Zuo *Journal of Inequalities and Applications* 2012, **2012**:107
http://www.journalofinequalitiesandapplications.com/content/2012/1/107

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

The Ptolemy constant of absolute normalized norms on $\mathbb{R}^2$

Zhanfei Zuo

Correspondence: zuozhanfei@139.com
Department of Mathematics and Statistics, Chongqing Three Gorges University, Wanzhou 404000, China

Abstract

We determine and estimate the Ptolemy constant of absolute normalized norms on $\mathbb{R}^2$ by means of their corresponding continuous convex functions on [0, 1]. Moreover, the exact values were calculated in some concrete Banach spaces.
2000 Mathematics Subject Classification: 46B20.

Keywords: Ptolemy constant, absolute normalized norm, Lorentz sequence space, the Cesàro sequence space

1. Introduction and preliminaries

There are several constants defined on Banach spaces such as the Gao [1] and von Neumann-Jordan constants [2]. It has been shown that these constants are very useful in geometric theory of Banach spaces, which enable us to classify several important concepts of Banach spaces such as uniformly non-squareness and uniform normal structure [3-8]. On the other hand, calculation of the constant for some concrete spaces is also of some interest [5,6,9].

Throughout this article, we assume that X is a real Banach space. By S_X and B_X we denote the unit sphere and the unit ball of a Banach space X, respectively. The notion of the Ptolemy constant of Banach spaces was introduced in [10] and recently it has been studied by Llorens-Fuster in [9].

Definition 1.1 For a normed space $(X, ||.||)$ the real number

$$C_p(X) := \sup\left\{\frac{\|x-y\|\,\|z\|}{\|x-z\|\,\|y\|+\|z-y\|\,\|x\|} : x, y, z \in X\backslash\{0\}, x \neq y \neq z \neq x\right\}$$

is called the Ptolemy constant of $(X, ||.||)$.

As we have already mentioned [10], $1 \leq C_p(X) \leq 2$ for all normed spaces X. The Ptolemy inequality shows that $C_p(H) = 1$ whenever $(H, ||.||)$ is an inner product space. It is obvious that if Y is a subspace of $(X, ||.||)$, then $C_p(Y) \leq C_p(X)$. Since $C_p(Y) = 2$ for $Y = (\mathbb{R}^2, ||.||_\infty)$, it follows that $C_p(X) = 2$ whenever X contains an isometric copy of $(\mathbb{R}^2, ||.||_\infty)$.

Recall that a norm on $\mathbb{R}^2$ is called absolute if $||(z, w)|| = ||(|z|, |w|)||$ for all $z, w \in \mathbb{R}$ and normalized if $||(1, 0)|| = ||(0, 1)|| = 1$. Let N_α denotes the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ denotes the family of all continuous convex functions on [0, 1] such that $\psi(1) = \psi(0) = 1$ and $\max\{1 - t, t\} \leq \psi(t) \leq 1 (0 \leq t \leq 1)$. It has been shown that N_α and Ψ are a one-to-one correspondence in view of the following proposition in [11].

Proposition 1.2 If $||.|| \in N_\alpha$, then $\psi(t) = ||(1 - t, t)|| \in \Psi$. On the other hand, if $\psi(t) \in \Psi$, defining the norm $||.||_\psi$ as

$$\left\|(z,\omega)\right\|_\psi := \begin{cases} (|z|+|\omega|)\psi\left(\dfrac{|\omega|}{|z|+|\omega|}\right), & (z,\omega) \neq (0,0); \\ 0, & (z,\omega) = (0,0). \end{cases}$$

then the norm $||.||_\psi \in N_\alpha$.

A simple example of absolute normalized norm is usual $l_p(1 \le p \le \infty)$ norm. From Proposition 1.2, one can easily get the corresponding function of the l_p norm:

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p}, & 1 \le p < \infty, \\ \max\{1-t, t\}, & p = \infty. \end{cases}$$

Also, the above correspondence enable us to get many non-l_p norms on $\mathbb{R}^2$. One of the properties of these norms is stated in the following result.

Proposition 1.3 Let $\psi, \phi \in \Psi$ and $\phi \le \psi$. Put $M = \max_{0\le t\le 1}\frac{\psi(t)}{\varphi(t)}$, then

$$\|.\|_\varphi \le \|.\|_\psi \le M\|.\|_\varphi.$$

The Cesàro sequence space was defined by Shue [12]. It is very useful in the theory of matrix operators and others. Let l be the space of real sequences. For $1 < p < \infty$, the Cesàro sequence space ces_p is defined by

$$ces_p = \left\{ x \in l : \|x\| = \left\|(x(i))\right\| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x(i)\right|\right)^p\right)^{1/p} < \infty \right\}$$

The geometry of Cesàro sequence spaces have been extensively studied in [13-21]. Let us restrict ourselves to the 2D Cesàro sequence space $ces_p^{(2)}$ which is just $\mathbb{R}^2$ equipped with the norm defined by norm defined by

$$\left\|(x,y)\right\| = \left(|x|^p + \left(\frac{|x|+|y|}{2}\right)^p\right)^{1/p}$$

2. Main results

In this section, we give a simple method to determine and estimate the Ptolemy constant of absolute normalized norms on $\mathbb{R}^2$. Moreover, the exact values were calculated in some concrete Banach spaces. For a norm $||.||$ on $\mathbb{R}^2$, we write $C_p(||.||)$ for $C_p(\mathbb{R}^2,||.||)$.

Proposition 2.1 Let $\phi \in \Psi$ and $\psi(t) = \phi(1 - t)$. Then $C_p(||.||_\phi) = C_p(||.||_\psi)$

Proof. For any $x = (a, b) \in \mathbb{R}^2$ and $a \neq 0$, $b \neq 0$, put $\tilde{x} = (b, a)$. Then

$$\|x\|_\varphi = (|a|+|b|)\varphi\left(\frac{|b|}{|a|+|b|}\right) = (|b|+|a|)\psi\left(\frac{|a|}{|a|+|b|}\right) = \left\|\tilde{x}\right\|_\psi.$$

Consequently, we have

$$\begin{aligned} C_p(\|.\|_\varphi) &= \sup\left\{\frac{\|x-y\|_\varphi\|z\|_\varphi}{\|x-z\|_\varphi\|y\|_\varphi+\|z-y\|_\varphi\|x\|_\varphi} : x,y,z\in X\backslash\{0\}, x\neq y\neq z\neq x\right\} \\ &= \sup\left\{\frac{\|\tilde{x}-\tilde{y}\|_\psi\|\tilde{z}\|_\psi}{\|\tilde{x}-\tilde{z}\|_\psi\|\tilde{y}\|_\psi+\|\tilde{z}-\tilde{y}\|_\psi\|\tilde{x}\|_\psi} : \tilde{x},\tilde{y},\tilde{z}\in X\backslash\{0\}, \tilde{x}\neq\tilde{y}\neq\tilde{z}\neq\tilde{x}\right\} \\ &= C_p(\|.\|_\psi). \end{aligned}$$

We now consider the Ptolemy constant of a class of absolute normalized norms on $\mathbb{R}^2$. Now let us put

$$M_1 = \max_{0\leq t\leq 1}\frac{\psi_2(t)}{\psi(t)} \text{ and } M_2 = \max_{0\leq t\leq 1}\frac{\psi(t)}{\psi_2(t)}$$

Theorem 2.2 Let $\psi \in \Psi$ and $\psi \leq \psi_2$, if the function $\frac{\psi_2(t)}{\psi(t)}$ attains its maximum at t = 1/2, then

$$C_p(\|.\|_\psi) = \frac{1}{2\psi^2(1/2)}.$$

Proof. By Proposition 1.3, we have $||.||_\psi \leq ||.||_2 \leq M_1||.||_\psi$. Let $x, y \in X$, $(x, y) \neq (0, 0)$, where $X = \mathbb{R}^2$. Then

$$\begin{aligned} \frac{\|x-y\|_\psi\|z\|_\psi}{\|x-z\|_\psi\|y\|_\psi+\|z-y\|_\psi\|x\|_\psi} &\leq \frac{\|x-y\|_2\|z\|_2}{(1/M_1^2)\|x-z\|_2\|y\|_2+\|z-y\|_2\|x\|_2} \\ &\leq M_1^2 C_p(\|.\|_2) \\ &\leq M_1^2 \end{aligned}$$

from the definition of $C_p(X)$, implies that

$$C_p(\|.\|_\psi) \leq M_1^2 = \max_{0\leq t\leq 1}\frac{\psi_2^2(t)}{\psi^2(t)}. \tag{1}$$

On the other hand, note that the function $\frac{\psi_2(t)}{\psi(t)}$ attains its maximum at t = 1/2, i.e., $M_1 = \frac{\psi_2(1/2)}{\psi(1/2)}$. Let us put x = (1/2, 1/2), y = (1/2, -1/2), z = (1, 0), then

$$\begin{aligned} \frac{\|x-y\|_\psi\|z\|_\psi}{\|x-z\|_\psi\|y\|_\psi+\|z-y\|_\psi\|x\|_\psi} &= \frac{\|(0,1)\|_\psi\|(1,0)\|_\psi}{\|(-1/2,1/2)\|_\psi\|(1/2,-1/2)\|_\psi+\|(1/2,1/2)\|_\psi\|(1/2,1/2)\|_\psi} \\ &= \frac{1}{2\psi^2(1/2)} \\ &= \frac{2\times 1/2\times(1-1/2)}{\psi^2(1/2)} = \frac{(1/2)^2+(1-1/2)^2}{\psi^2(1/2)} \\ &= \frac{\psi_2^2(1/2)}{\psi^2(1/2)} = M_1^2. \end{aligned}$$

From (1) and the above equality, we have

$$C_p(\|.\|_\psi) = M_1^2 = \frac{1}{2\psi^2(1/2)}.$$

Theorem 2.3 Let $\psi \in \Psi$ and $\psi \geq \psi_2$, if the function $\frac{\psi(t)}{\psi_2(t)}$ attains its maximum at t = 1/2, then

$$C_p(\|.\|_\psi) = 2\psi^2(1/2).$$

Proof. By Proposition 1.3, we have $||.||_2 \le ||.||_\psi \le M_2||.||_2$. Let $x, y \in X$, $(x, y) \neq (0, 0)$, where $X = \mathbb{R}^2$. Then

$$\begin{aligned}\frac{\|x-y\|_\psi\|z\|_\psi}{\|x-z\|_\psi\|y\|_\psi+\|z-y\|_\psi\|x\|_\psi} &\le \frac{M_2^2\|x-y\|_2\|z\|_2}{\|x-z\|_2\|y\|_2+\|z-y\|_2\|x\|_2}\\ &\le M_2^2 C_p(\|.\|_2)\\ &\le M_2^2\end{aligned}$$

from the definition of $C_p(X)$, implies that

$$C_p(\|.\|_\psi) \le M_2^2 = \max_{0\le t\le 1}\frac{\psi^2(t)}{\psi_2^2(t)}. \tag{2}$$

On the other hand, note that the function $\frac{\psi(t)}{\psi_2(t)}$ attains its maximum at $t = 1/2$, i.e., $M_2 = \frac{\psi(1/2)}{\psi_2(1/2)}$. Let us put $x = (1/2, 0)$, $y = (0, 1/2)$, $z = (1/2, 1/2)$, then

$$\begin{aligned}\frac{\|x-y\|_\psi\|z\|_\psi}{\|x-z\|_\psi\|y\|_\psi+\|z-y\|_\psi\|x\|_\psi} &= \frac{\|(1/2,-1/2)\|_\psi\|(1/2,1/2)\|_\psi}{\|(0,-1/2)\|_\psi\|(0,1/2)\|_\psi+\|(1/2,0)\|_\psi\|(1/2,0)\|_\psi}\\ &= 2\psi^2(1/2)\\ &= \frac{\psi^2(1/2)}{(1/2)^2+(1-1/2)^2}\\ &= \frac{\psi^2(1/2)}{\psi_2^2(1/2)} = M_2^2.\end{aligned}$$

From (2) and the above equality, we have

$$C_p(\|.\|_\psi) = M_2^2 = 2\psi^2(1/2).$$

Theorem 2.4 If X is the l_p $(1 \le p \le \infty)$ space, then

$$C_p(\|.\|_p) = \max\{2^{2/p-1}, 2^{2/q-1}\}.$$

In particular, $C_p(||.||_1) = C_p(||.||_\infty) = 2$.

Proof. Let $1 \le p \le 2$, then we have $\psi_p(t) \ge \psi_2(t)$ and $\psi_p(t)/\psi_2(t)$ attains s maximum at $t = 1/2$. Since

$$\psi_2(t) \le \psi_p(t) \le 2^{1/p-1/2}\psi_2(t)\ (0 \le t \le 1),$$

where the constant $2^{1/p-1/2}$ is the best possible. On the other hand, for $t = 1/2$, we have

$$\frac{\psi_p(1/2)}{\psi_2(1/2)} = \frac{((1-1/2)^p+(1/2)^p)^{1/p}}{((1-1/2)^2+(1/2)^2)^{1/2}} = 2^{1/p-1/2}$$

Therefore, by Theorem 2.3, we have

$$C_p(\|.\|_p) = 2\psi_p^2(1/2) = 2^{2/p-1}. \tag{3}$$

Similarly, for $2 < p < \infty$, then we have $1 < q < 2$ and $\psi_p(t) \le \psi_2(t)$. By Theorem 2.2, we have

$$C_p(\|.\|_p) = \frac{1}{2\psi_p^2(1/2)} = 2^{2/q-1}. \tag{4}$$

From (3) and (4), we have

$$C_p(\|.\|_p) = \max\{2^{2/p-1}, 2^{2/q-1}\}.$$

Lemma 2.5 Let $||.||$ and $|.|$ be two equivalent norms on a Banach space. If $a|.| \le ||.|| \le b|.|(b \ge a > 0)$, then

$$\frac{a^2 C_p(|.|)}{b^2} \le C_p(\|.\|) \le \frac{b^2 C_p(|.|)}{a^2}$$

Moreover, if $||x|| = a|x|$, then $C_p(||.||) = C_p(|.|)$.
Proof. From the definition of $C_p(X)$, we have

$$\begin{aligned}
C_p(\|.\|) &= \sup\left\{\frac{\|x-y\|\ \|z\|}{\|x-z\|\ \|y\| + \|z-y\|\ \|x\|} : x, y, z \in X\backslash\{0\}, x \neq y \neq z \neq x\right\} \\
&\le \sup\left\{\frac{b^2\ |x-y|\ |z|}{a^2\ |x-z|\ |y| + |z-y|\ |x|} : x, y, z \in X\backslash\{0\}, x \neq y \neq z \neq x\right\} \\
&= \frac{b^2}{a^2}\sup\left\{\frac{|x-y|\ |z|}{|x-z|\ |y| + |z-y|\ |x|} : x, y, z \in X\backslash\{0\}, x \neq y \neq z \neq x\right\} \\
&\le \frac{b^2}{a^2} C_p(|.|).
\end{aligned}$$

Similarly, we also have

$$\frac{a^2 C_p(|.|)}{b^2} \le C_p(\|.\|).$$

Example 2.6 Let $X = \mathbb{R}^2$ with the norm

$$\|x\| = \max\{\|x\|_2, \lambda\|x\|_1\}\ (1/\sqrt{2} \le \lambda \le 1).$$

Then

$$C_p(\|.\|) = 2\lambda^2.$$

Proof. It is very easy to check that $||x|| = \max\{||x||_2, \lambda||x||_1\} \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(t) = \|(1-t, t)\| = \max\{\psi_2(t), \lambda\} \ge \psi_2(t).$$

Therefore

$$\frac{\psi(t)}{\psi_2(t)} = \max\{1, \frac{\lambda}{\psi_2(t)}\}.$$

Since $\psi_2(t)$ attains minimum at $t = 1/2$ and hence $\frac{\psi(t)}{\psi_2(t)}$ attains maximum at $t = 1/2$. Therefore, from Theorem 2.3, we have

$$C_p(\|.\|) = 2\psi^2(1/2) = 2\lambda^2$$

Example 2.7 Let $X = \mathbb{R}^2$ with the norm

$$\|x\| = \max\{\|x\|_2, \lambda\|x\|_\infty\}\ (1 \le \lambda \le \sqrt{2}).$$

Then

$$C_p(\|.\|) = \lambda^2.$$

Proof. It is obvious to check that the norm $||x|| = \max\{||x||_2, \lambda||x||_\infty\}$ is absolute, but not normalized, since $||(1, 0)|| = ||(0, 1)|| = \lambda$. Let us put

$$|.| = \frac{\|.\|}{\lambda} = \max\left\{\frac{\|.\|_2}{\lambda}, \|.\|_\infty\right\}$$

Then $|.| \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(t) = \left\|(1-t,t)\right\| = \max\left\{\frac{\psi_2(t)}{\lambda}, \psi_\infty(t)\right\} \le \psi_2(t).$$

Thus

$$\frac{\psi_2(t)}{\psi(t)} = \min\left\{\lambda, \frac{\psi_2(t)}{\psi_\infty(t)}\right\}.$$

Consider the increasing continuous function $g(t) = \frac{\psi_2(t)}{\psi(t)} (0 \le t \le 1/2)$. Because $g(0) = 1$ and $g(1/2) = \sqrt{2}$, hence, there exists a unique $0 \le a \le 1$ such that $g(a) = \lambda$. In fact $g(t)$ is symmetric with respect to $t = 1/2$, then we have

$$g(t) = \begin{cases} \frac{\psi_2(t)}{\psi(t)}, & t \in [0,a] \cup [1-a,a]; \\ \lambda, & t \in [a, 1-a] \end{cases}$$

Obvious, $g(t)$ attains its maximum at $t = 1/2$. Hence, from Theorem 2.2 and Lemma 2.5, we have

$$C_p(\|.\|) = C_p(|.|) = \frac{1}{2\psi^2(1/2)} = \lambda^2.$$

Example 2.8 Let $X = \mathbb{R}^2$ with the norm

$$\|x\| = \left(\|x\|_2^2 + \lambda\,\|x\|_\infty^2\right)\ (\lambda \ge 0)$$

Then

$$C_p(\|.\|) = 2(1+\lambda)/\lambda + 2.$$

Proof. It is obvious to check that the norm $\|x\| = \left(\|x\|_2^2 + \lambda\,\|x\|_\infty^2\right)$ is absolute, but not normalized, since $||(1, 0)|| = ||(0, 1)|| = (1 + \lambda)^{1/2}$. Let us put

$$|.| = \frac{\|.\|}{\sqrt{1+\lambda}}.$$

Therefore $|.| \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(t) = \left\|(1-t,t)\right\| = \begin{cases} \left[(1-t)^2 + t^2/(1+\lambda)\right]^{1/2}, & t \in [0, 1/2], \\ \left[t^2 + (1-t)^2/(1+\lambda)\right]^{1/2}, & t \in [1/2, 1]. \end{cases}$$

Obvious $\psi(t) \leq \psi_2(t)$. Since $\lambda \geq 0$, $\frac{\psi_2(t)}{\psi(t)}$ is symmetric with respect to $t = 1/2$, it suffices to consider $\frac{\psi_2(t)}{\psi(t)}$ for $t \in [0, 1/2]$. Note that, for any $t \in [0, 1/2]$, put $g(t) = \frac{\psi_2(t)^2}{\psi(t)^2}$. Taking derivative of the function $g(t)$, then we have

$$g'(t) = \frac{2\lambda}{1+\lambda} \times \frac{t(1-t)}{[(1-t)^2 + t^2/(1+\lambda)]^2}.$$

We always have $g'(t) \geq 0$ for $0 \leq t \leq 1/2$, this implies that the function $g(t)$ is increased for $0 \leq t \leq 1/2$. Therefore, the function $\frac{\psi_2(t)}{\psi(t)}$ attains its maximum at $t = 1/2$, by Theorem 2.2 and Lemma 2.5, we have

$$C_p(\|.\|) = C_p(|.|) = \frac{1}{2\psi^2(1/2)} = 2(1+\lambda)/\lambda + 2.$$

Example 2.9 (Lorentz sequence spaces) Let $0 < a < 1$. Two-dimensional Lorentz sequence space, i.e., $\mathbb{R}^2$ with the norm

$$\|(z,\omega)\|_{a,2} = ((x_1^*)^2 + a(x_2^*)^2)^{1/2},$$

where (x_1^*, x_2^*) is the rearrangement of $(|z|, |\omega|)$ satisfying $(x_1^* \geq x_2^*)$, then

$$C_p(\|(z,\omega)\|_{a,2}) = \frac{2}{a+1}.$$

Proof. Indeed, $||(z, \omega)||_{a,2} \in \mathbb{N}_\alpha$, and the corresponding convex function is given by

$$\psi_{a,2}(t) = \|(1-t,t)\|_{a,2} = \begin{cases} [(1-t)^2 + at^2]^{1/2}, & t \in [0,1/2], \\ [t^2 + a(1-t)^2]^{1/2}, & t \in [1/2,1]. \end{cases}$$

Obvious $\psi_{a,2}(t) \leq \psi_2(t)$. Repeating the arguments in the proof of Example 2.8, we can easily get the conclusion that $\frac{\psi_2(t)}{\psi_{a,2}(t)}$ attains its maximum at $t = 1/2$. By Theorem 2.2, we have

$$C_p(\|(z,\omega)\|_{a,2}) = \frac{2}{a+1}.$$

Example 2.10 Let X be a 2D Cesàro space $ces_2^{(2)}$, then

$$C_p(ces_2^{(2)}) = 1 + \frac{1}{\sqrt{5}}.$$

Proof. We first define

$$|x,y| = \left\|\left(\frac{2x}{\sqrt{5}}, 2y\right)\right\|_{ces_2^{(2)}}$$

for $(x, y) \in \mathbb{R}^2$. It follows that $ces_2^{(2)}$ is isometrically isomorphic to $(\mathbb{R}^2,|.|)$ and $|.|$ is absolute and normalized norm, and the corresponding convex function is given by

$$\psi(t) = \left[\frac{4(1-t)^2}{5} + \left(\frac{1-t}{\sqrt{5}} + t\right)^2\right]^{\frac{1}{2}}$$

Indeed, $T : ces_2^{(2)} \to (\mathbb{R}^2, |.|)$ defined by $T(x, y) = \left(\frac{x}{\sqrt{5}}, 2y\right)$ is an isometric isomorphism. We prove that $\psi(t) \geq \psi_2(t)$. Note that

$$\left(\frac{1-t}{\sqrt{5}} + t\right)^2 \geq \left(\frac{1-t}{\sqrt{5}}\right)^2 + t^2$$

Consequently,

$$\psi(t) \geq ((1-t)^2 + t^2)^{1/2} = \psi_2(t)$$

Some elementary computation shows that $\frac{\psi(t)}{\psi_2(t)}$ attains its maximum at $t = 1/2$. Therefore, from Theorem 2.3, we have

$$C_p(ces_2^{(2)}) = 2\psi^2(1/2) = 1 + \frac{1}{\sqrt{5}}.$$

Acknowledgements
This research was supported by the fund of Scientific research in Southeast University (the support project of fundamental research) and NSF of CHINA, Grant No. 11126329.

Competing interests
The author declares that they have no competing interests.

Received: 19 October 2011 Accepted: 17 May 2012 Published: 17 May 2012

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doi:10.1186/1029-242X-2012-107
Cite this article as: Zuo: The Ptolemy constant of absolute normalized norms on $\mathbb{R}^2$. *Journal of Inequalities and Applications* 2012 **2012**:107.