Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Improved Heinz inequality and its application

Limin Zou* and Youyi Jiang

* Correspondence: limin-zou@163.com
School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing 404100, People's Republic of China

Abstract

We obtain an improved Heinz inequality for scalars and we use it to establish an inequality for the Hilbert-Schmidt norm of matrices, which is a refinement of a result due to Kittaneh.

Mathematical Subject Classification 2010: 26D07; 26D15; 15A18.

Keywords: Heinz inequality, convex function, Hilbert-Schmidt norm

1. Introduction

Let M_n be the space of $n \times n$ complex matrices and $||\cdot||$ stand for any unitarily invariant norm on M_n. So, $||UAV|| = ||A||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. If $A = [a_{ij}] \in M_n$, then

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

is the Hilbert-Schmidt norm of matrix A. It is known that the Hilbert-Schmidt norm is unitarily invariant.

The classical Young's inequality for nonnegative real numbers says that if $a, b \geq 0$ and $0 \leq \nu \leq 1$, then

$$a^{\nu} b^{1-\nu} \leq \nu a + (1 - \nu)\, b \tag{1.1}$$

with equality if and only if $a = b$. Young's inequality for scalars is not only interesting in itself but also very useful. If $\nu = \frac{1}{2}$, by (1.1), we obtain the arithmetic-geometric mean inequality

$$2\sqrt{ab} \leq a + b. \tag{1.2}$$

Kittaneh and Manasrah [1] obtained a refinement of Young's inequality as follows:

$$a^{\nu} b^{1-\nu} + r_0 \left(\sqrt{a} - \sqrt{b} \right)^2 \leq \nu a + (1 - \nu) b, \tag{1.3}$$

where $r_0 = \min\{\nu, 1 - \nu\}$.

Let $a, b \geq 0$ and $0 \leq \nu \leq 1$. The Heinz means are defined as follows:

$$H_{\nu}(a, b) = \frac{a^{\nu} b^{1-\nu} + a^{1-\nu} b^{\nu}}{2}.$$

It follows from the inequalities (1.1) and (1.2) that the Heinz means interpolate between the geometric mean and the arithmetic mean:

$$\sqrt{ab} \le H_v(a,\ b) \le \frac{a+b}{2}. \tag{1.4}$$

The second inequality of (1.4) is known as Heinz inequality for nonnegative real numbers.

As a direct consequence of the inequality (1.3), Kittaneh and Manasrah [1] obtained a refinement of the Heinz inequality as follows:

$$H_v(a,\ b) + r_0\left(\sqrt{a}-\sqrt{b}\right)^2 \le \frac{a+b}{2}, \tag{1.5}$$

where $r_0 = \min\{v,\ 1-v\}$.

Bhatia and Davis [2] proved that if $A, B, X \in M_n$ such that A and B are positive semidefinite and if $0 \le v \le 1$, then

$$2\left\|A^{1/2}XB^{1/2}\right\| \ \le\ \left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \ \le\ \|AX+XB\|. \tag{1.6}$$

This is a matrix version of the inequality (1.4). Kittaneh [3] proved that if $A, B, X \in M_n$ such that A and B are positive semidefinite and if $0 \le v \le 1$, then

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \ \le\ 4r_0\left\|A^{1/2}XB^{1/2}\right\| + \left(1-2r_0\right)\|AX+XB\|, \tag{1.7}$$

where $r_0 = \min\{v,\ 1-v\}$. This is a refinement of the second inequality in (1.6).

In this article, we first present a refinement of the inequality (1.5). After that, we use it to establish a refinement of the inequality (1.7) for the Hilbert-Schmidt norm.

2. A refinement of the inequality (1.5)

In this section, we give a refinement of the inequality (1.5). To do this, we need the following lemma.

Lemma 2.1. [4,5] Let $f(x)$ be a real valued convex function on an interval $[a, b]$. For any $x_1, x_2 \in [a, b]$, we have

$$f(x) \le \frac{f(x_2)-f(x_1)}{x_2-x_1}x - \frac{x_1 f(x_2)-x_2 f(x_1)}{x_2-x_1},\ x \in (x_1,\ x_2).$$

Theorem 2.1. Let $a, b \ge 0$ and $0 \le v \le 1$. If $r_0 = \min\{v,\ 1-v\}$, then

$$2H_v(a,\ b) \le \begin{cases} (1-4r_0)(a+b)+4r_0(a^{1/4}b^{3/4}+a^{3/4}b^{1/4}), & v \in [0,\ \frac{1}{4}] \cup [\frac{3}{4}, 1], \\ 2\,(4r_0-1)\sqrt{ab}+2(1-2r_0)(a^{1/4}b^{3/4}+a^{3/4}b^{1/4}), & v \in [\frac{1}{4},\ \frac{3}{4}]. \end{cases} \tag{2.1}$$

Proof. It is known that as a function of v, $H_v\,(a, b)$ is convex and attains its minimum at $v = \frac{1}{2}$. Let

$$f(v) = 2H_v(a,b) = a^v b^{1-v} + a^{1-v}b^v, \quad 0 \le v \le 1.$$

Obviously, $f(v)$ is convex. For $0 \le v \le \frac{1}{4}$, since $f(v)$ is convex on 0[1], by Lemma 2.1, we have

$$f(v) \le \frac{f\left(\frac{1}{4}\right) - f(0)}{\frac{1}{4} - 0} v - \frac{0 f\left(\frac{1}{4}\right) - \frac{1}{4} f(0)}{\frac{1}{4} - 0},$$

which is equivalent to

$$f(v) \le 4\left(f\left(\frac{1}{4}\right) - f(0)\right) v + f(0).$$

That is,

$$f(v) \le (1 - 4v) f(0) + 4v f\left(\frac{1}{4}\right).$$

So,

$$a^v b^{1-v} + a^{1-v} b^v \le (1 - 4r_0)(a + b) + 4r_0\left(a^{1/4} b^{3/4} + a^{3/4} b^{1/4}\right).$$

For $\frac{3}{4} \le v \le 1$, similarly, we have

$$f(v) \le \frac{f(1) - f\left(\frac{3}{4}\right)}{1 - \frac{3}{4}} v - \frac{\frac{3}{4} f(1) - f\left(\frac{3}{4}\right)}{1 - \frac{3}{4}},$$

which is equivalent to

$$f(v) \le 4\left(f(1) - f\left(\frac{3}{4}\right)\right) v - 3f(1) + 4f\left(\frac{3}{4}\right).$$

That is,

$$f(v) \le (4v - 3) f(1) + 4(1 - v) f\left(\frac{3}{4}\right).$$

So,

$$a^v b^{1-v} + a^{1-v} b^v \le (1 - 4r_0)(a + b) + 4r_0\left(a^{1/4} b^{3/4} + a^{3/4} b^{1/4}\right).$$

If $\frac{1}{4} \le v \le \frac{1}{2}$, then by Lemma 2.1, we have

$$f(v) \le \frac{f\left(\frac{1}{2}\right) - f\left(\frac{1}{4}\right)}{\frac{1}{2} - \frac{1}{4}} v - \frac{\frac{1}{4} f\left(\frac{1}{2}\right) - \frac{1}{2} f\left(\frac{1}{4}\right)}{\frac{1}{2} - \frac{1}{4}},$$

and so

$$f(v) \le (4v - 1) f\left(\frac{1}{2}\right) + 2(1 - 2v) f\left(\frac{1}{4}\right),$$

which is equivalent to

$$a^v b^{1-v} + a^{1-v} b^v \le 2(4r_0 - 1)\sqrt{ab} + 2(1 - 2r_0)\left(a^{1/4} b^{3/4} + a^{3/4} b^{1/4}\right).$$

If $\frac{1}{2} \le \nu \le \frac{3}{4}$, similarly, we have

$$f(\nu) \le \frac{f\left(\frac{3}{4}\right) - f\left(\frac{1}{2}\right)}{\frac{3}{4} - \frac{1}{2}}\nu - \frac{\frac{1}{2}f\left(\frac{3}{4}\right) - \frac{3}{4}f\left(\frac{1}{2}\right)}{\frac{3}{4} - \frac{1}{2}},$$

and so

$$f(\nu) \le (3 - 4\nu) f\left(\frac{1}{2}\right) + 2(2\nu - 1) f\left(\frac{3}{4}\right),$$

which is equivalent to

$$f(\nu) \le (4r_0 - 1) f\left(\frac{1}{2}\right) + 2(1 - 2r_0) f\left(\frac{3}{4}\right).$$

That is,

$$a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu} \le 2(4r_0 - 1)\sqrt{ab} + 2(1 - 2r_0)\left(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}\right).$$

This completes the proof. □

Now, we give a simple comparison between the upper bound for $a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}$ in (1.5) and (2.1). If $\nu \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, then

$$\begin{aligned} a + b - 2r_0\left(\sqrt{a} - \sqrt{b}\right)^2 &- (1 - 4r_0)(a + b) - 4r_0(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}) \\ &= 2r_0\left(a + b + 2\sqrt{ab} - 2\left(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}\right)\right) \\ &\ge 0. \end{aligned}$$

If $\nu \in \left[\frac{1}{4}, \frac{3}{4}\right]$, then

$$\begin{aligned} a + b - 2r_0\left(\sqrt{a} - \sqrt{b}\right)^2 &- 2(4r_0 - 1)\sqrt{ab} - 2(1 - 2r_0)(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}) \\ &= (1 - 2r_0)\left(a + b + 2\sqrt{ab} - 2\left(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}\right)\right) \\ &\ge 0. \end{aligned}$$

So, the inequality (2.1) is a refinement of the inequality (1.5).

3. An application

In this section, we give a refinement of the inequality (1.7) for the Hilbert-Schmidt norm based on the inequality (2.1).

Theorem 3.1. Let $A, B, X \in M_n$ such that A and B are positive semidefinite and suppose

that

$$\phi(\nu) = ||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||_2, \quad 0 \le \nu \le 1.$$

Then

$$\phi(\nu) \le \begin{cases} (1 - 4r_0)\phi(0) + 4r_0\phi\left(\frac{1}{4}\right), & \nu \in [0, \frac{1}{4}] \cup \left[\frac{3}{4}, 1\right] \\ (4r_0 - 1)\phi\left(\frac{1}{2}\right) + 2(1 - 2r_0)\phi\left(\frac{1}{4}\right), & \nu \in \left[\frac{1}{4}, \frac{3}{4}\right] \end{cases}, \tag{3.1}$$

where $r_0 = \min\{\nu, 1 - \nu\}$.

Proof. Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there exist unitary matrices U, $V \in M_n$ such that $A = U\Lambda_1 U^*$ and $B = V\Lambda_2 V^*$, where $\Lambda_1 = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_n)$, $\Lambda_2 = \mathrm{diag}(\mu_1,\ldots,\mu_n)$ and λ_i, $\mu_i \geq 0$, $i = 1,\ldots, n$. Let

$$Y = U^*XV = [y_{ij}].$$

If $v \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, then by (2.1) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
||A^v XB^{1-v} + A^{1-v}XB^v||_2^2 &= \sum_{i,j=1}^{n} (\lambda_i^v \mu_j^{1-v} + \lambda_i^{1-v}\mu_j^v)^2 |y_{ij}|^2 \\
&\leq \sum_{i,j=1}^{n} \left((1-4r_0)(\lambda_i + \mu_j) + 4r_0(\lambda_i^{1/4}\mu_j^{3/4} + \lambda_i^{3/4}\mu_j^{1/4})\right)^2 |y_{ij}|^2 \\
&= (1-4r_0)^2 \sum_{i,j=1}^{n} (\lambda_i + \mu_j)^2 |y_{ij}|^2 \\
&+ 16r_0^2 \sum_{i,j=1}^{n} \left(\lambda_i^{1/4}\mu_j^{3/4} + \lambda_i^{3/4}\mu_j^{1/4}\right)^2 |y_{ij}|^2 \\
&+ 8r_0(1-4r_0) \sum_{i,j=1}^{n} (\lambda_i + \mu_j) \left(\lambda_i^{1/4}\mu_j^{3/4} + \lambda_i^{3/4}\mu_j^{1/4}\right) |y_{ij}|^2 \\
&\leq\; (1-4r_0)^2\phi^2(0) + 16r_0^2\phi^2\left(\tfrac{1}{4}\right) + 8r_0(1-4r_0)\phi(0)\phi\left(\tfrac{1}{4}\right) \\
&= \left((1-4r_0)\phi(0) + 4r_0\phi\left(\tfrac{1}{4}\right)\right)^2.
\end{aligned}$$

If $v \in \left[\frac{1}{4}, \frac{3}{4}\right]$, the result follows from the inequality (2.1) and the same method above. This completes the proof. □

Remark. For the Hilbert-Schmidt norm, by the inequality (1.7), we have

$$\phi\,(v) \leq 2r_0\phi\left(\frac{1}{2}\right) + (1-2r_0)\,\phi\,(0)\,.$$

So, for $v \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, we have

$$\begin{aligned}
&2r_0\phi\left(\tfrac{1}{2}\right) + (1-2r_0)\phi\,(0) - (1-4r_0)\phi\,(0) - 4r_0\phi\left(\tfrac{1}{4}\right) \\
&\qquad = 2r_0\left(\phi\left(\tfrac{1}{2}\right) + \phi\,(0) - 2\phi\left(\tfrac{1}{4}\right)\right) \geq 0.
\end{aligned}$$

If $v \in \left[\frac{1}{4}, \frac{3}{4}\right]$, then

$$\begin{aligned}
&2r_0\phi\left(\tfrac{1}{2}\right) + (1-2r_0)\phi\,(0) - (4r_0-1)\phi\left(\tfrac{1}{2}\right) - 2(1-2r_0)\phi\left(\tfrac{1}{4}\right) \\
&\qquad = (1-2r_0)\left(\phi\left(\tfrac{1}{2}\right) + \phi\,(0) - 2\phi\left(\tfrac{1}{4}\right)\right) \geq 0.
\end{aligned}$$

So, the inequality (3.1) is a refinement of the inequality (1.7) for the Hilbert-Schmidt norm.

Acknowledgements
The authors wish to express their heartfelt thanks to the referees and Professor Gnana Bhaskar Tenali for their detailed and helpful suggestions for revising the manuscript. This research was supported by the Scientific Research Project of Chongqing Three Gorges University (No. 11QN-21).

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 30 October 2011 Accepted: 23 May 2012 Published: 23 May 2012

References
1. Kittaneh, F, Manasrah, Y: Improved Young and Heinz inequalities for matrices. J Math Anal Appl. **361**, 262–269 (2010). doi:10.1016/j.jmaa.2009.08.059
2. Bhatia, R, Davis, C: More matrix forms of the arithmetic-geometric mean inequality. SIAM J Matrix Anal Appl. **14**, 132–136 (1993). doi:10.1137/0614012
3. Kittaneh, F: On the convexity of the Heinz means. Integr Equ Oper Theory. **68**, 519–527 (2010). doi:10.1007/s00020-010-1807-6
4. Bhatia, R, Sharma, R: Some inequalities for positive linear maps. Linear Algebra Appl. **436**, 1562–1571 (2012). doi:10.1016/j.laa.2010.09.038
5. Wang, S, Zou, L, Jiang, Y: Some inequalities for unitarily invariant norms of matrices. J Inequal Appl. **2011**, 10 (2011). doi:10.1186/1029-242X-2011-10

doi:10.1186/1029-242X-2012-113
Cite this article as: Zou and Jiang: **Improved Heinz inequality and its application.** *Journal of Inequalities and Applications* 2012 **2012**:113.