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Hyers-Ulam stability of derivations on proper Jordan *CQ**-algebras

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Abstract

Eskandani and Vaezi proved the Hyers-Ulam stability of derivations on proper Jordan *CQ**-algebras associated with the following Pexiderized Jensen type functional equation

$$kf\left(\frac{x+y}{k}\right) = f_0(x) + f_1(y)$$

by using direct method. Using fixed point method, we prove the Hyers-Ulam stability of derivations on proper Jordan *CQ**-algebras. Moreover, we investigate the Pexiderized Jensen type functional inequality in proper Jordan *CQ**-algebras.

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1. Introduction and preliminaries

In 1940, Ulam [1] asked the first question on the stability problem. In 1941, Hyers [2] solved the problem of Ulam. This result was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an *unbounded Cauchy difference*. In 1994, a generalization of Rassias' Theorem was obtained by Găvruta [5]. Since then, several stability problems for various functional equations have been investigated by numerous mathematicians (see [6-25], M Eshaghi Gordji, unpublished work).

The Jensen equation is $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$, where f is a mapping between linear spaces. It is easy to see that a mapping $f : X \to Y$ between linear spaces with $f(0) = 0$ satisfies the Jensen equation if and only if it is additive [26]. Stability of the Jensen equation has been studied at first by Kominek [27].

We recall some basic facts concerning quasi *-algebras.

Definition 1.1. Let A be a linear space and let A_0 be a *-algebra contained in A as a subspace. We say that A is a quasi *-algebra over A_0 if

(*i*) the right and left multiplications of an element of A and an element of A_0 are defined and linear;

(*ii*) $x_1(x_2a) = (x_1x_2)a$, $(ax1)x2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

(*iii*) an involution *, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

Quasi *-algebras [28,29] arise in natural way as completions of locally convex *-algebras whose multiplication is not jointly continuous; in this case one has to deal with topological quasi *-algebras.

A quasi *-algebra (A, A_o) is said to be a locally convex quasi *-algebra if in A a locally convex topology τ is defined such that

(*i*) the involution is continuous and the multiplications are separately continuous;

(*ii*) A_o is dense in $A[\tau]$.

Throughout this article, we suppose that a locally convex quasi *-algebra (A, A_0) is complete. For an overview on partial *-algebra and related topics we refer to [30].

In a series of articles [31-35] many authors have considered a special class of quasi *algebras, called proper CQ^*-algebras, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a Banach module over the C^*-algebra A_0 with involution * and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a proper CQ^*-algebra if

(*i*) A_0 is dense in A with respect to its norm $\|\cdot\|$;

(*ii*) $(ab)^* = b^*a^*$ whenever the multiplication is defined;

(*iii*) $\| y \|_0 = sup_{a \in A, \| a \| \le 1} \| ay \|$ for all $y \in A_0$.

Definition 1.2. A proper CQ^*-algebra (A, A_0), endowed with the Jordan product

$$z \circ x = \frac{zx + xz}{2}$$

for all $x \in A$ and all $z \in A_0$, is called a proper Jordan CQ^*-algebra.

Definition 1.3. Let (A, A_0) be proper Jordan CQ^*-algebras.

A $\mathbb{C}$-linear mapping δ: $A_0 \to A$ is called a Jordan derivation if

$$\delta(x \circ y) = x \circ \delta(y) + \delta(x) \circ y$$

for all $x, y \in A_0$.

Park and Rassias [36] investigated homomorphisms and derivations on proper JCQ^*-triples.

Throughout this article, assume that k is a fixed positive integer.

Eskandani and Vaezi [37] proved the Hyers-Ulam stability of derivations on proper Jordan CQ^*-algebras associated with the following Pexiderized Jensen type functional equation

$$kf\left(\frac{x + y}{k}\right) = f_0(x) + f_1(y)$$

by using direct method.

In this article, using fixed point method, we prove the Hyers-Ulam stability of derivations on proper Jordan CQ^*-algebras.

Moreover, we investigate the Pexiderized Jensen type functional inequality in proper Jordan CQ^*-algebras.

2. Derivations on proper Jordan CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Jordan CQ^*-algebra with C^*-norm $\|\cdot\|_{A0}$ and norm $\|\cdot\|_A$.

Theorem 2.1. *Let $\phi : A_0 \times A_0 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x,\ 2^n y) = 0 \tag{2.1}$$

for all $x, y \in A_0$. Suppose that $f, f_0, f_1 : A_0 \to A$ are mappings with $f(0) = 0$ and

$$||\mu f(x) - f_0(y) - f_1(z)||_A \le \left\| kf\left(\frac{\mu x + y + z}{k}\right)\right\|_A \tag{2.2}$$

$$||f(x \circ y) + x \circ f_1(y) + f_0(x) \circ y||_A \le \varphi(x,\ y) \tag{2.3}$$

for all $\mu \in \mathbb{T}^1 : \{\mu \in \mathbb{C} :\ |\mu| = 1\}$ and all $x, y, z \in A_0$. Then the mapping $f : A_0 \to A$ is a Jordan derivation. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A_0$.

Proof. Letting $x = yz = 0$ in (2.2), we get $f_0(0) + f_1(0) = 0$.

Letting $\mu = 1$, $y = -x$ and $z = 0$ in (2.2), we get

$$f(x) = f_0(-x) + f_1(0) = f_0(-x) - f_0(0) \tag{2.4}$$

for all $x \in A_0$. Similarly, we have

$$f(x) = f_1(-x) + f_0(0) = f_1(-x) - f_1(0) \tag{2.5}$$

for all $x \in A_0$. By (2.2), we have

$$\begin{aligned}||f(x+y) - f(x) - f(y)||_A &= ||f(x+y) - (f_0(-x) + f_1(0)) - (f_1(-y) + f_0(0))||_A \\ &= ||f(x+y) - f_0(-x) - f_1(-y)||_A = 0\end{aligned}$$

for all $x, y \in A_0$. So the mapping $f : A_0 \to A$ is additive. Letting $y = -\mu x$ and $z = 0$ in (2.2), we get

$$\mu f(x) = f_0(-\mu x) + f_1(0) = f(\mu x)$$

for all $x \in A_0$. By the same reasoning as in the proof of [[38], Theorem 2.1], the mapping $f : A_0 \to A$ is $\mathbb{C}$-linear. By (2.1) and (2.3), we have

$$\begin{aligned}&||f(x \circ y) - x \circ f(y) - f(x) \circ y||_A \\ &= \lim_{n\to\infty} \frac{1}{4^n}||f(2^n x \circ 2^n y) - 2^n x \circ (f_1(-2^n y) - f_1(0)) - (f_0(-2^n x) - f_0(0)) \circ 2^n y||_A \\ &= \lim_{n\to\infty} \frac{1}{4^n}||f(2^n x \circ 2^n y) - 2^n x \circ f_1(-2^n y) - f_0(-2^n x) \circ 2^n y||_A \\ &\le \lim_{n\to\infty} \frac{\varphi(-2^n x, -2^n y)}{4^n} = 0\end{aligned}$$

for all $x, y \in A_0$. So

$$f(x \circ y) = x \circ f(y) + f(x) \circ y$$

for all $x, y \in A_0$. Therefore, the mapping $f : A_0 \to A$ is a Jordan derivation.

Since $f(-x) = -f(x)$ for all $x \in A_0$, it follows from (2.4) that

$$f(x) = -f(-x) = -(f_0(x) - f_0(0)) = f_0(0) - f_0(x)$$

for all $x \in A_0$. It follows from (2.5) that

$$f(x) = -f(-x) = -(f_1(x) - f_1(0)) = f_1(0) - f_1(x)$$

for all $x \in A_0$. This completes the proof.

□

Corollary 2.2. *Let* θ, r_0, r_1 *be nonnegative real numbers with* $r_0 + r_1 < 2$, *and let* $f, f_0, f_1 : A_0 \to A$ *be mappings satisfying* $f(0) = 0$, (2.2) *and*

$$||f(x \circ y) + x \circ f_1(y) + f_0(x) \circ y||_A \leq \theta ||x||_{A_0}^{r_0} ||y||_{A_0}^{r_1}$$

for all $x, y \in A_0$. *Then the mapping* $f : A_0 \to A$ *is a Jordan derivation. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A_0$.

Proof. The proof follows from Theorem 2.1.

□

Corollary 2.3. *Let* θ, r_0, r_1 *be nonnegative real numbers with* $r < 2$ *and let* $f, f_0, f_1 : A_0 \to A$ *be mappings satisfying* $f(0) = 0$, (2.2) *and*

$$||f(x \circ y) + x \circ f_1(y) + f_0(x) \circ y||_A \leq \theta (||x||_{A_0}^{r} + ||y||_{A_0}^{r})$$

for all $x, y \in A_0$. *Then the mapping* $f : A_0 \to A$ *is a Jordan derivation. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A$.

3. Hyers-Ulam stability of derivations on proper Jordan *CQ**-algebras

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [39,40]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [8].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies:

(GM_1) $d(x, y) = 0$ if and only if $x = y$;
(GM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(GM_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator.

We recall the following theorem by Diaz and Margolis [39].

Theorem 3.1. *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive function* $T : \Omega \to \Omega$ *with Lipschitz constant L. Then for each* *given* $x \in \Omega$, *either*

$$d(T^m x,\ T^{m+1}x) = \infty \quad \text{for all } m \geq 0,$$

or other exists a natural number m_0 *such that*

★ $d(T^m x,\ T^{m+1}x) < \infty$ *for all* $m \geq m_0$;
★ *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T;
★ y^* *is the unique fixed point of* T *in*

$$\Lambda = \{y \in \Omega : d(T^{m_0}x,\ y) < \infty\};$$

★ $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

Now we prove the Hyers-Ulam stability of derivations on proper Jordan CQ^*-algebras by using fixed point method.

Theorem 3.2. *Let* $f, f_0, f_1 : A_0 \to A$ *be mappings with* $f(0) = 0$ *for which there exists a function* $\varphi : A_0^2 \to [0, \infty)$ *with* $\phi(0, 0) = 0$ *such that*

$$\left\| kf\left(\frac{\mu x + \mu y}{k}\right) - \mu f_0(x) - \mu f_1(y) \right\|_A \leq \varphi(x,\ y), \tag{3.1}$$

$$\left\| kf\left(\frac{x \circ y}{k}\right) - x \circ f_1(y) - f_0(x) \circ y \right\|_A \leq \varphi(x,\ y) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x,\ y \in A_0$. *If there exists an* $L < 1$ *such that* $\varphi(x, y) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2})$ *for all* $x, y \in A_0$, *then there exists a unique Jordan derivation* $\delta : A_0 \to A$ *such that*

$$\begin{aligned} ||f(x) - \delta(x)||_A &\leq \frac{1}{2k - 2kL}\varphi(kx,\ kx), \\ ||f_0(x) - f_0(0) - \delta(x)||_A &\leq \frac{1}{2 - 2L}\varphi(x,\ x) \end{aligned} \tag{3.3}$$

for all $x \in A_0$. *Moreover,* $f_0(x) - f_0(0) = f_1(x) - f_1(0)$ *for all* $x \in A_0$.

Proof. Letting $x = y = 0$ and $\mu = 1$ in (3.1), we get $f_0(0) + f_1(0) = 0$.

Letting $y = 0$ and $\mu = 1$ in (3.1), we get

$$kf\left(\frac{x}{k}\right) = f_0(x) + f_1(0) = f_0(x) - f_0(0) \tag{3.4}$$

for all $x \in A_0$. Similarly, we get

$$kf\left(\frac{y}{k}\right) = f_1(y) + f_0(0) = f_1(y) - f_1(0) \tag{3.5}$$

for all $y \in A_0$. Using (3.4) and (3.5), we get

$$f_0(x) - f_0(0) = f_1(x) - f_1(0)$$

for all $x \in A_0$.

Let $H : A0 \to A$ be a mapping defined by

$$H(x) := f_0(x) - f_0(0) = f_1(x) - f_1(0) = kf\left(\frac{x}{k}\right)$$

for all $x \in A_0$. Then we have

$$||H(\mu x + \mu y) - \mu H(x) - \mu H(y)||_A \leq \varphi(x, y) \tag{3.6}$$

for all $\mu \in \mathbb{T}^1$ and $x, y \in A_0$.

Consider the set

$$X := \{g : A_0 \to A\}$$

and introduce the *generalized metric* on X:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : ||g(x) - h(x)||_A \leq C\varphi(x, x), \forall x \in A_0\}.$$

It is easy to show that (X, d) is complete (see [[41], Lemma 2.1]).

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in A$.

By [[41], Theorem 3.1],

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in X$.

Letting $\mu = 1$ and $y = x$ in (3.6), we get

$$||H(2x) - 2H(x)|| \leq \varphi(x, x) \tag{3.7}$$

and so

$$||H(x) - \frac{1}{2}H(2x)|| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in A_0$. Hence $d(H, JH) \leq \frac{1}{2}$.

By Theorem 3.1, there exists a mapping $\delta : A_0 \to A$ such that

(1) δ is a fixed point of J, i.e.,

$$\delta(2x) = 2\delta(x) \tag{3.8}$$

for all $x \in A_0$. The mapping δ is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that δ is a unique mapping satisfying (3.8) such that there exists $C \in (0, \infty)$ satisfying

$$||H(x) - \delta(x)||_A \leq C\varphi(x, x)$$

for all $x \in A_0$.

(2) $d(J^nH, \delta) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{H(2^n x)}{2^n} = \delta(x) \tag{3.9}$$

for all $x \in A_0$.

(3) $d(H, \delta) \leq \frac{1}{1-L} d(H, JH)$, which implies the inequality

$$d(H, \delta) \leq \frac{1}{2 - 2L}.$$

This implies that the inequality (3.3) holds.

It follows from (3.6) and (3.9) that

$$\begin{aligned} &||\delta(\mu x + \mu y) - \mu\delta(x) - \mu\delta(y)||_A \\ &= \lim_{n\to\infty} \frac{1}{2^n} ||H(2^n \mu x + 2^n \mu y) - \mu H(2^n x) - \mu H(2^n y)||_A \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x,\ 2^n y) = 0 \end{aligned}$$

for $\mu \in \mathbb{T}^1$ and all $x, y \in A_0$. So

$$\delta(\mu x + \mu y) = \mu\delta(x) + \mu\delta(y)$$

for $\mu \in \mathbb{T}^1$ and all $x, y \in A_0$. By the same reasoning as in the proof of [[38], Theorem 2.1], the mapping $\delta : A_0 \to A$ is $\mathbb{C}$-linear.

It follows from $\varphi(x, y) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2})$ that

$$\lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x,\ 2^n y) \leq \frac{1}{2^n} \varphi(2^n x,\ 2^n y) = 0 \tag{3.10}$$

for all $x, y \in A_0$.

It follows from (3.2) and (3.10) that

$$\begin{aligned} &||\delta(x \circ y) - x \circ \delta(y) - \delta(x) \circ y||_A \\ &\leq \lim_{n\to\infty} \frac{1}{4^n} \left\| kf\left(4^n \left(\frac{x \circ y}{k}\right)\right) - 2^n x \circ (f_1(2^n y) + f_0(0)) - (f_0(2^n x) + f_1(0)) \circ 2^n y \right\|_A \\ &\leq \lim_{n\to\infty} \frac{1}{4^n} \left\| kf\left(4^n \left(\frac{x \circ y}{k}\right)\right) - 2^n x \circ f_1(2^n y) - f_0(2^n x) \circ 2^n y \right\|_A \\ &\leq \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x,\ 2^n y) = 0 \end{aligned}$$

for all $x, y \in A_0$. Hence

$$\delta(x \circ y) = x \circ \delta(y) + \delta(x) \circ y$$

for all $x, y \in A_0$. So $\delta : A_0 \to A$ is a Jordan derivation, as desired.

□

Corollary 3.3. [[37], Theorem 3.1] *Let be a nonnegative real number and* r_0, r_1 *positive real numbers with* $\lambda := r_0 + r_1 < 1$ *and let* $f, f_0, f_1 : A_0 \to A$ *be mappings with* $f(0) = 0$ *such that*

$$\left\| kf\left(\frac{\mu x + \mu y}{k}\right) - \mu f_0(x) - \mu f_1(y) \right\|_A \leq \theta ||x||_{A_0}^{r_0} ||y||_{A_0}^{r_1}, \tag{3.11}$$

$$\left\| kf\left(\frac{x \circ y}{k}\right) - x \circ f_1(y) - f_0(x) \circ y \right\|_A \leq \theta ||x||_{A_0}^{r_0} ||y||_{A_0}^{r_1} \tag{3.12}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in A_0$. *Then there exists a unique Jordan derivation* $\delta : A_0 \to A$ *such that*

$$||f(x) - \delta(x)||_A \le \frac{k^{\lambda-1}\theta}{2 - 2^\lambda}||x||_{A_0}^\lambda,$$
$$||f_0(x) - f_0(0) - \delta(x)||_A \le \frac{\theta}{2 - 2^\lambda}||x||_{A_0}^\lambda$$

for all $x \in A_0$. *Moreover,* $f_0(x) - f_0(0) = f_1(x) - f_1(0)$ *for all* $x \in A_0$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y) := \theta||x||_{A_0}^{r_0}||y||_{A_0}^{r_1}$$

for all $x, y \in A$. Letting $L = 2^{\lambda-1}$, we get the desired result.

□

Corollary 3.4. [[37], Theorem 3.4] *Let* θ, r *be a nonnegative real numbers with* $0 < r < 1$, *and let* $f, f_0, f_1 : A_0 \to A$ *be mappings with* $f(0) = 0$ *such that*

$$\left\|kf\left(\frac{\mu x + \mu y}{k}\right) - \mu f_0(x) - \mu f_1(y)\right\|_A \le \theta(||x||_{A_0}^r + ||y||_{A_0}^r), \tag{3.13}$$

$$\left\|kf\left(\frac{x \circ y}{k}\right) - x \circ f_1(y) - f_0(x) \circ y\right\|_A \le \theta(||x||_{A_0}^r + ||y||_{A_0}^r) \tag{3.14}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in A_0$. *Then there exists a unique Jordan derivation* $\delta : A_0 \to A$ *such that*

$$||f(x) - \delta(x)||_A \le \frac{2k^{r-1}\theta}{2 - 2^r}||x||_{A_0}^r,$$
$$||f_0(x) - f_0(0) - \delta(x)||_A \le \frac{2\theta}{2 - 2^r}||x||_{A_0}^r$$

for all $x \in A_0$. *Moreover,* $f_0(x) - f_0(0) = f_1(x) - f_1(0)$ *for all* $x \in A_0$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y) := \theta(||x||_{A_0}^r + ||y||_{A_0}^r)$$

for all $x, y \in A$. Letting $L = 2^{r-1}$, we get the desired result.

□

Theorem 3.5. *Let* $f, f_0, f_1 : A_0 \to A$ *be mappings with* $f(0) = f_0(0) = f_1(0) = 0$ *for which there exists a function* $\varphi : A_0^2 \to [0, \infty)$ *satisfying* (3.1) *and* (3.2). *If there exists an* $L < 1$ *such that* $\varphi(x, y) \le \frac{L}{4}\varphi(2x, 2y)$ *for all* $x, y \in A_0$, *then there exists a unique Jordan derivation* δ: $A_0 \to A$ *such that*

$$\begin{aligned} ||f(x) - \delta(x)||_A &\le \frac{L}{4k - 4kL}\varphi(kx, kx), \\ ||f_0(x) - \delta(x)||_A &\le \frac{L}{4 - 4L}\varphi(x, x) \end{aligned} \tag{3.15}$$

for all $x \in A_0$. *Moreover,* $f_0(x) = f_1(x)$ *for all* $x \in A_0$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 3.2. Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $H(x) := f_0(x) = f_1(x) = kf(\frac{x}{k})$ for all $x \in A_0$. It follows from (3.7) that

$$\left\|H(x) - 2H\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{4}\varphi(x, y)$$

for all $x \in A_0$. Thus $d(H, JH) \le \frac{L}{4}$. One can show that there exists a mapping $\delta : A_0 \to A$ such that

$$d(H, \delta) \le \frac{L}{4 - 4L}.$$

Hence we obtain the inequality (3.15).

It follows from $\varphi(x, y) \le \frac{L}{4}\varphi(2x, 2y)$ that

$$\lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in A_0$. So

$$\begin{aligned} &||\delta(x \circ y) - x \circ \delta(y) - \delta(x) \circ y||_A \\ &\le \lim_{n\to\infty} 4^n \left\|kf\left(4^n \frac{x \circ y}{4^n k}\right) - \frac{x}{2^n} \circ f_1\left(\frac{y}{2^n}\right) - f_0\left(\frac{x}{2^n}\right) \circ \frac{y}{2^n}\right\|_A \\ &\le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned}$$

for all $x, y \in A_0$. Hence

$$\delta(x \circ y) = x \circ \delta(y) + \delta(x) \circ y$$

for all $x, y \in A_0$. So $\delta : A_0 \to A$ is a Jordan derivation, as desired.

The rest of the proof is similar to the proof of Theorem 3.2.

□

Corollary 3.6. [[37], Theorem 3.2] *Let θ be a nonnegative real number and r_0, r_1 positive real numbers with $\lambda := r_0 + r_1 > 2$ and let $f, f_0, f_1 : A_0 \to A$ be mappings satisfying $f(0) = f_0(0) = f_1(0) = 0$, (3.11) and (3.12). Then there exists a unique Jordan derivation δ: $A_0 \to A$ such that*

$$||f(x) - \delta(x)||_A \le \frac{k^{\lambda-1}\theta}{2^\lambda - 4}||x||_{A_0}^\lambda$$

$$||f_i(x) - \delta(x)||_A \le \frac{\theta}{2^\lambda - 4}||x||_{A_0}^\lambda$$

for all $x \in A_0$. Moreover, $f_0(x) = f_1(x)$ for all $x \in A_0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta||x||_{A_0}^{r_0}||y||_{A_0}^{r_1}$$

for all $x, y \in A$. Letting $L = 2^{2-\lambda}$, we get the desired result.

□

Corollary 3.7. [[37], Theorem 3.3] *Let θ, r be nonnegative real numbers with $r > 2$, and let $f, f_0, f_1 : A_0 \to A$ be mappings satisfying $f(0) = f_0(0) = f_1(0) = 0$, (3.13) and (3.14). Then there exists a unique Jordan derivation $\delta : A_0 \to A$ such that*

$$||f(x) - \delta(x)||_A \leq \frac{2k^{r-1}\theta}{2^r - 4}||x||_{A_0}^r,$$
$$||f_0(x) - \delta(x)||_A \leq \frac{2\theta}{2^r - 4}||x||_{A_0}^r$$

for all $x \in A_0$. *Moreover*, $f_0(x) = f_1(x)$ *for all* $x \in A_0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta\left(||x||_{A_0}^r + ||y||_{A_0}^r\right)$$

for all $x, y \in A$. Letting $L = 2^{2-r}$, we get the desired result.

□

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
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