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Sharp bounds for Seiffert mean in terms of root mean square

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Abstract

We find the greatest value α and least value β in (1/2,1) such that the double inequality

$$S(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < S(\beta a + (1-\beta)b, \beta b + (1-\beta)a)$$

holds for all $a,b > 0$ with $a \neq b$. Here, $T(a, b) = (a-b)/[2 \arctan((a-b)/(a+b))]$ and $S(a, b) = [(a^2 + b^2)/2]^{1/2}$ are the Seiffert mean and root mean square of a and b, respectively.

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1 Introduction

For $a,b > 0$ with $a \neq b$ the Seiffert mean $T(a, b)$ and root mean square $S(a, b)$ are defined by

$$T(a,b) = \frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)} \tag{1.1}$$

and

$$S(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \tag{1.2}$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities and properties for T and S can be found in the literature [1-14].

Let $A(a,b) = (a+b)/2, G(a,b) = \sqrt{ab}$, and $M_p(a, b) = ((a^p+b^p)/2)^{1/p}$ $(p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ be the arithmetic, geometric, and pth power means of two positive numbers a and b, respectively. Then it is well known that

$$G(a,b) = M_0(a,b) < A(a,b) = M_1(a,b) < T(a,b) < S(a,b) = M_2(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Seiffert [1] proved that inequalities

$$A(a,b) < T(a,b) < S(a,b)$$

hold for all a, $b > 0$ with $a \neq b$.

Chu et al. [5] found the greatest value p_1 and least value p_2 such that the double inequality $H_{p1}(a,b) < T(a,b) < H_{p2}(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $H_p(a, b) = ((a^p + (ab)^{p/2} + b^p)/3)^{1/p}$ $(p \neq 0)$ and $H_0(a,b) = \sqrt{ab}$ is the pth power-type Heron mean of a and b.

In [6], Wang et al. answered the question: What are the best possible parameters λ and μ such that the double inequality $L_\lambda(a,b) < T(a,b) < L_\mu(a,b)$ holds for all $a,b > 0$ with $a \neq b$? where $L_r(a,b) = (a^{r+1} + b^{r+1})/(a^r + b^r)$ is the rth Lehmer mean of a and b.

Chu et al. [7] proved that inequalities

$$pT(a,b) + (1-p)G(a,b) < A(a,b) < qT(a,b) + (1-q)G(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $p \leq 3/5$ and $q \geq \pi/4$.

Hou and Chu [9] gave the best possible parameters α and β such that the double inequality

$$\alpha S(a,b) + (1-\alpha)\overline{H}(a,b) < T(a,b) < \beta S(a,b) + (1-\beta)\overline{H}(a,b)$$

holds for all a, $b > 0$ with $a \neq b$.

For fixed a, $b > 0$ with $a \neq b$, let $x \in [1/2,1]$ and

$$f(x) = S(xa + (1-x)b, xb + (1-x)a).$$

Then it is not difficult to verify that $f(x)$ is continuous and strictly increasing in $[1/2,1]$. Note that $f(1/2) = A(a,b) < T(a,b)$ and $f(1) = S(a, b) > T(a, b)$. Therefore, it is natural to ask what are the greatest value α and least value β in $(1/2,1)$ such that the double inequality

$$S(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < S(\beta a + (1-\beta)b, \beta b + (1-\beta)a)$$

holds for all a, $b > 0$ with $a \neq b$. The main purpose of this article is to answer these questions. Our main result is the following Theorem 1.1.

Theorem 1.1. If $\alpha, \beta \in (1/2,1)$, then the double inequality

$$S(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < S(\beta a + (1-\beta)b, \beta b + (1-\beta)a) \quad (1.3)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 + \sqrt{16/\pi^2 - 1})/2$ and $\beta \geq (3 + \sqrt{6})/6$.

2 Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (1 + \sqrt{16/\pi^2 - 1})/2$ and $\mu = (3 + \sqrt{6})/6$.

We first proof that inequalities

$$T(a,b) > S(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) \quad (2.1)$$

and

$$T(a,b) < S(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \quad (2.2)$$

hold for all a, $b > 0$ with $a \neq b$.

From (1.1) and (1.2), we clearly see that both $T(a, b)$ and $S(a, b)$ are symmetric and homogenous of degree 1. Without loss of generality we assume that $a > b$. Let $t = a/b > 1$ and $p \in (1/2,1)$, then from (1.1) and (1.2) one has

$$\begin{aligned} & S(pa+(1-p)b, pb+(1-p)a)-T(a,b) \\ &= b\frac{\sqrt{[pt+(1-p)]^2+[(1-p)t+p]^2}}{2\arctan\left(\frac{t-1}{t+1}\right)} \\ &\times\left\{\sqrt{2}\arctan\left(\frac{t-1}{t+1}\right)-\frac{t-1}{\sqrt{[pt+(1-p)]^2+[(1-p)t+p]^2}}\right\}. \end{aligned} \tag{2.3}$$

Let

$$f(t)=\sqrt{2}\arctan\left(\frac{t-1}{t+1}\right)-\frac{t-1}{\sqrt{[pt+(1-p)]^2+[(1-p)t+p]^2}}, \tag{2.4}$$

then simple computations lead to

$$f(1)=0, \tag{2.5}$$

$$\lim_{t\to+\infty} f(t)=\frac{\sqrt{2}\pi}{4}-\frac{1}{\sqrt{p^2+(1-p)^2}}, \tag{2.6}$$

$$f'(t)=\frac{f_1(t)}{\{[pt+(1-p)]^2+[(1-p)t+p]^2\}^{\frac{3}{2}}(t^2+1)}, \tag{2.7}$$

where

$$f_1(t)=\sqrt{2}\{[pt+(1-p)]^2+[(1-p)t+p]^2\}^{\frac{3}{2}}-(t+1)(t^2+1). \tag{2.8}$$

Note that

$$\begin{aligned} & \{\sqrt{2}\{[pt+(1-p)]^2+[(1-p)t+p]^2\}^{\frac{3}{2}}\}^2-[(t+1)(t^2+1)]^2 \\ &= (t-1)^2 g_1(t), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} g_1(t) &= (16p^6-48p^5+72p^4-64p^3+36p^2-12p+1)t^4-16p^2 \\ & (4p^2-4p+3)(p-1)^2t^3+2(48p^6-144p^5+168p^4-96p^3+36p^2-12p+1) \\ & \times t^2-16p^2(4p^2-4p+3)(p-1)^2t+16p^6-48p^5+72p^4-64p^3+36p^2 \\ & -12p+1, \end{aligned} \tag{2.10}$$

$$g_1(1)=4(12p^2-12p+1). \tag{2.11}$$

Let $g_2(t)=g_1'(t)/4$, $g_3(t)=g_2'(t)$, $g_4(t)=g_3'(t)/6$. Then simple computations lead to

$$\begin{aligned} g_2(t) &= (16p^6-48p^5+72p^4-64p^3+36p^2-12p+1)t^3-12p^2 \\ & (4p^2-4p+3)(p-1)^2t^2+(48p^6-144p^5+168p^4-96p^3+36p^2-12p+1) \\ & t-4p^2(4p^2-4p+3)(p-1)^2, \end{aligned} \tag{2.12}$$

$$g_2(1) = 4(6p^2 - 12p + 1), \tag{2.13}$$

$$g_3(t) = 3(16p^6 - 48p^5 + 72p^4 - 64p^3 + 36p^2 - 12p + 1)t^2 - 24p^2(4p^2 - 4p + 3)(p - 1)^2 t + 48p^6 - 144p^5 + 168p^4 - 96p^3 + 36p^2 - 12p + 1, \tag{2.14}$$

$$g_3(1) = 4(6p^4 - 12p^3 + 18p^2 - 12p + 1), \tag{2.15}$$

$$g_4(t) = (16p^6 - 48p^5 + 72p^4 - 64p^3 + 36p^2 - 12p + 1)t - 4p^2(4p^2 - 4p + 3)(p - 1)^2, \tag{2.16}$$

$$g_4(1) = 12p^4 - 24p^3 + 24p^2 - 12p + 1. \tag{2.17}$$

We divide the proof into two cases.

Case 1. $p = \lambda = (1 + \sqrt{16/\pi^2 - 1})/2$. Then equations (2.6), (2.11), (2.13), (2.15), and (2.17) lead to

$$\lim_{t\to+\infty} f(t) = 0, \tag{2.18}$$

$$g_1(1) = -\frac{4(5\pi^2 - 48)}{\pi^2} < 0, \tag{2.19}$$

$$g_2(1) = -\frac{2(5\pi^2 - 48)}{\pi^2} < 0, \tag{2.20}$$

$$g_3(1) = -\frac{2(7\pi^4 - 48\pi^2 - 192)}{\pi^4} < 0, \tag{2.21}$$

$$g_4(1) = -\frac{2(\pi^4 - 96)}{\pi^4} < 0. \tag{2.22}$$

Note that

$$16p^6 - 48p^5 + 72p^4 - 64p^3 + 36p^2 - 12p + 1 = \frac{1024 - \pi^6}{\pi^6} > 0. \tag{2.23}$$

From (2.10), (2.12), (2.14), (2.16), and (2.23) we clearly see that

$$\lim_{t\to+\infty} g_1(t) = +\infty \tag{2.24}$$

$$\lim_{t\to+\infty} g_2(t) = +\infty \tag{2.25}$$

$$\lim_{t\to+\infty} g_3(t) = +\infty \tag{2.26}$$

$$\lim_{t\to+\infty} g_4(t) = +\infty \tag{2.27}$$

From equation (2.16) and inequality (2.23) we clearly see that $g_4(t)$ is strictly increasing in $[1, +\infty)$, then inequality (2.22) and equation (2.27) lead to the conclusion that there exists $t_0 > 1$ such that $g_4(t) < 0$ for $t \in (1,t_0)$ and $g_4(t) > 0$ for $t \in (t_0,+\infty)$. Hence, $g_3(t)$ is strictly decreasing in $[1, t_0]$ and strictly increasing in $[t_0, +\infty)$.

It follows from (2.21) and (2.26) together with the piecewise monotonicity of $g_3(t)$ that there exists $t_1 > t_0 > 1$ such that $g_2(t)$ is strictly decreasing in $[1,t_1]$ and strictly increasing in $[t_1,+\infty)$.

From (2.20) and (2.25) together with the piecewise monotonicity of $g_2(t)$ we conclude that there exists $t_2 > t_1 > 1$ such that $g_1(t)$ is strictly decreasing in $[1,t_2]$ and strictly increasing in $[t_2,+\infty)$.

Equations (2.7)-(2.9), (2.19), and (2.24) together with the piecewise monotonicity of $g_1(t)$ imply that there exists $t_3 > t_2 > 1$ such that $f(t)$ is strictly decreasing in $[1,t_3]$ and strictly increasing in $[t_3, +\infty)$.

Therefore, inequality (2.1) follows from equations (2.3)-(2.5) and (2.18) together with the piecewise monotonicity of $f(t)$.

Case 2. $p = \mu = (3+\sqrt{6})/6$. Then equation (2.10) becomes

$$g_1(t) = \frac{(17t^2+2t+17)}{108}(t-1)^2 > 0 \tag{2.28}$$

for $t > 1$.

Equations (2.7)-(2.10) and inequality (2.28) lead to the conclusion that $f(t)$ is strictly increasing in $[1, +\infty)$.

Therefore, inequality (2.2) follows from equations (2.3)-(2.5) and the monotonicity of $f(t)$.

From the monotonicity of $f(x) = S(xa + (1-x)b, xb + (1-x)a)$ in $[1/2,1]$ and inequalities (2.1) and (2.2) we know that inequality (1.3) holds for all $\alpha \le (1+\sqrt{16/\pi^2-1})/2$, $\beta \ge (3+\sqrt{6})/6$ and $a, b > 0$ with $a \ne b$.

Next, we prove that $\lambda = (1+\sqrt{16/\pi^2-1})/2$ is the best possible parameter in $(1/2,1)$ such that inequality (2.1) holds for all $a, b > 0$ with $a \ne b$.

For any $1 > p > \lambda = (1+\sqrt{16/\pi^2-1})/2$, from (2.6) one has

$$\lim_{t\to+\infty} f(t) = \frac{\pi}{2} - \frac{1}{p^2+(1-p)^2} > 0. \tag{2.29}$$

Equations (2.3) and (2.4) together with inequality (2.29) imply that for any $1 > p > \lambda = (1+\sqrt{16/\pi^2-1})/2$ there exists $T_0 = T_0(p) > 1$ such that

$$S(pa+(1-p)b, pb+(1-p)a) > T(a,b)$$

for $a/b \in (T_0, +\infty)$.

Finally, we prove that $\mu = (3+\sqrt{6})/6$ is the best possible parameter in $(1/2,1)$ such that inequality (2.2) holds for all $a, b > 0$ with $a \ne b$.

For any $1/2 < p < \mu = (3+\sqrt{6})/6$, from (2.11) one has

$$g_1(1) = 4(12p^2 - 12p + 1) < 0. \tag{2.30}$$

From inequality (2.30) and the continuity of $g_1(t)$ we know that there exists $\delta = \delta(p) > 0$ such that

$$g_1(t) < 0 \tag{2.31}$$

for $t \in (1, 1+\delta)$.

Equations (2.3)-(2.5) and (2.7)-(2.10) together with inequality (2.31) imply that for any $1/2 < p < \mu = (3 + \sqrt{6})/6$ there exists $\delta = \delta(p) > 0$ such that

$$T(a, b) > S(pa + (1 - p)b, pb + (1 - p)a)$$

for $a/b \in (1,1 + \delta)$.

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Authors' contributions
Y-MC provided the main idea in this article. S-WH carried out the proof of inequality (2.1) in this article. Z-HS carried out the proof of inequality (2.2) in this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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