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Some sharp integral inequalities involving partial derivatives

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Abstract

The main purpose of the present article is to establish some new sharp integral inequalities in $2n$ independent variables. Our results in special cases yield some of the recent results on Pachpatter, Agarwal and Sheng's inequalities and provide some new estimates on such types of inequalities.

Mathematics Subject Classification 2000: 26D15.

Keywords: Cauchy-Schwarz's inequality, Pachpatte's inequality, Hölder integral inequality, the arithmetic-geometric means inequality

1 Introduction

Inequalities involving functions of n independent variables, their partial derivatives, integrals play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [1-10]. Especially, in view of wider applications, inequalities due to Agarwal, Opial, Pachpatte, Wirtinger, Poincaréand et al. have been generalized and sharpened from the very day of their discover. As a matter of fact these now have become research topic in their own right [11-14]. In the present article we shall use the same method of Agarwal and Sheng [15], establish some new estimates on these types of inequalities involving $2n$ independent variables. We further generalize these inequalities which lead to result sharp than those currently available. An important characteristic of our results is that the constants in the inequalities are explicit.

2 Main results

Let R be the set of real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space. Let E, E' be a bounded domain in R^n defined by $E \times E' = \prod_{i=1}^{n} [a_i, b_i] \times [c_i, d_i], i = 1, \ldots, n$. For $x_i, y_i \in R$, $i = 1, \ldots, n$, $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a variable point in $E \times E'$ and $dxdy = dx_1 \ldots dx_n\, dy_1 \ldots dy_n$. For any continuous real-valued function $u(x, y)$ defined on $E \times E'$ we denote by $\int_E \int_{E'} u(x, y)\, dxdy$ the $2n$-fold integral

$$\int_{a_1}^{b_1} \cdots \int a_n{}^{b_n} \int c_1{}^{d_1} \cdots \int c_n{}^{d_n} u(x_1, \ldots, x_n, y_1, \ldots, y_n) dx_1 \ldots dx_n dy_1 \ldots dy_n,$$

and for any $(x, y) \in E \times E'$, $\int E(x) \int E'(x) u(s, t)\, dsdt$ is the $2n$-fold integral

$$\int a_1{}^{x_1} \cdots \int a_n{}^{x_n} \int c_1{}^{y_1} \cdots \int c_n{}^{y_n} u(s_1, \ldots, s_n, t_1, \ldots, t_n) dx_1 \ldots ds_n dt_1 \ldots dt_n,$$

We represent by $F(E \times E')$ the class of continuous functions $u(x, y) : E \times E' \to \mathbb{R}$, for each $i, 1 \le i \le n$,

$$u(x,y)\big|_{x_i=a_i} = 0, u(x,y)\big|_{y_i=c_i} = 0, u(x,y)\big|_{x_i=b_i} = 0\ , u(x,y)\big|_{y_i=d_i} = 0, (i = 1, \ldots, n)$$

the class $F(E \times E')$ is denoted as $G(E \times E')$.

Theorem 2.1. *Let $l, \mu, \lambda \ge 1$, be given real numbers such that $\frac{1}{\mu} + \frac{1}{\lambda} = 1$. Further, let $u(x, y) \in G(E \times E')$. Then, the following inequality holds*

$$\begin{aligned} \int E \int E' |u(x,y)|^l dxdy \le \frac{1}{2n} \left(\sum_{i=1}^{n} [(b_i - a_i)(c_i - d_i)]^{\mu} \right)^{1/\mu} \left(\int E \int E' |u(x,y)|^{(l-1)\mu} dxdy \right)^{1/\mu} \\ \times \left(\int E \int E' \left\| \text{grad } u(x,y) \right\|_{\lambda}^{\lambda} dxdy \right)^{1/\lambda}, \end{aligned} \tag{2.1}$$

where

$$\left\| \text{grad } u(x,y) \right\|_{\lambda} = \left(\sum_{i=1}^{n} \left| \frac{\partial^2}{\partial x_i \partial y_i} u(x,y) \right|^{\lambda} \right)^{1/\lambda}.$$

Proof. For each fixed i, $1 \le i \le n$, in view of

$$u(x,y)\big|_{x_i=a_i} = 0, u(x,y)\big|_{y_i=c_i} = 0, u(x,y)\big|_{x_i=b_i} = 0, u(x,y)\big|_{y_i=d_i} = 0, (i = 1, \ldots, n)$$

we have

$$u^l(x,y) = u^{l-1}(x,y) \int a_i{}^{x_i} \int c_i{}^{y_i} \frac{\partial^2}{\partial s_i \partial t_i} u(x,y; s_i, t_i) ds_i dt_i, \tag{2.2}$$

and

$$u^l(x,y) = u^{l-1}(x,y) \int x_i{}^{b_i} \int y_i{}^{d_i} \frac{\partial^2}{\partial s_i \partial t_i} u(x,y; s_i, t_i) ds_i dt_i, \tag{2.3}$$

where

$$u(x,y; s_i, t_i) = u(x_1, \ldots, x_{i-1}, s_i, x_{i+1}, \ldots, x_n, y_1, \ldots, y_{i-1}, t_i, y_{i+1}, \ldots, y_n).$$

Hence, from (2.2) and (2.3) and in view of the arithmetic-geometric means inequality and Hölder inequality with indices μ and λ, it follows that

$$\begin{aligned} |u(x,y)|^l &\le \frac{1}{2} |u(x,y)|^{l-1} \int a_i{}^{b_i} \int c_i{}^{d_i} \left| \frac{\partial^2}{\partial s_i \partial t_i} u(x,y; s_i, t_i) \right| ds_i dt_i \\ &\le \frac{1}{2} |u(x,y)|^{l-1} [(b_i - a_i)(c_i - d_i)]^{1/\mu} \left(\int a_i{}^{x_i} \int c_i{}^{y_i} \left| \frac{\partial^2}{\partial s_i \partial t_i} u(x,y; s_i, t_i) \right|^{\lambda} ds_i dt_i \right)^{1/\lambda}. \end{aligned} \tag{2.4}$$

Now, summing the inequalities (2.4) for $1 \le i \le n$, integrating over $E \times E'$ and applying Holder inequality with indices μ and λ two times, we get

$$\int E \int E' \left|u(x,y)\right|^{l} dxdy \le \frac{1}{2n}\sum_{i=1}^{n}[(b_i-a_i)(c_i-d_i)]^{1/\mu}$$

$$\times \int E \int E' \left|u(x,y)\right|^{l-1}\left(\int a_i^{b_i}\int c_i^{d_i}\left|\frac{\partial^2}{\partial s_i \partial t_i}u(x,y;s_i,t_i)\right|^{\lambda} ds_i dt_i\right)^{1/\lambda} dxdy$$

$$\le \frac{1}{2n}\left(\int E\int E'\left|u(x,y)\right|^{(l-1)\mu}dxdy\right)^{1/\mu}\sum_{i=1}^{n}[(b_i-a_i)(c_i-d_i)]^{1/\mu}$$

$$\times\left(\int E\int E'\int a_i^{b_i}\int c_i^{d_i}\left|\frac{\partial^2}{\partial s_i\partial t_i}u(x,y;s_i,t_i)\right|^{\lambda}ds_i dt_i dxdy\right)^{1/\lambda}$$

$$\le \frac{1}{2n}\left(\int E\int E'\left|u(x,y)\right|^{(l-1)\mu}dxdy\right)^{1/\mu}\sum_{i=1}^{n}[(b_i-a_i)(c_i-d_i)]^{1/\mu+1/\lambda}$$

$$\times\left(\int E\int E'\left|\frac{\partial^2}{\partial x_i\partial y_i}u(x,y)\right|^{\lambda}dxdy\right)^{1/\lambda}$$

$$\le \frac{1}{2n}\left(\int E\int E'\left|u(x,y)\right|^{(l-1)\mu}dxdy\right)^{1/\mu}\left(\sum_{i=1}^{n}[(b_i-a_i)(c_i-d_i)]^{\mu}\right)^{1/\mu}$$

$$\times\left(\int E\int E'\left\|\text{grad } u(x,y)\right\|_{\lambda}^{\lambda}dxdy\right)^{1/\lambda},$$

where

$$\left\|\text{grad } u(x,y)\right\|_{\lambda} = \left(\sum_{i=1}^{n}\left|\frac{\partial^2}{\partial x_i\partial y_i}u(x,y)\right|^{\lambda}\right)^{1/\lambda}.$$

The proof is complete.

Remark 2.1. Let $u(x, y)$ reduce to $u(x)$ in (2.1) and with suitable modifications, then (2.1) becomes

$$\int E\left|u(x)\right|^{(l)\mu}dx \le \frac{1}{2n}\left(\int E\left|u(x)\right|^{(l-1)\mu}dx\right)^{1/\mu}\left(\sum_{i=1}^{n}(b_i-a_i)^{\mu}\right)^{1/\mu}$$

$$\times\left(\int E\left\|\text{grad } u(x)\right\|_{\lambda}^{\lambda}dx\right)^{1/\lambda},$$

where

$$\left\|\text{grad } u(x)\right\|_{\lambda} = \left(\sum_{i=1}^{n}\left|\frac{\partial}{\partial x_i}u(x)\right|^{\lambda}\right)^{1/\lambda}.$$

This is just a important inequality which was given by Agarwal and Sheng [15].

Remark 2.2. For the given real numbers $l_k \ge 0$, $1 \le k \le r$, such that $rl_k \ge 1$, the arithmetic-geometric means inequality and (2.1) gives

$$\int E\int E'\prod_{k=1}^{r}\left|u_k(x,y)\right|^{l_k}dxdy \le \frac{1}{r}\sum_{k=1}^{r}\int E\int E'\left|u_k(x,y)\right|^{rl_k}dxdy$$

$$\le \frac{1}{2nr}\left(\sum_{i=1}^{n}[(b_i-a_i)(c_i-d_i)]^{\mu}\right)^{1/\mu}\sum_{k=1}^{r}\left(\int E\int E'\left|u_k(x,y)\right|^{(rl_k-1)\mu}dxdy\right)^{1/\mu} \tag{2.5}$$

$$\times\left(\int E\int E'\left\|\text{grad } u_k(x,y)\right\|_{\lambda}^{\lambda}dxdy\right)^{1/\lambda}.$$

This is just a general form of the following result which was given by Agarwal and Sheng [15].

$$\int E \prod_{k=1}^{r} |u_k(x)|^{l_k} dx \leq \frac{1}{2nr} \left(\sum_{i=1}^{n} (b_i - a_i)^{\mu} \right)^{1/\mu} \sum_{k=1}^{r} \left(\int E |u_k(x)|^{(rl_k - 1)\mu} dx \right)^{1/\mu} \times \left(\int E \, \| \text{grad } u_k(x) \|_{\lambda}^{\lambda} dx \right)^{1/\lambda},$$

where

$$\| \text{grad } u_k(x) \|_{\lambda} = \left(\sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} u(x) \right|^{\lambda} \right)^{1/\lambda}.$$

Remark 2.3. In particular, for $l_k = (p_k + 2)/(2r)$, $p_k \geq 1, 1 \leq k \leq r$, $\mu = \lambda = 2$, the inequality (2.5) reduces to

$$\int E \int E' \left(\prod_{k=1}^{r} |u_k(x, y)|^{(p_k+2)/2} \right)^{1/r} dxdy$$
$$\leq \frac{1}{2nr} \left(\sum_{i=1}^{n} [(b_i - a_i)(c_i - d_i)]^2 \right)^{1/2} \sum_{k=1}^{r} \left(\int E \int E' |u(x, y)|^{p_k} dxdy \right)^{1/2} \times \left(\int E \int E' \, \| \text{grad } u_k(x, y) \|_2^2 \, dxdy \right)^{1/2}.$$

This is just a general form of the following result which was given by Agarwal and Sheng [15].

$$\int E \left(\prod_{k=1}^{r} |u_k(x)|^{(p_k+2)/2} \right)^{1/r} dx$$
$$\leq \frac{1}{2nr} \left(\sum_{i=1}^{n} (b_i - a_i)^2 \right)^{1/2} \sum_{k=1}^{r} \left(\int E |u(x)|^{p_k} dx \right)^{1/2} \left(\int E \, \| \text{grad } u_k(x) \|_2^2 \, dx \right)^{1/2}.$$

On the other hand, the above inequality with the right-hand side multiplied by $\left(\prod_{k=1}^{r} ((p_k + 2)/2)\right)^{1/r}$ and the term $\left(\sum_{i=1}^{n} (b_i - a_i)^2\right)^{1/2}$ replace by $\sqrt{n}\beta$ has been proved by Pachpatte [16].

Remark 2.4. If $u(x, y)$ reduce to $u(x)$ in (2.1), then the inequality (2.1) and its particular case $l \geq 2$, $\mu = \lambda = 2$ with the right-hand side multiplied by l have been separately proved by Pachpatte in [17].

Theorem 2.2. *Let* $\lambda \geq 1$ *and* $u(x, y) \in G(E \times E')$. *Then, the following inequality holds*

$$\int E \int E' |u(x, y)|^{2\lambda} dxdy \leq \frac{\pi \lambda^2 \beta^2 \alpha^2}{128n} \left(\int E \int E' |u(x, y)|^{2\lambda} dxdy \right)^{(\lambda-1)/\lambda} \times \left(\int E \int E' \sum_{i=1}^{n} \left| \frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda - 1) \frac{1}{u(x, y)} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial y_i} \right|^{2\lambda} dxdy \right)^{1/\lambda}, \quad (2.6)$$

where $\beta = \max_{1 \leq i \leq n}(b_i - a_i)$ *and* $\alpha = \max_{1 \leq i \leq n}(d_i - c_i)$.

Proof. For each fixed i, $1 \le i \le n$, we obtain that

$$u^{\lambda}(x,y) = \lambda \int a_i^{x_i} \int c_i^{y_i} \left[u^{\lambda-1}(x,y;s_i,t_i)\frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda-1)u^{\lambda-2}(x,y;s_i,t_i)\frac{\partial u}{\partial s_i}\frac{\partial u}{\partial t_i}\right] ds_i dt_i,$$

and hence from the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \left|u(x,y)\right|^{\lambda} &\le \lambda^2 (x_i - a_i)(y_i - c_i) \\ \times \int a_i^{x_i} \int c_i^{y_i} &\left| y^{\lambda-1}(x,y;s_i,t_i)\frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda-1)u^{\lambda-2}(x,y;s_i,t_i)\frac{\partial u}{\partial s_i}\frac{\partial u}{\partial t_i}\right|^2 ds_i dt_i, \end{aligned} \tag{2.7}$$

and similarly,

$$\begin{aligned} \left|u(x,y)\right|^{\lambda} &\le \lambda^2 (b_i - x_i)(d_i - y_i) \\ \times \int x_i^{b_i} \int y_i^{d_i} &\left| u^{\lambda-1}(x,y;s_i,t_i)\frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda-1)u^{\lambda-2}(x,y;s_i,t_i)\frac{\partial u}{\partial s_i}\frac{\partial u}{\partial t_i}\right|^2 ds_i dt_i, \end{aligned} \tag{2.8}$$

Hence, multiplying (2.7) and (2.8) and in view of using the arithmetic-geometric means inequality, summing the resulting inequalities for $1 \le i \le n$, and then integrating over $E \times E'$, to obtain

$$\begin{aligned} \int E \int E' \left|u(x,y)\right|^{2\lambda} dxdy \le \frac{\lambda^2}{2n} \int E \int E' &\left\{ \sum_{i=1}^{n} [(x_i - a_i)(y_i - c_i)(b_i - x_i)(d_i - y_i)]^{1/2} \right. \\ \times \int a_i^{b_i} \int c_i^{d_i} &\left. \left| u^{\lambda-1}(x,y;s_i,t_i)\frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda-1)u^{\lambda-2}(x,y;s_i,t_i)\frac{\partial u}{\partial s_i}\frac{\partial u}{\partial t_i}\right|^2 ds_i dt_i \right\} dxdy \\ = \frac{\lambda^2}{2n} \sum_{i=1}^{n} \int a_i^{b_i} \int c_i^{d_i} &[(x_i - a_i)(y_i - c_i)(b_i - x_i)(d_i - y_i)]^{1/2} dx_i dy_i \\ \times \int E \int E' &\left| y^{\lambda-1}(x,y)\frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda-1)u^{\lambda-2}(x,y)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial y_i}\right|^2 dxdy \\ \le \frac{\pi\lambda^2\beta^2\alpha^2}{128n} \int E \int E' &\sum_{i=1}^{n} \left| u^{\lambda-1}(x,y)\frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda-1)u^{\lambda-2}(x,y)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial y_i}\right|^2 dxdy, \end{aligned}$$

where $\beta = \max_{1 \le i \le n}(b_i - a_i)$ and $\alpha = \max_{1 \le i \le n}(d_i - c_i)$.

Hence, using Hölder inequality with indices λ and $\lambda/(\lambda - 1)$ in right-hand side of above inequality, we have

$$\begin{aligned} \int E \int E' \left|u(x,y)\right|^{2\lambda} dxdy &\le \frac{\pi\lambda^2\beta^2\alpha^2}{128n} \left(\int E \int E' \left|u(x,y)\right|^{2\lambda} dxdy \right)^{(\lambda-1)/\lambda} \\ &\times \left(\int E \int E' \sum_{i=1}^{n} \left| \frac{\partial^2 u}{\partial s_i \partial t_i} + (\lambda-1)\frac{1}{u(x,y)}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial y_i}\right|^{2\lambda} dxdy \right)^{1/\lambda}. \end{aligned}$$

The proof is complete.

Remark 2.5. Let $u(x, y)$ reduce to $u(x)$ in (2.6) and with suitable modifications, then (2.6) becomes the following Agarwal and Sheng [15] inequality.

$$\int E \left|u(x)\right|^{2\lambda} dx \le \frac{\pi\lambda^2\beta^2}{16n} \left(\int E \left|u(x)\right|^{2\lambda} dx \right)^{(\lambda-1)/\lambda} \left(\int E \left\| \text{grad } u(x) \right\|_2^{2\lambda} dx \right)^{1/\lambda},$$

where $\beta = \max_{1 \le i \le n}(b_i - a_i)$.

Theorem 2.3. *Let $l \geq 0$, $m \geq 1$ be given real numbers, and let $u(x, y) \in G(E \times E')$. Then, the following inequality holds*

$$\begin{aligned}\int E\int E'\left|u(x,y)\right|^{l+m}dxdy \leq \frac{1}{n}\left(\frac{m+l}{2m}\right)^m\sum_{i=1}^{n}[(b_i-a_i)(d_i-c_i)]^m\\ \times\int E\int E'\left|u^{l/m}(x,y)\frac{\partial^2 u}{\partial x_i\partial y_i}+\frac{l}{m}u^{(l/m-1)}(x,y)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial y_i}\right|^m dxdy.\end{aligned} \tag{2.8a}$$

Proof. For each fixed i, $1 \leq i \leq n$, we obtain that

$$\begin{aligned}u^{l+m}(x,y) = \frac{m+l}{m}[u(x,y)]^{(m-1)(l+m)/m}\\ \times\int a_i^{x_i}\int c_i^{y_i}\left[u^{l/m}(x,y;s_i,t_i)\frac{\partial^2 u}{\partial s_i\partial t_i}+\frac{l}{m}u^{(l/m-1)}(x,y;s_i,t_i)\frac{\partial u}{\partial s_i}\frac{\partial u}{\partial t_i}\right]ds_i dt_i,\end{aligned}$$

and, hence, it follows that

$$\begin{aligned}\left|u(x,y)\right|^{l+m} \leq \frac{m+l}{m}\left|u(x,y)\right|^{(m-1)(l+m)/m}\\ \times\int a_i^{x_i}\int c_i^{y_i}\left|u^{l/m}(x,y;s_i,t_i)\frac{\partial^2 u}{\partial s_i\partial t_i}+\frac{l}{m}u^{(l/m-1)}(x,y;s_i,t_i)\frac{\partial u}{\partial s_i}\frac{\partial u}{\partial t_i}\right|ds_i dt_i,\end{aligned} \tag{2.9}$$

and, similarly,

$$\begin{aligned}\left|u(x,y)\right|^{l+m} \leq \frac{m+l}{m}\left|u(x,y)\right|^{(m-1)(l+m)/m}\\ \times\int x_i^{b_i}\int y_i^{d_i}\left|u^{l/m}(x,y;s_i,t_i)\frac{\partial^2 u}{\partial s_i\partial t_i}+\frac{l}{m}u^{(l/m-1)}(x,y;s_i,t_i)\frac{\partial u}{\partial s_i}\frac{\partial u}{\partial t_i}\right|ds_i dt_i.\end{aligned} \tag{2.10}$$

Now, adding (2.9) and (2.10) and integrating the resulting inequality from a_i to b_i and c_i to d_i, respectively. Then

$$\begin{aligned}\int a_i^{b_i}\int c_i^{d_i}\left|u(x,y)\right|^{l+m}dx_i dy_i \leq \frac{m+l}{2m}\left(\int a_i^{b_i}\int c_i^{d_i}\left|u(x,y)\right|^{(m-1)(l+m)/m}dx_i dy_i\right)\\ \times\int a_i^{b_i}\int c_i^{d_i}\left|u^{l/m}(x,y)\frac{\partial^2 u}{\partial x_i\partial y_i}+\frac{l}{m}u^{(l/m-1)}(x,y)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial y_i}\right|dx_i dy_i.\end{aligned}$$

Next in each integral of the right-hand side of the above inequality we apply Hölder inequality with indices m and $m/(m - 1)$, to get

$$\begin{aligned}\int a_i^{b_i}\int c_i^{d_i}\left|u(x,y)\right|^{l+m}dx_i dy_i \leq \frac{m+l}{2m}\left(\int a_i^{b_i}\int c_i^{d_i}\left|u(x,y)\right|^{l+m}dx_i dy_i\right)^{(m-1)/m}\\ \times[(b_i-a_i)(d_i-c_i)]^{1/m}[(b_i-a_i)(d_i-c_i)]^{(m-1)/m}\\ \times\left(\int a_i^{b_i}\int c_i^{d_i}\left|u^{l/m}(x,y)\frac{\partial^2 u}{\partial x_i\partial y_i}+\frac{l}{m}u^{(l/m-1)}(x,y)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial y_i}\right|^m dx_i dy_i\right)^{1/m},\end{aligned}$$

which is unless $\int a_i^{b_i}\int c_i^{d_i}\left|u(x,y)\right|^{l+m}dx_i dy_i = 0$ (for which the inequality (2.8) is obvious), is the same as

$$\left(\int a_i^{b_i} \int c_i^{d_i} \left| u(x,y) \right|^{l+m} dx_i dy_i \right)^{1/m} \leq \frac{m+l}{2m} [(b_i - a_i)(d_i - c_i)]$$
$$\times \left(\int a_i^{b_i} \int c_i^{d_i} \left| u^{l/m}(x,y) \frac{\partial^2 u}{\partial x_i \partial y_i} + \frac{l}{m} u^{(l/m-1)}(x,y) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial y_i} \right|^m dx_i dy_i \right)^{1/m}.$$

Finally, raising m-th power both sides of the above inequality, integrating the resulting inequality from a_j to b_j and c_j to d_j, respectively, then summing the n inequalities $1 \leq i \leq n$, we find the desired inequality (2.8).

Remark 2.6. Let $u(x, y)$ reduce to $u(x)$ in (2.8) and with suitable modifications, then (2.8) becomes the following Agarwal and Sheng [15] inequality.

$$\int E \left| u(x) \right|^{l+m} dx \leq \frac{1}{n} \left(\frac{m+l}{2m} \right)^m \sum_{i=1}^{n} (b_i - a_i)^m \int E \left| u(x) \right|^l \left| \frac{\partial}{\partial x_i} u(x) \right|^m dx.$$

Remark 2.7. The inequality (2.8) for $u(x, y)$ reduce to $u(x)$, with the right-hand sides multiplied by m^m and $(b_i - a_i)^m$ replaced by $(\alpha\beta)^m$ has been obtained by Pachpatte [18].

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Authors' contributions
C-JZ, W-SC and MB jointly contributed to the main results Theorems 2.1, 2.2, and 2.3. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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