\le 2^{-2j} \sum_k \int_{2^{-j-k-1}\le\rho(\zeta)\le 2^{-j-k+1}} |\widehat{f}(\xi)|^2 d\xi \\
&= C2^{-2j}||f||_{L^2(\mathbb{R}^{n+1})}.
\end{aligned} \tag{2.10}$$

Similar to the proof of (2.10), using Plancherel's theorem and (2.3), if $j<0$ we get

$$||T_jf||_{L^2(\mathbb{R}^{n+1})} \leq C\big(1+|j|\big)^{-(1+\beta)}||f||_{L^2(\mathbb{R}^{n+1})}. \tag{2.11}$$

In short

$$||T_jf||_{L^2(\mathbb{R}^{n+1})} \leq C\big(1+|j|\big)^{-(1+\beta)}||f||_{L^2(\mathbb{R}^{n+1})}, for j\in\mathbb{Z}. \tag{2.12}$$

By interpolating between (2.9) and (2.12), we obtain

$$||T_jf||_{L^p(\mathbb{R}^{n+1})} \leq C\big(1+|j|\big)^{-(1+\beta)}||f||_{L^p(\mathbb{R}^{n+1})}. \tag{2.13}$$

for

$$p\in\left(\frac{2+2\beta}{1+2\beta}, 2+2\beta\right)$$

and some $\beta>0$. Thus, (2.5) follows from (2.13). One may then use a randomization argument to derive (2.6). Lemma 2.1 is proved.

## 3 Proof of Theorem 1

The main idea of the proof of Theorem 1 is taken from [10]and [11]. Let $\Omega$ satisfies (1.1), (1.2), and (1.3) for some $\beta>0$. Let $\Phi(y)=(y, \varphi(\rho(y)))$, where $\phi(t)=\sum_{j=0}^{m} a_jt^j$, $m\in\mathbb{N}$. Let $D_k=\{y\in\mathbb{R}^n : 2^k<\rho(y)\leq 2^{k+1}\}$ and define the family of measures $\sigma_k$ on $\mathbb{R}^{n+1}$ by

$$\int_{\mathbb{R}^{n+1}} f\left(y, y_{n+1}\right)d\sigma_k=\int_{D_k} f\left(y,\phi\left(\rho\left(y\right)\right)\right)\frac{\Omega\left(y\right)}{\rho(y)^{\alpha}}dy, \tag{3.1}$$

and $\sigma^* f(x)=\sup_{k\in\mathbb{Z}}(|\sigma_k| * |f|(x)$.

It is easy to see that

$$||\sigma_k||=\int_{D_k}\frac{|\Omega\left(y'\right)|}{\rho(y)^{\alpha}}dy=\int_{S_{n-1}}\int_{2^k}^{2^{k+1}}|\Omega\left(y'\right)|J\left(y'\right)|\frac{d\rho}{\rho}d\sigma\left(y'\right)\leq C. \tag{3.2}$$

In light of (3.2) and Lemma 2.3, it suffices to show that $\sigma_k$ satisfies (2.3) and (2.4).

For $(\xi, \xi_{n+1})\in\mathbb{R}^n\times\mathbb{R}$, $y'\in S^{n-1}$, and $\lambda\in\mathbb{Z}$. Let

$$I_\lambda\left(\xi,\xi_{n+1},y'\right)=\int_1^2 e^{i\left[A_{\lambda\rho}\xi\cdot y'+\xi_{n+1}\phi(\lambda\rho)\right]}d\rho.$$

Set $\Lambda=\{\alpha_i : \alpha_i$ is the positive integers which is less than $m$ in $\{\alpha_1,\ldots,\alpha_n\}$ and $\overline{\Lambda}=\{1,2,\ldots,m\}\setminus\Lambda$. Then $\frac{d^{\alpha_i}\phi\left(t\right)}{dt^{\alpha_i}}|_{t=0}=0$, where $\alpha_i\in\Lambda$, and $\overline{\Lambda}$ is not a subset of $\{\alpha_1,\ldots,\alpha_n\}$. Therefore, we get

$$A_{\lambda\rho}\xi\cdot y'+\xi_{n+1}\phi\left(\lambda\rho\right)=\rho^{\alpha_1}\lambda^{\alpha_1}\xi_1y_1'+\ldots+\rho^{\alpha_n}\lambda^{\alpha_n}\xi_ny_n'+\xi_{n+1}\sum_{j\in\bar{\Lambda}}a_j(\lambda\rho)^j.$$

Without loss of generality, we may assume $\Lambda$ consists of $r$ distinct numbers and let $\overline{\Lambda}=\{i_1,i_2,\ldots,i_{m-r}\}$ If $\alpha_j's$ are all distinct, by Lemma 2.1, we get immediately

$$\begin{aligned}|I_\lambda\left(\xi,\xi_{n+1},y'\right)| &\le \left(|\lambda^{\alpha_1}\xi_1 y_1'| + \cdots + |\lambda^{\alpha_n}\xi_n y_n'| + (m--r)\,|\lambda\xi_{n+1}|\right)^{-1/(n+m-r)} \\ &\le \left(|\lambda^{\alpha_1}\xi_1 y_1' + \cdots + \lambda^{\alpha_n}\xi_n y_n'|\right)^{-1/(n+m-r)} = |A_\lambda \xi\cdot y'|^{-1/(n+m-r)}.\end{aligned} \tag{3.3}$$

If $\{\alpha j\}$ only consists of $s$ distinct numbers, we suppose that

$$\begin{aligned}&\alpha_1 = \alpha_2 = \cdots = \alpha_{l_1},\\ &\alpha_{l_1+1} = \cdots = \alpha_{l_1+l_2},\\ &\cdots\\ &\alpha_{l_1+\cdots+l_{s-1}+1} = \cdots = \alpha_n,\end{aligned}$$

where $s$ is a positive integer with $1 \le s \le n$, $l_1$, $l_2$,..., $l_s$ are positive integers such that $l_1 + l_2 + \cdots + l_s = n$ and $\alpha_1, \alpha_{l_1+l_2}, \cdots, \alpha_{l_1+\cdots+l_{s-1}}, \alpha_n$ are distinct. Obviously,

$$\gamma(t) = \left(t^{\alpha_1}, t^{\alpha_{l_1+l_2}}, \ldots, t^{\alpha_{l_1+\cdots+l_{s-1}}}, t^{\alpha_n}, t^{i_1}, t^{i_2}, \ldots, t^{i_{m-r}}\right)$$

does not lie in an affine hyperplane in $\mathbb{R}^{s+m-r}$. Then using Lemma 2.1 again, there exists $C > 0$ such that for any vector $\eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n$,

$$\begin{aligned}&\int_1^2 e^{2i\left(\eta_1+\cdots+\eta_{l_1}\right)t^{\alpha_{l_1}}+\left(\eta_{l_1+1}+\cdots+\eta_{l_1+l_2}\right)t^{\alpha_{l_1+l_2}}+\cdots+\left(\eta_{l_1+\cdots+l_{s-1}+1}+\cdots+\eta_n\right)t^{\alpha_n}+\lambda\xi_{n+1}\sum_{j\in\bar{\Lambda}}t^j}dt\\ &\le C\left(|\eta_1+\cdots+\eta_{l_1}|^2 + |\eta_{l_1+1}+\cdots+\eta_{l_1+l_2}|^2 + \cdots\right.\\ &\quad\left.+|\eta_{l_1+\cdots+l_{s-1}+1}+\cdots+\eta_n|^2 + (m-r)\,|\lambda\xi_{n+1}|^2\right)^{-1/2(s+m-r)}\\ &\le C\left(|\eta_1+\cdots+\eta_{l_1}| + |\eta_{l_1+1}+\cdots+\eta_{l_1+l_2}| + \cdots + |\eta_{l_1+\cdots+l_{s-1}+1}+\cdots+\eta_n|\right)^{-1/(s+m-r)}\\ &\le C\left|\sum_{j=1}^n \eta_j\right|^{-1/(s+m-r)}.\end{aligned}$$

Let $\eta_j = \lambda^{\alpha_j}\xi_j y_j'$, we have

$$\begin{aligned}|I_\lambda\left(\xi,\xi_{n+1},y\right)| &\le \left(|\lambda^{\alpha_1}\xi_1 y_1'| + \cdots + |\lambda^{\alpha_n}\xi_n y_n'|\right)^{-1/(s+m-r)} \\ &\le \left(|\lambda^{\alpha_1}\xi_1 y_1' + \cdots + \lambda^{\alpha_n}\xi_n y_n'|\right)^{-1/(s+m-r)} = |A_\lambda \xi\cdot y'|^{-1/(s+m-r)}.\end{aligned} \tag{3.3a}$$

On the other hand, it is easy to see that

$$|I_\lambda\left(\xi,\xi_{n+1},y'\right)| \le 1. \tag{3.4}$$

From (3.3), (3.3') and (3.4), we get

$$|I_\lambda\left(\xi,\xi_{n+1},y'\right)| \le \frac{C\left[\ln\left(1/|\eta'\cdot y'|\right)\right]^{1+\beta}}{\left(\ln|A_\lambda\xi|\right)^{1+\beta}}, \quad \text{for} \quad |A_\lambda\xi| \ge 2,$$

where $\eta' = \dfrac{A_\lambda\xi}{|A_\lambda\xi|}$. Thus, by (1.3), we get

$$\int_{S^{n-1}} |I_\lambda\left(\xi,\xi_{n+1},y'\right)\Omega\left(y'\right)|d\sigma\left(y'\right) \le C(\ln|A_\lambda\xi|)^{-(1+\beta)}.$$

Therefore,

$$\begin{aligned}
&|\widehat{\sigma_k}(\xi,\xi_{n+1})| \\
&= \left| \int_{D_k} e^{i(\xi\cdot y+\xi_{n+1}\phi(\rho(y)))} \frac{\Omega(y)}{\rho(y)^{\alpha}} dy \right| \\
&= \left| \int_{S^{n-1}} \int_{2^k}^{2^{k+1}} e^{i(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho))} \Omega(y') J(y') \frac{d\rho}{\rho} d\sigma(y') \right| \\
&= \left| \int_{S^{n-1}} \int_{1}^{2} e^{i\left(\xi\cdot A_{2^k\rho} y'+\xi_{n+1}\phi(2^k\rho)\right)} \Omega(y') J(y') \frac{d\rho}{\rho} d\sigma(y') \right| \\
&\le C \int_{S^{n-1}} |I_{2^k}(\xi,\xi_{n+1},y')||\Omega(y')|d\sigma(y') \\
&\le C(\ln|A_{2^k}\xi|)^{-(1+\beta)}.
\end{aligned} \tag{3.5}$$

On the other hand, by (1.2), we can obtain

$$\begin{aligned}
&|\widehat{\sigma_k}(\xi,\xi_{n+1})| \\
&= \left| \int_{D_k} e^{i(\xi\cdot y+\xi_{n+1}\phi(\rho(y)))} \frac{\Omega(y)}{\rho(y)^{\alpha}} dy \right| \\
&= \left| \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} e^{i(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho))} \Omega(y') J(y') d\sigma(y') \frac{d\rho}{\rho} \right| \\
&= \left| \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} \left( e^{i(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho))} - e^{i\xi_{n+1}\phi(\rho)} \right) \Omega(y') J(y') d\sigma(y') \frac{d\rho}{\rho} \right| \\
&\le C \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} |e^{i(\xi\cdot A_\rho y'+\xi_{n+1}\phi(\rho))} - e^{i\xi_{n+1}\phi(\rho)}||\Omega(y')||J(y')|d\sigma(y') \frac{d\rho}{\rho} \\
&\le C \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} |\xi\cdot A_\rho y'||\Omega(y')||J(y')|d\sigma(y') \frac{d\rho}{\rho} \\
&\le C \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} |A_{2^k}\xi\cdot y'||\Omega(y')||J(y')|d\sigma(y) \frac{d\rho}{\rho} \\
&\le C|A_{2^k}\xi| \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} |\Omega(y')||J(y')| \left| \frac{A_{2^{k+1}}\xi}{A_{2^{k+1}}\xi}\cdot y' \right| d\sigma(y') d\rho \\
&\le C|A_{2^k}\xi|.
\end{aligned} \tag{3.6}$$

Clearly, (3.5) and (3.6) imply (2.3) holds. Finally, we shall show that (2.4) holds.

$$\begin{aligned}
&\sigma_k^*\left(f\right)(x)\\
&=\sup_{k\in\mathbb{R}}\left(|\sigma_k| * |f|\right)(x)\\
&=\int_{D_k} |f\left(x-\Phi\left(y\right)\right)|\frac{|\Omega\left(y\right)|}{\rho\left(y\right)^{\alpha}}dy\\
&=\int_{S^{n-1}}\int_{2^k}^{2^{k+1}} |f\left(x-\Phi\left(A_\rho y'\right)\right)||\Omega\left(y'\right)|\frac{d\rho}{\rho}d\sigma\left(y'\right)\\
&\le \frac{1}{2^k}\int_{S^{n-1}} |\Omega\left(y'\right)|\left(\int_{2^k}^{2^{k+1}} |f\left(x-\Phi\left(A_\rho y'\right)\right)|d\rho\right)d\sigma\left(y'\right)\\
&\le C\int_{S^{n-1}} |\Omega\left(y'\right)|M_\Phi\left(f\right)(x)\,d\sigma\left(y'\right).
\end{aligned}$$

By Lemma 2.2, we obtain $||M_\Phi(f)||_p \le C||f||_p$, where $C>0$ is independent of $k$, the coefficient of $\varphi(t)$ and $f$, since $\Omega$ is integrable on $S^{n-1}$, thus $||\sigma^*(f)||_p \le C||f||_p$. This shows (2.4) holds. This completes the proof of the Theorem 1.

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**Authors' contributions**
YC carried out the parabolic singular integral operator studies and drafted the manuscript. WY participated in the study of Littlewood-Paley theory. FW conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

**Competing interests**
The authors declare that they have no competing interests.

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