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Refinement of an integral inequality

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Abstract

In this study, we generalize and sharpen an integral inequality raised in theory for convex and star-shaped sets and relax the conditions on the integrand.
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1 Introduction

In the study [1], which investigated convex and star-shaped sets, the following interesting result was obtained.

Theorem 1. *([1, Lemma 2.1]) Let $p : [0, T\,] \to \mathbb{R}$ be a nonnegative convex function such that $p(0) = 0$. Then for $0 < a \le b \le T$ and $k \in \mathbb{N}^+$ the inequality*

$$\int_0^b t^k p(t)dt \ge \left(\frac{b}{a}\right)^{k+2} \int_0^a t^k p(t)dt \tag{1}$$

holds.

In this note, we shall show that the convexity of the function $p(t)$ may be replaced by the condition that $\frac{p(t)}{t}$ is increasing, sharpen inequality (1), and obtain the following a general result using a monotone form of l'Hospital's rule, a elementary method, and Mitrinović-Pečarić inequality, respectively.

Theorem 2. *Let $p : [0, T\,] \to \mathbb{R}$ be a nonnegative continuous function such that $p(0) = 0$ and $\frac{p(t)}{t}$be a monotone function on $(0, T\,]$. Let $A = \lim_{x\to 0^+} \frac{p(x)}{x}$.Then for $0 < x \le b \le T$ and $k \ge 0$ the double inequality*

$$\alpha \le \left(\frac{b}{x}\right)^{k+2} \int_0^x t^k p(t)dt \le \beta \tag{2}$$

holds so that

(i) when $\frac{p(t)}{t}$is increasing, we have$\alpha = \frac{b^{k+2}A}{k+2}$, $\beta = \int_0^b t^k p(t)dt$;

(ii) when $\frac{p(t)}{t}$is decreasing, we have $\alpha = \int_0^b t^k p(t)dt$, $\beta = \frac{b^{k+2}A}{k+2}$.

Furthermore, these paired numbers α *and* β *defined in (i) and (ii) are the best constants in (2).*

2 Two lemmas

Lemma 1. *([2-5], A Monotone form of L'Hospital's rule) Let f, g* : $[a, b] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (a, b). *Further, let* $g' \neq 0$ *on* (a, b). *If* f'/g'*is increasing (or decreasing) on* (a, b), *then the functions* $\frac{f(x)-f(b)}{g(x)-g(b)}$ *and* $\frac{f(x)-f(a)}{g(x)-g(a)}$ *are also increasing (or decreasing) on (a, b).*

Lemma 2. *([6], Mitrinović-Pečarić inequality]) If f is increasing function and p satisfies the conditions* $0 \leq \int_a^x p(t)dt \leq \int_a^b p(t)dt$*for* $x \in [a, b]$, *and for some* $c \in [a, b]$, $\int_c^b p(t)dt > 0$, $\int_c^b p(t)dt > 0$, *then we have*

$$\frac{\int_a^c p(t)f(t)dt}{\int_a^c p(t)dt} \leq \frac{\int_a^b p(t)f(t)dt}{\int_a^b p(t)dt} \leq \frac{\int_c^b p(t)f(t)dt}{\int_c^b p(t)dt}. \tag{3}$$

If f is decreasing the inequalities (3) are reversed.

3 A concise proof of Theorem 2

Let $H(t) = \frac{\int_0^t b^{k+1}s^k p(bs)ds}{t^{k+2}} = \frac{f_1(t)}{g_1(t)}$, where $f_1(t) = \int_0^t b^{k+1}s^k p(bs)ds$, $g_1(t) = t^{k+2}$, and $0 < t \leq 1$. Then $\frac{f_1'(t)}{g_1'(t)} = \frac{b^{k+1}p(bt)}{(k+2)t}$.

(a) When $\frac{p(t)}{t}$ is increasing, we have $\frac{f_1'(t)}{g_1'(t)}$ is also increasing, and $H(t) = \frac{f_1(t)}{g_1(t)} = \frac{f_1(t)-f_1(0)}{g_1(t)-g_1(0)}$ is increasing by Lemma 1. At the same time, $\lim_{t\to 0^+} H(t) = \lim_{t\to 0^+} \frac{b^{k+1}p(bt)}{(k+2)t} = \frac{b^{k+2}A}{k+2}$, and $\lim_{t\to 1} H(t) = \int_0^1 b^{k+1}t^k p(bt)dt = \int_0^b u^k p(u)du$. So we obtain

$$\frac{b^{k+2}A}{k+2} \leq \frac{\int_0^t b^{k+1}s^k p(bs)ds}{t^{k+2}} \leq \int_0^b u^k p(u)du, \tag{4}$$

$\frac{b^{k+2}A}{k+2}$ and $\int_0^b u^k p(u)du$ are the best constants in (4).

Replacing t with x/b in (4), we have $\int_0^t b^{k+1}s^k p(bs)ds = \int_0^{\frac{x}{b}} b^{k+1}s^k p(bs)ds$. Then let $bs = u$, we obtain $\int_0^{\frac{x}{b}} b^{k+1}s^k p(bs)ds = \int_0^x u^k p(u)du$, and

$$\frac{b^{k+2}A}{k+2} \leq \left(\frac{b}{x}\right)^{k+2} \int_0^x t^k p(t)dt \leq \int_0^b u^k p(u)du \tag{5}$$

holds. Furthermore $\alpha = \frac{b^{k+2}A}{k+2}$ and $\beta = \int_0^b u^k p(u)du$ are the best constants in (5).

(b) When $\frac{p(t)}{t}$ is decreasing, we obtain corresponding result by the same way.

4 New elementary proof of Theorem 2

Let $F(x) = \frac{\int_0^x t^k p(t)dt}{x^{k+2}}$ for $x \in$ (0, b]. Assume that $\frac{p(t)}{t}$ is increasing. By a simple calculation and the inequality $p(t) \le t\frac{p(x)}{x}$ for $0 < t \le x \le b$ we have that

$$xF'(x) = \frac{p(x)}{x} - (k+2)\frac{\int_0^x t^k p(t)dt}{x^{k+2}} \ge \frac{p(x)}{x} - (k+2)\frac{\int_0^x t^k t\frac{p(x)}{x}dt}{x^{k+2}} = 0.$$

So $F(x)$ is increasing and the chain inequality

$$\begin{aligned} \alpha &= \frac{b^{k+2}}{k+2}\lim_{x\to 0^+}\frac{p(x)}{x} = \lim_{x\to 0^+} b^{k+2}F(x) \le \inf_{x\in(0,b]} b^{k+2}F(x) \\ &\le \left(\frac{b}{x}\right)^{k+2}\int_0^x t^k p(t)dt \\ &\le \sup_{x\in(0,b]} b^{k+2}F(x) = \lim_{x\to b} b^{k+2}F(x) = \int_0^b t^k p(t)dt = \beta \end{aligned} \tag{6}$$

holds. Then the double inequality (5) holds, α and β are the best constants in (6) or (2).

The decreasing case can be proved similarly.

5 Other proof of Theorem 2

In what follows, we also assume that $\frac{p(t)}{t}$ is increasing.

Let $p(t) = t^{k+1}, f(t) = \frac{p(t)}{t}, c = x$, and $a = 0$ in Lemma 2, we can obtain

$$\frac{\int_0^x t^k p(t)dt}{\int_0^x t^{k+1}dt} \le \frac{\int_0^b t^k p(t)dt}{\int_0^b t^{k+1}dt} \le \frac{\int_x^b t^k p(t)dt}{\int_x^b t^{k+1}dt}. \tag{7}$$

(i) The left-side inequality of (7) deduces

$$\left(\frac{b}{x}\right)^{k+2}\int_0^x t^k p(t)dt \le \int_0^b t^k p(t)dt,$$

then the right-side inequality of (2) holds.

(ii) Let $b \to 0^+$ in the right-side inequality of (7), we can obtain

$$\lim_{b\to 0^+}\frac{\int_0^b t^k p(t)dt}{\int_0^b t^{k+1}dt} = \lim_{b\to 0^+}\frac{p(b)}{b} \le \frac{\int_x^0 t^k p(t)dt}{\int_x^0 t^{k+1}dt} = \frac{\int_0^x t^k p(t)dt}{\int_0^x t^{k+1}dt},$$

then the left-side inequality of (2) holds.

Let $G(x) = (\frac{b}{x})^{k+2}\int_0^x t^k p(t)dt$. Since $\lim_{x\to 0+} G(x) = \alpha$ and $G(b) = \beta$, we obtain that is α and β are the best constants in (2).

Competing interests
The author declares that they have no competing interests.

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