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Convergence of iterative sequences for fixed points of an infinite family of nonexpansive mappings based on a hybrid steepest descent methods

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Abstract

The propose of this article is to consider the strong convergence of an iterative sequences for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of the variational inequalities for inverse strongly monotone mappings, and the set of solutions of system of equilibrium problems in Hilbert spaces by using a hybrid steepest descent methods. Our results improve and generalize many known corresponding results.

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1. Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $||\cdot||$, respectively. Let C be a nonempty closed convex subset of H and let $F: C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F: C \times C \to \mathbb{R}$ is to find $x^* \in C$ such that

$$F(x^*, \gamma) \geq 0, \quad \forall \gamma \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$.

Let $\{F_i, i = 1, 2, ..., N\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\{F_1, F_2, ..., F_N\}$ is to find a common element $x^* \in C$ such that

$$\begin{cases} F_1(x^*, \gamma) \geq 0, & \forall \gamma \in C, \\ F_2(x^*, \gamma) \geq 0, & \forall \gamma \in C, \\ \quad \vdots \\ F_N(x^*, \gamma) \geq 0, & \forall \gamma \in C. \end{cases} \tag{1.2}$$

We denote the set of solutions of (1.2) by $\cap_{i=1}^{N} SEP(F_i)$, where $SEP(F_i)$ is the set of solutions to the equilibrium problems, that is,

$$F_i(x^*, \gamma) \geq 0, \quad \forall \gamma \in C. \tag{1.3}$$

If $N = 1$, then the problem (1.2) is reduced to the equilibrium problems.

If $N = 1$ and $F(x^*, y) = \langle Bx^*, y - x^* \rangle$, then the problem (1.2) is reduced to the variational inequality problems of finding $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $VI(C, B)$.

Many problems in applied sciences, such as monotone inclusion problems, saddle point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases. Some methods have been proposed to solve $VI(C, B)$, $EP(F)$, and $SEP(F_i)$; see, for example [1-22] and references therein. The above formulations (1.2) extends this formulism to such problems, covering in particular various forms of feasibility problems [23,24].

Let P_C be the metric projection of H onto the closed convex subset C. Let $S: C \to C$ be a *nonexpansive* mapping, that is, $||Sx - Sy|| \leq ||x - y||$ for all $x, y \in C$. The set of fixed points of S is denoted by $F(S) = \{x \in C: Sx = x\}$. If $C \subset H$ is nonempty, bounded, closed and convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty; see, for example, [25,26]. A mapping $f: C \to C$ is a *contraction* on C if there exists a constant $\eta \in (0,1)$ such that $||f(x) - f(y)|| \leq \eta ||x - y||$ for all $x, y \in C$.

Definition 1.1. Let $B: C \to H$ be nonlinear mappings. Then B is called

(1) *monotone* if $\langle Bx - By, x - y \rangle \geq 0$, $\forall x, y \in C$,

(2) *ξ-inverse-strongly monotone* if there exists a constant $\xi > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \xi \left\| Bx - By \right\|^2, \quad \forall x, y \in C.$$

(3) A set-valued mapping $Q: H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in Qx$ and $g \in Qy$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $Q: H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, f) \in H \times H$

$$\langle x - y, f - g \rangle \geq 0, \quad \forall (y, g) \in Graph(Q)$$

(the graph of mapping Q) implies that $f \in Qx$.

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping defined on a real Hilbert space H:

$$\min_{x \in F} \left[\frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle \right],$$

where F is the fixed point set of a nonexpansive mapping S defined on H and b is a given point in H.

A linear bounded operator A is *strong positive* if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H.$$

Marino and Xu [27] introduced a new iterative scheme by the *viscosity approximation method*:

$$x_{n+1} = \varepsilon_n \gamma f(x_n) + (1 - \varepsilon_n A) Sx_n. \tag{1.5}$$

They proved that the sequences $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality:

$$\langle \gamma fz - Az, x - z \rangle \leq 0, \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem:

$$\min_{x \in F(S)} \left[\frac{1}{2} \langle Ax, x \rangle - h(x) \right], \tag{1.6}$$

where h is a potential function for γf.

In order to find a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequalities for a ξ-inverse-strongly monotone mapping, Takahashi and Toyoda [28] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n) S P_C(x_n - \alpha_n B x_n), \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where B is a ξ-inverse-strongly monotone mapping, $\{\gamma_n\}$ is a sequence in $(0, 1)$, and $\{\alpha_n\}$ is a sequence in $(0, 2\xi)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.7) converges weakly to some $z \in F(S) \cap VI(C, B)$.

In order to find a common element of $F(S) \cap VI(C, B)$, let $S: H \to H$ be a nonexpansive mapping, Yamada [29] introduced the following iterative scheme called the *hybrid steepest descent method*:

$$x_{n+1} = S x_n - \alpha_n \mu B S x_n, \quad \forall n \geq 1, \tag{1.8}$$

where $x_1 = x \in H$, $\{\alpha_n\} \subset (0, 1)$, let $B: H \to H$ be a strongly monotone and Lipschitz continuous mapping and μ is a positive real number. He proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution of the $F(S) \cap VI(C, B)$.

Let C be a nonempty closed convex subset of H. Given $r > 0$ the operators $J_r^F : H \to C$ defined by

$$J_r^F(x) = \left\{ z \in C : F(z, \gamma) + \frac{1}{r} \langle \gamma - z, z - x \rangle \geq 0, \quad \forall \gamma \in C \right\},$$

is called the resolvent of F (see [3]). It is shown in [3] that under suitable hypotheses on F (to be stated precisely in Section 2), $J_r^F : H \to C$ is single-valued and firmly nonexpansive and satisfied

$$F(J_r^F) = EP(F), \quad \forall r > 0.$$

Using the result, in 2009, Colao et al. [10] introduced and considered an implicit iterative scheme for finding a common element of the set of solutions of the system equilibrium problems and the set of common fixed points of an infinite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{z_n\}$ recursively by

$$x_n = \varepsilon_n \gamma f(x_n) + (1 - \varepsilon_n A) W_n J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \tag{1.9}$$

where $\{\epsilon_n\}$ be a sequences in $(0, 1)$. It is proved [10] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.9) converges

strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{M} SEP(F_k))$, where z is the unique solution of the variational inequality and which is the optimality condition for the minimization problem.

In 2010, Colao and Marino [30] introduced the following explicit viscosity scheme with respect to W-mappings for an infinite family of nonexpansive mappings

$$x_{n+1} = \varepsilon_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \varepsilon_n A)W_n J_{r_n}^F x_n. \tag{1.10}$$

They prove that sequence $\{x_n\}$ and $\{J_{r_n}^F\}$ converge strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$, where z is an equilibrium point for F and is the unique solution of the variational inequality:

$$\langle \gamma f z - Az, x - z \rangle \le 0, \quad \forall x \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)} \left[\frac{1}{2} \langle Ax, x \rangle - h(x) \right],$$

where h is a potential function for γf. Recently, Chantarangsi et al. [11] introduced some iterative processes based on the viscosity hybrid steepest descent method for finding a common solutions of a generalized mixed equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem in a real Hilbert space.

In this article, motivated by above results, we introduce an iterative scheme for finding a common element of the set of solutions of system of equilibrium problems, the set of fixed points of an infinite family of nonexpansive mapping, and the set of solutions of variational inequality problems for inverse strongly monotone mapping in a real Hilbert space by using a new hybrid steepest descent methods. The results shown in this article improve and extend the recent ones announced by many others.

2. Preliminaries

Let H be a real Hilbert space, when $\{x_n\}$ is a sequence in H, we denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. Let C be nonempty closed convex subset of H. The nearest point projection $P_C: H \to C$ defined from H onto C is the function which assigns to each $x \in H$ its nearest point denoted by $P_C x$ in C. Thus, $P_C x$ is the unique point in C such that $||x - P_C x|| \le ||x - y||$, $\forall y \in C$. It easy to see that P_C is nonexpansive and

$$x^* \in VI(C, B) \Leftrightarrow x^* = P_C(x^* - \lambda B x^*), \quad \lambda > 0. \tag{2.1}$$

Lemma 2.1. [26]*Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let $\xi > 0$ and let $A: C \to H$ be ξ-inverse strongly monotone. If $0 < \varrho \le 2\xi$, then $I - \varrho B$ is a nonexpansive mapping of C into H.*

Lemma 2.2. [26]*Let H be a real Hilbert spaces, there hold the following identities:*

(i) *for each $x \in H$ and $x^* \in C$, $x^* = P_C x \Leftrightarrow \langle x - x^*, y - x^* \rangle \le 0$ for all $y \in C$;*
(ii) *$P_C: H \to C$ is nonexpansive, that is, $||P_C x - P_C y|| \le ||x - y||$ for all $x, y \in H$;*
(iii) *P_C is firmly nonexpansive, that is, $||P_C x - P_C y||^2 \le \langle P_C x - P_C y, x - y \rangle$ for all $x, y \in H$;*

(iv) $||tx + (1 - t)y||^2 = t||x||^2 + (1 - t)||y||^2 - t(1 - t)||x - y||^2$, $\forall t \in [0,1]$, *for all* $x,y \in H$;

(v) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle$.

Lemma 2.3. [31]*Each Hilbert space H satisfies Opial's condition, that is, for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,$$

hold for each $y \in H$ *with* $y \neq x$.

Lemma 2.4. [27]*Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with* $\eta \in (0,1)$, *and A be a strongly positive linear bounded operator on H with coefficient* $\bar{\gamma} > 0$. *Then, for* $0 < \gamma < \dfrac{\bar{\gamma}}{\eta}$,

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\bar{\gamma} - \eta\gamma) \|x - y\|^2, \quad \forall x, y \in H.$$

That is, A - γf is a strongly monotone with coefficient $\bar{\gamma} - \gamma\eta$.

Lemma 2.5. [27]*Assume A be a strongly positive linear bounded operator on H with coefficient* $\bar{\gamma} > 0$*and* $0 < \rho \leq ||A||^{-1}$. *Then* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Throughout this article, we assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Then, we have the following lemmas.

Lemma 2.6. [1]*Let C be a nonempty closed convex subset of H and let F be a bifunction of C × C into* $\mathbb{R}$ *satisfying (A1)-(A4). Let r >0 and x* $\in$ *H. Then, there exists z* $\in$ *C such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7. [3]*Assume that F : C × C* $\to \mathbb{R}$ *satisfies (A1)-(A4). For r > 0 and x* $\in$ *H, define a mapping* $J_r^F : H \to C$*as follows:*

$$J_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}$$

for all z $\in$ *H. Then, the following hold:*

(1) J_r^F*is single-valued;*
(2) J_r^F*is firmly nonexpansive, that is, for any x,y* $\in$ *H,*

$$\left\| J_r^F x - J_r^F y \right\|^2 \leq \langle J_r^F x - J_r^F y, x - y \rangle;$$

(3) $F(J_r^F) = EP(F)$;

(4) *EP(F) is closed and convex.*

Lemma 2.8. [32]*Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (||l_{n+1} - l_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n\to\infty} ||l_n - x_n|| = 0$.*

Lemma 2.9. [33]*Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + c_n, n \geq 0,$$

where $\{b_n\}$ is a sequence in (0,1) and $\{c_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} b_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{c_n}{b_n} \leq 0 \, or \, \sum_{n=1}^{\infty} |c_n| < \infty$,

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a family of infinitely of nonexpansive mappings of C into itself and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in [0,1]. For any $n \geq 1$, define a mapping $W_n: C \to C$ as follows:

$$
\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1 - \mu_n)I, \\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1})I, \\
&\vdots \\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1 - \mu_k)I, \\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1})I, \\
&\vdots \\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1 - \mu_2)I, \\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1 - \mu_1)I,
\end{aligned}
\tag{3.1}
$$

such a mappings W_n is nonexpansive from C to C and it is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\mu_1, \mu_2, ..., \mu_n$ (see [34]).

Lemma 3.1. [34,35]*Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...,$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then,*

(1) *for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*

(2) *the mapping W of C into itself as follows:*

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C \tag{3.2}$$

is a nonexpansive mapping satisfying $F(W) = \cap_{n=1}^{\infty} F(T_n)$, which it is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$;

(3) $F(W_n) = \cap_{n=1}^{\infty} F(T_n)$, for each $n \geq 1$;

(4) If E is any bounded subset of C, then $\lim_{n \to \infty} \sup_{x \in E} \|Wx - W_n x\| = 0$.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H, let F_k, $k \in \{1, 2, 3, \ldots, M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be an infinite family of nonexpansive mappings of C into itself and let B be ξ-inverse strongly monotone such that

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap \left(\cap_{k=1}^{M} SEP(F_k) \right) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\eta \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \dfrac{\bar{\gamma}}{\eta}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ y_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \alpha_n B x_n), \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} y_n, \\ x_{n+1} = \epsilon_n \gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \epsilon_n A)P_C(W_n u_n - \lambda_n B W_n u_n), \quad \forall n \geq 1, \end{cases} \quad (3.3)$$

where $\{W_n\}$ is the sequence generated by (3.1) and $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in (0,1) and $\{r_{k,n}\}$, $k \in \{1,2,3,\ldots, M\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

(C1) $\lim_{n \to \infty} \epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

(C3) $\{\alpha_n\}, \{\lambda_n\} \subset [e, g] \subset (0, 2\xi)$, $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} \lambda_n = 0$,

(C4) $\{\delta_n\} \subset [0, b]$, for some $b \in (0,1)$ and $\lim_{n \to \infty} |\delta_{n+1} - \delta_n| = 0$,

(C5) $\liminf_{n \to \infty} r_{k,n} > 0$ and $\lim_{n \to \infty} |r_{k,n+1} - r_{k,n}| = 0$ for each $k \in \{1, 2, 3, \ldots, M\}$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$, which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in \Theta. \quad (3.4)$$

Equivalently, we have $z = P_{\Theta}(I - A + \gamma f)(z)$.

Proof. From the restrictions on control sequence, without loss of generality, that $\epsilon_n \leq (1 - \beta_n)\|A\|^{-1}$ for all $n \geq 1$. From Lemma 2.5, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$. We will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup \{ |\langle Ax, x \rangle| : x \in H, \|x\| = 1 \}.$$

Observe that

$$\langle ((1 - \beta_n)I - \epsilon_n A)x, x \rangle = 1 - \beta_n - \epsilon_n \langle Ax, x \rangle \geq 1 - \beta_n - \epsilon_n \|A\| \geq 0,$$

this show that $(1 - \beta_n)I - \epsilon_n A$ is positive. It follows that

$$
\begin{aligned}
\|(1 - \beta_n)I - \varepsilon_n A\| &= \sup\left\{\left|\langle((1 - \beta_n)I - \varepsilon_n A)x, x\rangle\right| : x \in H, \|x\| = 1\right\} \\
&= \sup\left\{1 - \beta_n - \varepsilon_n\langle Ax, x\rangle : x \in H, \|x\| = 1\right\} \\
&\leq 1 - \beta_n - \varepsilon_n\bar{\gamma}.
\end{aligned}
$$

We divide the proof of Theorem 3.2 into seven steps.

Step 1. We show that the mapping $P_\Theta(\gamma f + (I - A))$ has a unique fixed point.

Since f be a contraction of C into itself with coefficient $\eta \in (0,1)$. Then, we have

$$
\begin{aligned}
\|P_\Theta(\gamma f + (I - A))(x) - P_\Theta(\gamma f + (I - A))(y)\| &\leq \|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\| \\
&\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\
&\leq \gamma\eta\|x - y\| + (1 - \bar{\gamma})\|x - y\| \\
&= (1 - (\bar{\gamma} - \eta\gamma))\|x - y\|, \quad \forall x, y \in C.
\end{aligned}
$$

Since $0 < 1 - (\bar{\gamma} - \eta\gamma) < 1$, it follows that $P_\Theta(\gamma f + (I - A))$ is a contraction of C into itself. Therefore, by the Banach Contraction Mapping Principle, has a unique fixed point, say $z \in C$, that is,

$$
z = P_\Theta(\gamma f + (I - A))(z).
$$

Step 2. We show that $W_n - \lambda_n B W_n$ is nonexpansive.

For all $x, y \in C$, let W_n is the sequence defined by (3.1) and $\lambda_n \in (0, 2\xi)$, we obtain $W_n - \lambda_n B W_n$ is a nonexpansive. Indeed,

$$
\begin{aligned}
&\|(W_n - \lambda_n B W_n)x - (W_n - \lambda_n B W_n)y\|^2 \\
&= \|(W_n x - W_n y) - \lambda_n(B W_n x - B W_n y)\|^2 \\
&= \|W_n x - W_n y\|^2 - 2\lambda_n\langle W_n x - W_n y, B W_n x - B W_n y\rangle + \lambda_n^2\|B W_n x - B W_n y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_n\xi\|B W_n x - B W_n y\| + \lambda_n^2\|B W_n x - B W_n y\|^2 \\
&= \|x - y\|^2 - \lambda_n(\lambda_n - 2\xi)\|B W_n x - B W_n y\|^2 \\
&\leq \|x - y\|^2,
\end{aligned}
\tag{3.5}
$$

which implies that $W_n - \lambda_n B W_n$ is a nonexpansive.

Step 3. We show that the sequence $\{x_n\}$ is bounded.

In fact, let $\tilde{x} \in \Theta$, then

$$
\tilde{x} = P_C(\tilde{x} - \alpha_n B\tilde{x}).
$$

Setting $v_n = P_C(x_n - \alpha_n B x_n)$ and $I - \alpha_n B$ is a nonexpansive mapping (Lemma 2.1), we obtain

$$
\begin{aligned}
\|v_n - \tilde{x}\| &= \|P_C(x_n - \alpha_n B x_n) - P_C(\tilde{x} - \alpha_n B\tilde{x})\| \\
&\leq \|(x_n - \alpha_n B x_n) - (\tilde{x} - \alpha_n B\tilde{x})\| \\
&= \|(I - \alpha_n B)x_n - (I - \alpha_n B)\tilde{x}\| \\
&\leq \|x_n - \tilde{x}\|
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
\left\| y_n - \tilde{x} \right\| &\le (1 - \delta_n) \left\| x_n - \tilde{x} \right\| + \delta_n \left\| v_n - \tilde{x} \right\| \\
&\le (1 - \delta_n) \left\| x_n - \tilde{x} \right\| + \delta_n \left\| x_n - \tilde{x} \right\| \\
&= \left\| x_n - \tilde{x} \right\|.
\end{aligned}
\tag{3.7}
$$

Let $\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$ for $k \in \{1, 2, 3, \dots, M\}$ and $\Im_n^0 = I$ for all n. Because $J_{r_{k,n}}^{F_k}$ is nonexpansive for each $k = 1, 2, 3, \dots, M$, $\tilde{x} = \Im_n^k \tilde{x}$ and (3.7), we note that $u_n = \Im_n^M y_n$. It follows that

$$
\left\| u_n - \tilde{x} \right\| = \left\| \Im_n^M y_n - \Im_n^M \tilde{x} \right\| \le \left\| y_n - \tilde{x} \right\| \le \left\| x_n - \tilde{x} \right\|.
\tag{3.8}
$$

Let $e_n = P_C(W_n u_n - \lambda_n B W_n u_n)$, we can prove that

$$
\begin{aligned}
\left\| e_n - \tilde{x} \right\| &= \left\| P_C(W_n u_n - \lambda_n B W_n u_n) - P_C(W_n \tilde{x} - \lambda_n B W_n \tilde{x}) \right\| \\
&\le \left\| (W_n u_n - \lambda_n B W_n u_n) - (W_n \tilde{x} - \lambda_n B W_n \tilde{x}) \right\| \\
&= \left\| (W_n - \lambda_n B W_n) u_n - (W_n - \lambda_n B W_n) \tilde{x} \right\| \\
&\le \left\| u_n - \tilde{x} \right\| \le \left\| x_n - \tilde{x} \right\|,
\end{aligned}
\tag{3.9}
$$

which yields that

$$
\begin{aligned}
\left\| x_{n+1} - \tilde{x} \right\| &= \left\| \varepsilon_n (\gamma f(u_n) - A\tilde{x}) + \beta_n (x_n - \tilde{x}) + ((1 - \beta_n)I - \varepsilon_n A)(e_n - \tilde{x}) \right\| \\
&\le \varepsilon_n \left\| \gamma f(u_n) - A\tilde{x} \right\| + \beta_n \left\| x_n - \tilde{x} \right\| + \left\| (1 - \beta_n)I - \varepsilon_n A \right\| \left\| e_n - \tilde{x} \right\| \\
&\le \varepsilon_n \gamma \left\| f(u_n) - f(\tilde{x}) \right\| + \varepsilon_n \left\| \gamma f(\tilde{x}) - A\tilde{x} \right\| + \beta_n \left\| x_n - \tilde{x} \right\| + (1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\| e_n - \tilde{x} \right\| \\
&\le \varepsilon_n \gamma \eta \left\| u_n - \tilde{x} \right\| + \varepsilon_n \left\| \gamma f(\tilde{x}) - A\tilde{x} \right\| + \beta_n \left\| x_n - \tilde{x} \right\| + (1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\| x_n - \tilde{x} \right\| \\
&\le \varepsilon_n \gamma \eta \left\| x_n - \tilde{x} \right\| + \varepsilon_n \left\| \gamma f(\tilde{x}) - A\tilde{x} \right\| + \beta_n \left\| x_n - \tilde{x} \right\| + (1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\| x_n - \tilde{x} \right\| \\
&= (1 - (\bar{\gamma} - \gamma \eta)\varepsilon_n) \left\| x_n - \tilde{x} \right\| + \frac{(\bar{\gamma} - \gamma \eta)\varepsilon_n}{(\bar{\gamma} - \gamma \eta)} \left\| \gamma f(\tilde{x}) - A\tilde{x} \right\|.
\end{aligned}
$$

By induction, we have

$$
\left\| x_n - \tilde{x} \right\| \le \max \left\{ \left\| x_1 - \tilde{x} \right\|, \frac{\left\| \gamma f(\tilde{x}) - A\tilde{x} \right\|}{\bar{\gamma} - \gamma \eta} \right\}, \quad \forall n \in \mathbb{N}.
\tag{3.10}
$$

This implies that $\{x_n\}$ is bounded, and hence so are $\{u_n\}$, $\{e_n\}$, $\{y_n\}$, $\{BW_n u_n\}$, $\{Bx_n\}$, $\{Ae_n\}$, $\{v_n - x_n\}$, and $\{f(u_n)\}$.

Step 4. We show that $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$.

We claim that if ω_n be a bounded sequence in C, then

$$
\lim_{n \to \infty} \left\| \Im_n^k \omega_n - \Im_{n+1}^k \omega_n \right\| = 0,
\tag{3.11}
$$

for every $k \in \{1, 2, 3, \dots, M\}$. From Step 2 of the proof of Theorem 3.1 in [10], we have that for $k \in \{1, 2, 3, \dots, M\}$,

$$
\lim_{n \to \infty} \left\| J_{r_{k,n+1}}^{F_k} \omega_n - J_{r_{k,n}}^{F_k} \omega_n \right\| = 0.
\tag{3.12}
$$

Note that for every $k \in \{1,2,3,...,M\}$, we obtain

$$\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \dots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} = J_{r_{k,n}}^{F_k} \Im_n^{k-1}.$$

Thus,

$$
\begin{aligned}
&\left\| \Im_n^k \omega_n - \Im_{n+1}^k \omega_n \right\| \\
&= \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} \omega_n - J_{r_{k,n+1}}^{F_k} \Im_{n+1}^{k-1} \omega_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} \omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} \omega_n \right\| + \left\| J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} \omega_n - J_{r_{k,n+1}}^{F_k} \Im_{n+1}^{k-1} \omega_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} \omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} \omega_n \right\| + \left\| \Im_n^{k-1} \omega_n - \Im_{n+1}^{k-1} \omega_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} \omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} \omega_n \right\| + \left\| J_{r_{k-1,n}}^{F_{k-1}} \Im_n^{k-2} \omega_n - J_{r_{k-1,n+1}}^{F_{k-1}} \Im_n^{k-2} \omega_n \right\| \\
&\quad + \left\| \Im_n^{k-2} \omega_n - \Im_{n+1}^{k-2} \omega_n \right\| \\
&\leq \left\| J_{r_{k,n}}^{F_k} \Im_n^{k-1} \omega_n - J_{r_{k,n+1}}^{F_k} \Im_n^{k-1} \omega_n \right\| + \left\| J_{r_{k-1,n}}^{F_{k-1}} \Im_n^{k-2} \omega_n - J_{r_{k-1,n+1}}^{F_{k-1}} \Im_n^{k-2} \omega_n \right\| \\
&\quad + \dots + \left\| J_{r_{2,n}}^{F_2} \Im_n^1 \omega_n - J_{r_{2,n+1}}^{F_2} \Im_n^1 \omega_n \right\| + \left\| J_{r_{1,n}}^{F_1} \omega_n - J_{r_{1,n+1}}^{F_1} \omega_n \right\|.
\end{aligned}
\tag{3.13}
$$

Now, apply (3.12) to conclude (3.11).

Since T_n and $U_{n,n}$ are nonexpansive, we have

$$
\begin{aligned}
\|W_{n+1}x_n - W_n x_n\| &= \left\| \mu_1 T_1 U_{n+1,2} x_n - \mu_1 T_1 U_{n,2} x_n \right\| \\
&\leq \mu_1 \left\| U_{n+1,2} x_n - U_{n,2} x_n \right\| \\
&= \mu_1 \left\| \mu_2 T_2 U_{n+1,3} x_n - \mu_2 T_2 U_{n,3} x_n \right\| \\
&\leq \mu_1 \mu_2 \left\| U_{n+1,3} x_n - U_{n,3} x_n \right\| \\
&\leq \dots \\
&\leq \mu_1 \mu_2 \dots \mu_n \left\| U_{n+1,n+1} x_n - U_{n,n+1} x_n \right\| \\
&\leq M_1 \prod_{i=1}^n \mu_i,
\end{aligned}
\tag{3.14}
$$

where $M_1 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M_1$ for all $n \geq 0$. From $I - \alpha_n B$ is nonexpansive, we have

$$
\begin{aligned}
\|v_{n+1} - v_n\| &= \left\| P_C(x_{n+1} - \alpha_{n+1} B x_{n+1}) - P_C(x_n - \alpha_n B x_n) \right\| \\
&\leq \left\| (x_{n+1} - \alpha_{n+1} B x_{n+1}) - (x_n - \alpha_n B x_n) \right\| \\
&\leq \left\| (x_{n+1} - \alpha_{n+1} B x_{n+1}) - (x_n - \alpha_{n+1} B x_n) \right\| + |\alpha_{n+1} - \alpha_n| \|B x_n\| \\
&\leq \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| \|B x_n\|.
\end{aligned}
\tag{3.15}
$$

From (3.3) and (3.15), we have

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \left\| (1 - \delta_{n+1})(x_{n+1} - x_n) + \delta_{n+1}(v_{n+1} - v_n) + (\delta_{n+1} - \delta_n)(v_n - x_n) \right\| \\
&\leq (1 - \delta_{n+1}) \|x_{n+1} - x_n\| + \delta_{n+1} \|v_{n+1} - v_n\| + |\delta_{n+1} - \delta_n| \|v_n - x_n\| \\
&\leq (1 - \delta_{n+1}) \|x_{n+1} - x_n\| + \delta_{n+1} \{ \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| \|B x_n\| \} \\
&\quad + |\delta_n - \delta_{n+1}| \|x_n - v_n\| \\
&= \|x_{n+1} - x_n\| + \delta_{n+1} |\alpha_{n+1} - \alpha_n| \|B x_n\| + |\delta_n - \delta_{n+1}| \|x_n - v_n\|.
\end{aligned}
\tag{3.16}
$$

Now, we compute $||u_{n+1} - u_n||$ and $||e_{n+1} - e_n||$. Consider the following computation:

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \left\| \Im_{n+1}^M \gamma_{n+1} - \Im_n^M \gamma_n \right\| \\
&\leq \left\| \Im_{n+1}^M \gamma_{n+1} - \Im_{n+1}^M \gamma_n \right\| + \left\| \Im_{n+1}^M \gamma_n - \Im_n^M \gamma_n \right\| \\
&\leq \left\| \gamma_{n+1} - \gamma_n \right\| + \left\| \Im_{n+1}^M \gamma_n - \Im_n^M \gamma_n \right\| \\
&\leq \|x_{n+1} - x_n\| + \delta_{n+1} |\alpha_{n+1} - \alpha_n| \|Bx_n\| + |\delta_n - \delta_{n+1}| \|x_n - v_n\| \\
&\quad + \left\| \Im_{n+1}^M \gamma_n - \Im_n^M \gamma_n \right\|
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
\|e_{n+1} - e_n\| &= \left\| p_C(W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - P_C(W_n u_n - \lambda_n BW_n u_n) \right\| \\
&\leq \left\| (W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - (W_n u_n - \lambda_n BW_n u_n) \right\| \\
&= \left\| (W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - (W_{n+1}u_n - \lambda_{n+1}BW_{n+1}u_n) \right. \\
&\quad \left. + (W_{n+1}u_n - \lambda_{n+1}BW_{n+1}u_n) - (W_n u_n - \lambda_n BW_n u_n) \right\| \\
&\leq \left\| (W_{n+1}u_{n+1} - \lambda_{n+1}BW_{n+1}u_{n+1}) - (W_{n+1}u_n - \lambda_{n+1}BW_{n+1}u_n) \right\| \\
&\quad + \|W_{n+1}u_n - W_n u_n\| + \|\lambda_n BW_n u_n - \lambda_{n+1}BW_{n+1}u_n\| \\
&\leq \|u_{n+1} - u_n\| + M_1 \prod_{i=1}^{n} \mu_i + \lambda_n \|BW_n u_n\| + \lambda_{n+1} \|BW_{n+1}u_n\| \\
&\leq \|x_{n+1} - x_n\| + \delta_{n+1} |\alpha_{n+1} - \alpha_n| \|Bx_n\| + |\delta_n - \delta_{n+1}| \|x_n - v_n\| \\
&\quad + \left\| \Im_{n+1}^M \gamma_n - \Im_n^M \gamma_n \right\| + M_1 \prod_{i=1}^{n} \mu_i + \lambda_n \|BW_n u_n\| + \lambda_{n+1} \|BW_{n+1}u_n\|.
\end{aligned}
\tag{3.18}
$$

Setting

$$
l_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\varepsilon_n \gamma f(u_n) + ((1 - \beta_n)I - \varepsilon_n A)e_n}{1 - \beta_n},
$$

we have $x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n$, $n \geq 1$. It follows that

$$
\begin{aligned}
l_{n+1} - l_n &= \frac{\varepsilon_{n+1}\gamma f(u_{n+1}) + ((1 - \beta_{n+1})I - \varepsilon_{n+1}A)e_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{\varepsilon_n \gamma f(u_n) + ((1 - \beta_n)I - \varepsilon_n A)e_n}{1 - \beta_n} \\
&= \frac{\varepsilon_{n+1}}{1 - \beta_{n+1}} \left(\gamma f(u_{n+1}) - Ae_{n+1} \right) + \frac{\varepsilon_n}{1 - \beta_n} \left(Ae_n - \gamma f(u_n) \right) + (e_{n+1} - e_n).
\end{aligned}
\tag{3.19}
$$

It follows from (3.18) and (3.19) that

$$
\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| &\leq \frac{\varepsilon_{n+1}}{1 - \beta_{n+1}} \left\| \gamma f(u_{n+1}) - Ae_{n+1} \right\| + \frac{\varepsilon_n}{1 - \beta_n} \left\| Ae_n - \gamma f(u_n) \right\| \\
&\quad + \delta_{n+1} |\alpha_{n+1} - \alpha_n| \|Bx_n\| + |\delta_n - \delta_{n+1}| \|x_n - v_n\| \\
&\quad + \left\| \Im_{n+1}^M \gamma_n - \Im_n^M \gamma_n \right\| + M_1 \prod_{i=1}^{n} \mu_i + \lambda_n \|BW_n u_n\| \\
&\quad + \lambda_{n+1} \|BW_{n+1}u_n\| \\
&\leq \frac{\varepsilon_{n+1}}{1 - \beta_{n+1}} \left(\left\| \gamma f(u_{n+1}) \right\| + \|Ae_{n+1}\| \right) + \frac{\varepsilon_n}{1 - \beta_n} \left(\|Ae_n\| + \left\| \gamma f(u_n) \right\| \right) \\
&\quad + \delta_{n+1} |\alpha_{n+1} - \alpha_n| \|Bx_n\| + |\delta_n - \delta_{n+1}| \|x_n - v_n\| \\
&\quad + \left\| \Im_{n+1}^M \gamma_n - \Im_n^M \gamma_n \right\| + M_1 \prod_{i=1}^{n} \mu_i + \lambda_n \|BW_n u_n\| \\
&\quad + \lambda_{n+1} \|BW_{n+1}u_n\|.
\end{aligned}
\tag{3.20}
$$

This together with conditions (C1)-(C4) and (3.11) imply that

$$\limsup_{n \to \infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0.$$

By Lemma 2.8, we obtain

$$\lim_{n \to \infty} \|l_n - x_n\| = 0.$$

Consequently,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty}(1 - \beta_n)\|l_n - x_n\| = 0. \tag{3.21}$$

Applying (3.11), (3.21) and conditions (C3), (C4) to (3.15) and (3.17), we obtain that

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = \lim_{n \to \infty} \|v_{n+1} - v_n\| = 0. \tag{3.22}$$

Step 5. We show that $\lim_{n \to \infty} \|W_n e_n - e_n\| = 0$.

For any $\tilde{x} \in \Theta$ and (3.5), we obtain

$$
\begin{aligned}
\|v_n - \tilde{x}\|^2 &= \|P_C(x_n - \alpha_n B x_n) - P_C(\tilde{x} - \alpha_n B \tilde{x})\|^2 \\
&\le \|(x_n - \alpha_n B x_n) - (\tilde{x} - \alpha_n B \tilde{x})\|^2 \\
&\le \|x_n - \tilde{x}\|^2 + (\alpha_n^2 - 2\alpha_n \xi)\|B x_n - B \tilde{x}\|^2.
\end{aligned} \tag{3.23}
$$

By Lemma 2.2(iv) and (3.23), we have

$$
\begin{aligned}
\|y_n - \tilde{x}\|^2 &\le (1 - \delta_n)\|x_n - \tilde{x}\|^2 + \delta_n \|v_n - \tilde{x}\|^2 \\
&\le (1 - \delta_n)\|x_n - \tilde{x}\|^2 + \delta_n\{\|x_n - \tilde{x}\|^2 + (\alpha_n^2 - 2\alpha_n \xi)\|B x_n - B \tilde{x}\|^2\} \\
&= \|x_n - \tilde{x}\|^2 + (\alpha_n^2 - 2\alpha_n \xi)\delta_n \|B x_n - B \tilde{x}\|^2.
\end{aligned} \tag{3.24}
$$

So, from (3.8) and (3.24), we derive

$$\|e_n - \tilde{x}\|^2 \le \|u_n - \tilde{x}\|^2 \le \|y_n - \tilde{x}\|^2 \le \|x_n - \tilde{x}\|^2 + (\alpha_n^2 - 2\alpha_n \xi)\delta_n \|B x_n - B \tilde{x}\|^2. \tag{3.25}$$

From (3.3), we have

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|((1 - \beta_n)I - \varepsilon_n A)(e_n - \tilde{x}) + \beta_n(x_n - \tilde{x}) + \varepsilon_n(\gamma f(u_n) - A\tilde{x})\|^2 \\
&= \|((1 - \beta_n)I - \varepsilon_n A)(e_n - \tilde{x}) + \beta_n(x_n - \tilde{x})\|^2 \\
&\quad + \varepsilon_n^2 \|\gamma f(u_n) - A\tilde{x}\|^2 + 2\beta_n \varepsilon_n \langle x_n - \tilde{x}, \gamma f(u_n) - A\tilde{x}\rangle \\
&\quad + 2\varepsilon_n \langle((1 - \beta_n)I - \varepsilon_n A)(e_n - \tilde{x}), \gamma f(u_n) - A\tilde{x}\rangle \\
&\le ((1 - \beta_n - \varepsilon_n \bar{\gamma})\|e_n - \tilde{x}\| + \beta_n \|x_n - \tilde{x}\|)^2 + \varepsilon_n L_n \\
&\le (1 - \beta_n - \varepsilon_n \bar{\gamma})^2 \|e_n - \tilde{x}\|^2 + \beta_n^2 \|x_n - \tilde{x}\|^2 \\
&\quad + 2(1 - \beta_n - \varepsilon_n \bar{\gamma})\beta_n \|e_n - \tilde{x}\| \|x_n - \tilde{x}\| + \varepsilon_n L_n \\
&\le \left[(1 - \varepsilon_n \bar{\gamma})^2 - 2(1 - \varepsilon_n \bar{\gamma})\beta_n + \beta_n^2\right]\|e_n - \tilde{x}\|^2 \\
&\quad + (1 - \beta_n - \varepsilon_n \bar{\gamma})\beta_n \left\{\|e_n - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2\right\} + \beta_n^2 \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&= (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\|e_n - \tilde{x}\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\le (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\left\{\|x_n - \tilde{x}\|^2 + (\alpha_n^2 - 2\alpha_n \xi)\delta_n \|B x_n - B \tilde{x}\|^2\right\} \\
&\quad + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&= (1 - \varepsilon_n \bar{\gamma})^2 \|x_n - \tilde{x}\|^2 \\
&\quad + (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})(\alpha_n^2 - 2\alpha_n \xi)\delta_n \|B x_n - B \tilde{x}\|^2 + \varepsilon_n L_n \\
&\le \|x_n - \tilde{x}\|^2 + (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})(\alpha_n^2 - 2\alpha_n \xi)\delta_n \|B x_n - B \tilde{x}\|^2 + \varepsilon_n L_n.
\end{aligned} \tag{3.26}
$$

It follows that

$$
\begin{aligned}
&(1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})(2g\xi - e^2)b \|Bx_n - B\tilde{x}\|^2 \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})(2\alpha_n \xi - \alpha_n^2)\delta_n \|Bx_n - B\tilde{x}\|^2 \\
&\leq \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq \|x_n - x_{n+1}\| \left(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\| \right) + \varepsilon_n L_n,
\end{aligned}
$$

where

$$
\begin{aligned}
L_n &= \varepsilon_n \|\gamma f(u_n) - A\tilde{x}\|^2 + 2\beta_n \langle x_n - \tilde{x}, \gamma f(u_n) - A\tilde{x} \rangle \\
&\quad + 2\langle ((1 - \beta_n)I - \varepsilon_n A)(e_n - \tilde{x}), \gamma f(u_n) - A\tilde{x} \rangle.
\end{aligned}
$$

By conditions (C1), (C2) and (3.21), we obtain

$$
\lim_{n \to \infty} \|Bx_n - B\tilde{x}\| = 0. \tag{3.27}
$$

Since P_C is firmly nonexpansive mapping, we have

$$
\begin{aligned}
\|v_n - \tilde{x}\|^2 &= \|P_C(x_n - \alpha_n Bx_n) - P_C(\tilde{x} - \alpha_n B\tilde{x})\|^2 \\
&\leq \langle (x_n - \alpha_n Bx_n) - (\tilde{x} - \alpha_n B\tilde{x}), v_n - \tilde{x} \rangle \\
&= \frac{1}{2} \left\{ \|(x_n - \alpha_n Bx_n) - (\tilde{x} - \alpha_n B\tilde{x})\|^2 + \|v_n - \tilde{x}\|^2 \right. \\
&\quad \left. - \|(x_n - \alpha_n Bx_n) - (\tilde{x} - \alpha_n B\tilde{x}) - (v_n - \tilde{x})\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \|x_n - \tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 - \|(x_n - v_n) - \alpha_n(Bx_n - B\tilde{x})\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \|x_n - \tilde{x}\|^2 + \|v_n - \tilde{x}\|^2 - \|x_n - v_n\|^2 \right. \\
&\quad \left. - \alpha_n^2 \|Bx_n - B\tilde{x}\|^2 + 2\alpha_n \|x_n - v_n\| \|Bx_n - B\tilde{x}\| \right\}.
\end{aligned}
$$

Hence, we have

$$
\|v_n - \tilde{x}\|^2 \leq \|x_n - \tilde{x}\|^2 - \|x_n - v_n\|^2 + 2\alpha_n \|x_n - v_n\| \|Bx_n - B\tilde{x}\|
$$

and so

$$
\begin{aligned}
\|y_n - \tilde{x}\|^2 &\leq (1 - \delta_n)\|x_n - \tilde{x}\|^2 + \delta_n \|v_n - \tilde{x}\|^2 \\
&\leq (1 - \delta_n)\|x_n - \tilde{x}\|^2 + \delta_n \{ \|x_n - \tilde{x}\|^2 - \|x_n - v_n\|^2 + 2\alpha_n \|x_n - v_n\| \|Bx_n - B\tilde{x}\| \} \tag{3.28} \\
&= \|x_n - \tilde{x}\|^2 - \delta_n \|x_n - v_n\|^2 + 2\delta_n \alpha_n \|x_n - v_n\| \|Bx_n - B\tilde{x}\|.
\end{aligned}
$$

Using (3.26) and (3.28), we also have

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\|e_n - \tilde{x}\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\|u_n - \tilde{x}\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\|y_n - \tilde{x}\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\{ \|x_n - \tilde{x}\|^2 - \delta_n \|x_n - v_n\|^2 + 2\delta_n \alpha_n \|x_n - v_n\| \|Bx_n - B\tilde{x}\| \right\} \\
&\quad + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq \|x_n - \tilde{x}\|^2 - (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\delta_n \|x_n - v_n\|^2 \\
&\quad + 2(1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\delta_n \alpha_n \|x_n - v_n\| \|Bx_n - B\tilde{x}\| + \varepsilon_n L_n.
\end{aligned}
$$

It follow that

$$
\begin{aligned}
(1 - \varepsilon_n \bar{\gamma})&(1 - \beta_n - \varepsilon_n \bar{\gamma}) \delta_n \|x_n - v_n\|^2 \\
&\leq \|x_n - x_{n+1}\| \left(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\| \right) \\
&\quad + 2(1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \delta_n \alpha_n \|x_n - v_n\| \left\| Bx_n - B\tilde{x} \right\| + \varepsilon_n L_n.
\end{aligned}
$$

From conditions (C1), C(4), (3.21) and (3.27), we obtain

$$
\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{3.29}
$$

Observe also that if $e_n = P_C(W_n u_n - \lambda_n B W_n u_n)$, then

$$
\begin{aligned}
\left\| e_n - \tilde{x} \right\|^2 &= \left\| P_C(W_n u_n - \lambda_n B W_n u_n) - P_C(\tilde{x} - \lambda_n B\tilde{x}) \right\|^2 \\
&\leq \left\| (W_n u_n - \lambda_n B W_n u_n) - (\tilde{x} - \lambda_n B\tilde{x}) \right\|^2 \\
&= \left\| (W_n u_n - \lambda_n B W_n u_n) - (W_n \tilde{x} - \lambda_n B W_n \tilde{x}) \right\|^2 \\
&\leq \left\| u_n - \tilde{x} \right\|^2 + (\lambda_n^2 - 2\lambda_n \xi) \left\| B W_n u_n - B\tilde{x} \right\|^2 \\
&\leq \left\| x_n - \tilde{x} \right\|^2 + (\lambda_n^2 - 2\lambda_n \xi) \left\| B W_n u_n - B\tilde{x} \right\|^2.
\end{aligned} \tag{3.30}
$$

Substituting (3.30) in (3.26), we have

$$
\begin{aligned}
\left\| x_{n+1} - \tilde{x} \right\|^2 &\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\| e_n - \tilde{x} \right\|^2 + (1 - \varepsilon_n \bar{\gamma}) \beta_n \left\| x_n - \tilde{x} \right\|^2 + \varepsilon_n L_n \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\{ \left\| x_n - \tilde{x} \right\|^2 + (\lambda_n^2 - 2\lambda_n \xi) \left\| B W_n u_n - B\tilde{x} \right\|^2 \right\} \\
&\quad + (1 - \varepsilon_n \bar{\gamma}) \beta_n \left\| x_n - \tilde{x} \right\|^2 + \varepsilon_n L_n \\
&\leq \left\| x_n - \tilde{x} \right\|^2 + (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})(\lambda_n^2 - 2\lambda_n \xi) \left\| B W_n u_n - B\tilde{x} \right\|^2 + \varepsilon_n L_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(1 - \varepsilon_n \bar{\gamma})&(1 - \beta_n - \varepsilon_n \bar{\gamma})(2g\xi - e^2) \left\| B W_n u_n - B\tilde{x} \right\|^2 \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})(2\lambda_n \xi - \lambda_n^2) \left\| B W_n u_n - B\tilde{x} \right\|^2 \\
&\leq \|x_n - x_{n+1}\| \left(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\| \right) + \varepsilon_n L_n
\end{aligned}
$$

Since $||x_{n+1} - x_n|| \to 0 \ (n \to \infty)$ and conditions (C1) and (C2), we obtain

$$
\lim_{n \to \infty} \left\| B W_n u_n - B\tilde{x} \right\| = 0. \tag{3.31}
$$

Since P_C is firmly nonexpansive (Lemma 2.2 (iii)), we have

$$
\begin{aligned}
\left\| e_n - \tilde{x} \right\|^2 &= \left\| P_C(W_n u_n - \lambda_n B W_n u_n) - P_C(\tilde{x} - \lambda_n B\tilde{x}) \right\|^2 \\
&\leq \left\langle (W_n u_n - \lambda_n B W_n u_n) - (\tilde{x} - \lambda_n B\tilde{x}), e_n - \tilde{x} \right\rangle \\
&= \frac{1}{2} \left\{ \left\| (W_n u_n - \lambda_n B W_n u_n) - (\tilde{x} - \lambda_n B\tilde{x}) \right\|^2 + \left\| e_n - \tilde{x} \right\|^2 \right. \\
&\quad \left. - \left\| (W_n u_n - \lambda_n B W_n u_n) - (\tilde{x} - \lambda_n B\tilde{x}) - (e_n - \tilde{x}) \right\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \left\| u_n - \tilde{x} \right\|^2 + \left\| e_n - \tilde{x} \right\|^2 - \left\| (W_n u_n - e_n) - \lambda_n (B W_n u_n - B\tilde{x}) \right\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \left\| x_n - \tilde{x} \right\|^2 + \left\| e_n - \tilde{x} \right\|^2 - \|W_n u_n - e_n\|^2 \right. \\
&\quad \left. - \lambda_n^2 \left\| B W_n u_n - B\tilde{x} \right\|^2 + 2\lambda_n \|W_n u_n - e_n\| \left\| B W_n u_n - B\tilde{x} \right\| \right\}.
\end{aligned}
$$

Hence, we have

$$\left\| e_n - \tilde{x} \right\|^2 \le \left\| x_n - \tilde{x} \right\|^2 - \left\| W_n u_n - e_n \right\|^2 + 2\lambda_n \left\| W_n u_n - e_n \right\| \left\| BW_n u_n - B\tilde{x} \right\|. \quad (3.32)$$

Using (3.26) and (3.32), we also have

$$
\begin{aligned}
\left\| x_{n+1} - \tilde{x} \right\|^2 &\le (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\left\| e_n - \tilde{x} \right\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \left\| x_n - \tilde{x} \right\|^2 + \varepsilon_n L_n \\
&\le (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\left\{ \left\| x_n - \tilde{x} \right\|^2 - \left\| W_n u_n - e_n \right\|^2 \right. \\
&\quad \left. + 2\lambda_n \left\| W_n u_n - e_n \right\| \left\| BW_n u_n - B\tilde{x} \right\| \right\} + (1 - \varepsilon_n \bar{\gamma})\beta_n \left\| x_n - \tilde{x} \right\|^2 + \varepsilon_n L_n \\
&\le \left\| x_n - \tilde{x} \right\|^2 - (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\left\| W_n u_n - e_n \right\|^2 \\
&\quad + 2(1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\lambda_n \left\| W_n u_n - e_n \right\| \left\| BW_n u_n - B\tilde{x} \right\| + \varepsilon_n L_n.
\end{aligned}
$$

It follow that

$$
\begin{aligned}
&(1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\left\| W_n u_n - e_n \right\|^2 \\
&\le \left\| x_n - x_{n+1} \right\| \left(\left\| x_n - \tilde{x} \right\| + \left\| x_{n+1} - \tilde{x} \right\| \right) \\
&\quad + 2(1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma})\lambda_n \left\| W_n u_n - e_n \right\| \left\| BW_n u_n - B\tilde{x} \right\| + \varepsilon_n L_n.
\end{aligned}
$$

From conditions (C1), (3.21), and (3.31), we obtain

$$\lim_{n \to \infty} \left\| W_n u_n - e_n \right\| = 0. \qquad (3.33)$$

For any $\tilde{x} \in \Theta$, note that $J_{r_{k,n}}^{F_k}$ is firmly nonexpansive (Lemma 2.7(2)) for $k \in \{1, 2, 3,\dots, M\}$, then we have

$$
\begin{aligned}
\left\| \Im_n^k \gamma_n - \tilde{x} \right\|^2 &= \left\| J_{r_{k,n}}^{F_n} \Im_n^{k-1} \gamma_n - J_{r_{k,n}}^{F_k} \tilde{x} \right\|^2 \\
&\le \left\langle J_{r_{k,n}}^{F_n} \Im_n^{k-1} \gamma_n - J_{r_{k,n}}^{F_k} \tilde{x}, \Im_n^{k-1} \gamma_n - \tilde{x} \right\rangle \\
&= \left\langle \Im_n^k \gamma_n - \tilde{x}, \Im_n^{k-1} \gamma_n - \tilde{x} \right\rangle \\
&= \frac{1}{2}\left(\left\| \Im_n^k \gamma_n - \tilde{x} \right\|^2 + \left\| \Im_n^{k-1} \gamma_n - \tilde{x} \right\|^2 - \left\| \Im_n^k \gamma_n - \Im_n^{k-1} \gamma_n \right\|^2 \right).
\end{aligned}
$$

So, we obtain

$$\left\| \Im_n^k \gamma_n - \tilde{x} \right\|^2 \le \left\| \Im_n^{k-1} \gamma_n - \tilde{x} \right\|^2 - \left\| \Im_n^k \gamma_n - \Im_n^{k-1} \gamma_n \right\|^2, \quad k = 1, 2, 3, \dots, M$$

which implies that for each $k \in \{1, 2, 3,\dots, M - 1\}$,

$$
\begin{aligned}
\left\| \Im_n^k \gamma_n - \tilde{x} \right\|^2 &\le \left\| \Im_n^0 \gamma_n - \tilde{x} \right\|^2 - \left\| \Im_n^k \gamma_n - \Im_n^{k-1} \gamma_n \right\|^2 \\
&\quad - \left\| \Im_n^{k-1} \gamma_n - \Im_n^{k-2} \gamma_n \right\|^2 - \cdots - \left\| \Im_n^2 \gamma_n - \Im_n^1 \gamma_n \right\|^2 - \left\| \Im_n^1 \gamma_n - \Im_n^0 \gamma_n \right\|^2 \\
&\le \left\| \gamma_n - \tilde{x} \right\|^2 - \left\| \Im_n^k \gamma_n - \Im_n^{k-1} \gamma_n \right\|^2 \\
&\le \left\| x_n - \tilde{x} \right\|^2 - \left\| \Im_n^k \gamma_n - \Im_n^{k-1} \gamma_n \right\|^2.
\end{aligned}
$$

Consequently, from (3.26) we derive that

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \|e_n - \tilde{x}\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \|u_n - \tilde{x}\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&= (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\|\mathfrak{I}_n^k \gamma_n - \tilde{x}\right\|^2 + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\{ \|x_n - \tilde{x}\|^2 - \left\|\mathfrak{I}_n^k \gamma_n - \mathfrak{I}_n^{k-1} \gamma_n\right\|^2 \right\} \\
&\quad + (1 - \varepsilon_n \bar{\gamma})\beta_n \|x_n - \tilde{x}\|^2 + \varepsilon_n L_n \\
&\leq \|x_n - \tilde{x}\|^2 - (1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\|\mathfrak{I}_n^k \gamma_n - \mathfrak{I}_n^{k-1} \gamma_n\right\|^2 + \varepsilon_n L_n.
\end{aligned}
$$

Thus, we have

$$
(1 - \varepsilon_n \bar{\gamma})(1 - \beta_n - \varepsilon_n \bar{\gamma}) \left\|\mathfrak{I}_n^k \gamma_n - \mathfrak{I}_n^{k-1} \gamma_n\right\|^2 \leq \|x_n - x_{n+1}\| \left(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|\right) + \varepsilon_n L_n.
$$

By $\liminf\limits_{n \to \infty} \beta_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$ and (3.21), so we deduce that

$$
\lim_{n \to \infty} \left\|\mathfrak{I}_n^k \gamma_n - \mathfrak{I}_n^{k-1} \gamma_n\right\| = 0, \quad k = 1, 2, \ldots, M - 1, \tag{3.34}
$$

that is,

$$
\left\|u_n^{(k)} - u_n^{(k-1)}\right\| \to 0 \text{ as } n \to \infty.
$$

Therefore, we have

$$
\begin{aligned}
\|\gamma_n - u_n\| &= \left\|\mathfrak{I}_n^0 \gamma_n - \mathfrak{I}_n^k \gamma_n\right\| \\
&\leq \left\|\mathfrak{I}_n^0 \gamma_n - \mathfrak{I}_n^1 \gamma_n\right\| + \left\|\mathfrak{I}_n^1 \gamma_n - \mathfrak{I}_n^2 \gamma_n\right\| + \cdots + \left\|\mathfrak{I}_n^{M-1} \gamma_n - \mathfrak{I}_n^M \gamma_n\right\|.
\end{aligned}
$$

From (3.34), we have

$$
\lim_{n \to \infty} \|\gamma_n - u_n\| = 0. \tag{3.35}
$$

Since $x_{n+1} = \epsilon_n \gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \epsilon_n A)e_n$, we have

$$
\begin{aligned}
\|x_n - e_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - e_n\| \\
&= \|x_n - x_{n+1}\| + \|\varepsilon_n \gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \varepsilon_n A)e_n - e_n\| \\
&= \|x_n - x_{n+1}\| + \|\varepsilon_n (\gamma f(u_n) - Ae_n) + \beta_n(x_n - e_n)\| \\
&\leq \|x_n - x_{n+1}\| + \varepsilon_n \left(\|\gamma f(u_n)\| + \|Ae_n\|\right) + \beta_n \|x_n - e_n\|,
\end{aligned}
$$

that is,

$$
\|x_n - e_n\| \leq \frac{1}{1 - \beta_n} \|x_n - x_{n+1}\| + \frac{\varepsilon_n}{1 - \beta_n} \left(\|\gamma f(u_n)\| + \|Ae_n\|\right).
$$

By conditions (C1), (C2), and (3.21) it follows that

$$
\lim_{n \to \infty} \|x_n - e_n\| = 0. \tag{3.36}
$$

On the other hand, from (3.3), we have

$$
\|\gamma_n - x_n\| = \delta_n \|v_n - x_n\|.
$$

Since $\lim_{n\to\infty} \|x_n - v_n\| = 0$, we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.37}$$

We observe that

$$
\begin{aligned}
\|W_n e_n - e_n\| &\leq \|W_n e_n - W_n u_n\| + \|W_n u_n - e_n\| \\
&\leq \|e_n - x_n + x_n - y_n + y_n - u_n\| + \|W_n u_n - e_n\| \\
&\leq \|e_n - x_n\| + \|x_n - y_n\| + \|y_n - u_n\| + \|W_n u_n - e_n\|.
\end{aligned}
$$

Consequently, we obtain

$$\lim_{n\to\infty} \|W_n e_n - e_n\| = 0. \tag{3.38}$$

Let W be the mapping defined by (3.2). Since $\{e_n\}$ is bounded, applying Lemma 3.1(4) and (3.38), we have

$$\|W e_n - e_n\| \leq \|W e_n - W_n e_n\| + \|W_n e_n - e_n\| \to 0 \text{ as } n \to \infty. \tag{3.39}$$

Step 6. We show that $q \in \Theta$, where $\Theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{M} SEP(F_k)) \cap VI(C, B)$.

Since $\{x_n\}$ is bounded, we see that there exits a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to q. It follows from (3.37) and (3.36) that $y_{n_i} \rightharpoonup q$ and $e_{n_i} \rightharpoonup q$. From (3.35), we obtain that $\Im_{n_i}^k y_{n_i} \rightharpoonup q$ for $k = 1, 2,...,M$.

First, we show that $q \in \cap_{k=1}^{M} SEP(F_k)$. Since $u_n = \Im_n^k y_n$ for $k = 1, 2, 3,..., M$, we also have

$$F_k(\Im_n^k y_n, \gamma) + \frac{1}{r_n}\langle \gamma - \Im_n^k y_n, \Im_n^k y_n - \Im_n^{k-1} y_n \rangle \geq 0, \quad \forall \gamma \in C.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle \gamma - \Im_n^k y_n, \Im_n^k y_n - \Im_n^{k-1} y_n \rangle \geq -F_k(\Im_n^k y_n, \gamma) \geq F_k(\gamma, \Im_n^k y_n).$$

Replacing n by n_i, we have

$$\left\langle \gamma - \Im_{n_i}^k y_{n_i}, \frac{\Im_{n_i}^k y_{n_i} - \Im_{n_i}^{k-1} y_{n_i}}{r_{n_i}} \right\rangle \geq F_k(\gamma, \Im_{n_i}^k y_{n_i}).$$

Since $\frac{\Im_{n_i}^k y_{n_i} - \Im_{n_i}^{k-1} y_{n_i}}{r_{n_i}} \to 0$ and $\Im_{n_i}^k y_{n_i} \rightharpoonup q$, it follows by (A4) that

$$F_k(\gamma, q) \leq 0 \quad \forall \gamma \in C,$$

for each $k = 1,2,3, ...,M$.

For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1 - t)q$. Since $y \in C$ and $q \in C$, we have $y_t \in C$ and hence $F_k(y_t, q) \leq 0$. So, from (A1) and (A4) we have

$$0 = F_k(y_t, y_t) \leq t F_k(y_t, \gamma) + (1 - t)F_k(y_t, q) \leq t F_k(y_t, \gamma)$$

and hence $F_k(y_t, y) \geq 0$. From (A3), we have $F_k(q, y) \geq 0$ for all $y \in C$ and hence $q \in SEP(F_k)$ for $k = 1, 2, 3,..., M$, that is, $q \in \cap_{k=1}^{M} SEP(F_k)$.

Next, we show that $q \in \cap_{n=1}^{\infty} F(T_n)$. By Lemma 3.1(2), we have $F(W) = \cap_{n=1}^{\infty} F(T_n)$. Assume $q \notin F(W)$. Since $e_{n_i} \rightharpoonup q$ and $q \neq Wq$, it follows by the Opial's condition (Lemma 2.3) that

$$\liminf_{i \to \infty} \|e_{n_i} - q\| < \liminf_{i \to \infty} \|e_{n_i} - Wq\|$$

$$\leq \liminf_{i \to \infty} \{\|e_{n_i} - We_{n_i}\| + \|We_{n_i} - Wq\|\}$$

$$\leq \liminf_{i \to \infty} \|e_{n_i} - q\|$$

which derives a contradiction. Thus, we have $q \in F(W) = \cap_{n=1}^{\infty} F(T_n)$.

Finally, now we prove that $q \in VI(C, B)$.

We define the maximal monotone operator

$$Qq_1 = \begin{cases} Bq_1 + N_C q_1, & q_1 \in C, \\ \emptyset, & q_1 \notin C. \end{cases}$$

For any given $(q_1, q_2) \in G(Q)$, hence $q_2 - Bq_1 \in N_C q_1$. Since $e_n \in C$ we see from the definition of N_C that

$$\langle q_1 - e_n, q_2 - Bq_1 \rangle \geq 0.$$

On the other hand, from $e_n = P_C(W_n u_n - \alpha_n B W_n u_n)$, we have

$$\langle q_1 - e_n, e_n - (W_n u_n - \alpha_n B W_n u_n) \rangle \geq 0,$$

that is

$$\left\langle q_1 - e_n, \frac{e_n - W_n u_n}{\alpha_n} + B W_n u_n \right\rangle \geq 0.$$

Therefore, we obtain

$$\begin{aligned}
\langle q_1 - e_{n_i}, q_2 \rangle &\geq \langle q_1 - e_{n_i}, Bq_1 \rangle \\
&\geq \langle q_1 - e_{n_i}, Bq_1 \rangle - \left\langle q_1 - e_{n_i}, \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}} + B W_n u_{n_i} \right\rangle \\
&= \left\langle q_1 - e_{n_i}, Bq_1 - B W_n u_{n_i} - \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}} \right\rangle \\
&= \langle q_1 - e_{n_i}, Bq_1 - Be_{n_i} \rangle + \langle q_1 - e_{n_i}, Be_{n_i} - B W_n u_{n_i} \rangle \\
&\quad - \left\langle q_1 - e_{n_i}, \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}} \right\rangle \\
&\geq \langle q_1 - e_{n_i}, Be_{n_i} - B W_n u_{n_i} \rangle - \left\langle q_1 - e_{n_i}, \frac{e_{n_i} - W_n u_{n_i}}{\alpha_{n_i}} \right\rangle,
\end{aligned} \tag{3.40}$$

Since $\|e_{n_i} - W_n u_{n_i}\| \to 0$ as $i \to \infty$ and B is Lipschitz continuous we obtain that

$$\langle q_1 - q, q_2 \rangle \geq 0.$$

Notice that Q is maximal monotone, we obtain that $q \in Q^{-1}0$ and hence $q \in VI(C,B)$. This implies $q \in \Theta$. Since $z = P_{\Theta}(\gamma f + (I - A))(z)$, we have

$$\limsup_{n \to \infty} \langle x_n - z, \gamma f(z) - Az \rangle = \lim_{i \to \infty} \langle x_{n_i} - z, \gamma f(z) - Az \rangle = \langle q - z, \gamma f(z) - Az \rangle \leq 0. \tag{3.41}$$

On the other hand, we have

$$\begin{aligned}
\langle e_n - z, \gamma f(z) - Az \rangle &= \langle e_n - x_n, \gamma f(z) - Az \rangle + \langle x_n - z, \gamma f(z) - Az \rangle \\
&\leq \|e_n - x_n\| \|\gamma f(z) - Az\| + \langle x_n - z, \gamma f(z) - Az \rangle.
\end{aligned}$$

From (3.36) and (3.41), we obtain that

$$\limsup_{n\to\infty} \langle e_n - z, \gamma f(z) - Az \rangle \le 0. \tag{3.42}$$

Step 7. Finally, we show that $\{x_n\}$ converges strongly to $z = P_\Theta(I - A + \gamma f)(z)$. Indeed, from (3.3), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\| ((1-\beta_n)I - \varepsilon_n A)(e_n - z) + \beta_n(x_n - z) + \varepsilon_n(\gamma f(u_n) - Az) \right\|^2 \\
&= \left\| ((1-\beta_n)I - \varepsilon_n A)(e_n - z) + \beta_n(x_n - z) \right\|^2 \\
&\quad + \varepsilon_n^2 \|\gamma f(u_n) - Az\|^2 + 2\beta_n \varepsilon_n \langle x_n - z, \gamma f(u_n) - Az \rangle \\
&\quad + 2\varepsilon_n \langle ((1-\beta_n)I - \varepsilon_n A)(e_n - z), \gamma f(u_n) - Az \rangle \\
&\le \left((1 - \beta_n - \varepsilon_n \bar\gamma) \|e_n - z\| + \beta_n \|x_n - z\| \right)^2 + \varepsilon_n^2 \|\gamma f(u_n) - Az\|^2 \\
&\quad + 2\beta_n \varepsilon_n \gamma \langle x_n - z, f(u_n) - f(z) \rangle + 2\beta_n \varepsilon_n \langle x_n - z, \gamma f(z) - Az \rangle \\
&\quad + 2(1-\beta_n)\varepsilon_n \gamma \langle e_n - z, f(u_n) - f(z) \rangle + 2\varepsilon_n \langle e_n - z, \gamma f(z) - Az \rangle \\
&\quad - 2\beta_n \varepsilon_n \langle e_n - z, \gamma f(z) - Az \rangle - 2\varepsilon_n^2 \langle (A(e_n - z), \gamma f(u_n) - Az \rangle \\
&\le (1 - \beta_n - \varepsilon_n \bar\gamma)^2 \|e_n - z\|^2 + \beta_n^2 \|x_n - z\|^2 + 2(1 - \beta_n - \varepsilon_n\bar\gamma)\beta_n \|e_n - z\| \|x_n - z\| \\
&\quad + \varepsilon_n^2 \|\gamma f(u_n) - Az\|^2 + 2\beta_n \varepsilon_n \gamma \|x_n - z\| \|f(u_n) - f(z)\| \\
&\quad + 2(1-\beta_n)\varepsilon_n \gamma \|e_n - z\| \|f(u_n) - f(z)\| + 2\beta_n \varepsilon_n \|x_n - z\| \|\gamma f(z) - Az\| \\
&\quad - 2\beta_n \varepsilon_n \|e_n - z\| \|\gamma f(z) - Az\| - 2\varepsilon_n^2 \|A(e_n - z)\| \|\gamma f(u_n) - Az\| \\
&\quad + 2\varepsilon_n \langle e_n - z, \gamma f(z) - Az \rangle \\
&\le \left[(1 - \varepsilon_n \bar\gamma)^2 - 2(1 - \varepsilon_n \bar\gamma)\beta_n + \beta_n^2 \right] \|e_n - z\|^2 + \beta_n^2 \|x_n - z\|^2 \\
&\quad + (1 - \beta_n - \varepsilon_n\bar\gamma)\beta_n \left\{ \|e_n - z\|^2 + \|x_n - z\|^2 \right\} + \varepsilon_n^2 \|\gamma f(u_n) - Az\|^2 \\
&\quad + 2\beta_n \varepsilon_n \gamma \eta \|x_n - z\| \|u_n - z\| + 2(1-\beta_n)\varepsilon_n \gamma \eta \|e_n - z\| \|u_n - z\| \\
&\quad + 2\beta_n \varepsilon_n \|x_n - z\| \|\gamma f(z) - Az\| - 2\beta_n \varepsilon_n \|e_n - z\| \|\gamma f(z) - Az\| \\
&\quad - 2\varepsilon_n^2 \|A(e_n - z)\| \|\gamma f(u_n) - Az\| + 2\varepsilon_n \langle e_n - z, \gamma f(z) - Az \rangle \\
&\le (1 - \varepsilon_n \bar\gamma)(1 - \beta_n - \varepsilon_n \bar\gamma)\|e_n - z\|^2 + (1 - \varepsilon_n \bar\gamma)\beta_n \|x_n - z\|^2 \\
&\quad + \varepsilon_n^2 \|\gamma f(u_n) - Az\|^2 + 2\beta_n \varepsilon_n \gamma \eta \|x_n - z\|^2 + 2(1-\beta_n)\varepsilon_n \gamma \eta \|x_n - z\|^2 \\
&\quad + 2\beta_n \varepsilon_n \|x_n - z\| \|\gamma f(z) - Az\| - 2\beta_n \varepsilon_n \|e_n - z\| \|\gamma f(z) - Az\| \\
&\quad + 2\varepsilon_n^2 \|A(e_n - z)\| \|\gamma f(u_n) - Az\| + 2\varepsilon_n \langle e_n - z, \gamma f(z) - Az \rangle \\
&\le (1 - \varepsilon_n \bar\gamma)(1 - \beta_n - \varepsilon_n \bar\gamma)\|x_n - z\|^2 + (1 - \varepsilon_n \bar\gamma)\beta_n \|x_n - z\|^2 \\
&\quad + \varepsilon_n^2 \|\gamma f(u_n) - Az\|^2 + 2\varepsilon_n \gamma \eta \|x_n - z\|^2 \\
&\quad + 2\beta_n \varepsilon_n \|x_n - z\| \|\gamma f(z) - Az\| - 2\beta_n \varepsilon_n \|x_n - z\| \|\gamma f(z) - Az\| \\
&\quad + 2\varepsilon_n^2 \|A(e_n - z)\| \|\gamma f(u_n) - Az\| + 2\varepsilon_n \langle e_n - z, \gamma f(z) - Az \rangle \\
&= (1 - 2\varepsilon_n \bar\gamma + \varepsilon_n^2 \bar\gamma^2 + 2\varepsilon_n \gamma \eta)\|x_n - z\|^2 + \varepsilon_n^2 \|\gamma f(u_n) - Az\|^2 \\
&\quad + 2\varepsilon_n^2 \|A(e_n - z)\| \|\gamma f(u_n) - Az\| + 2\varepsilon_n \langle e_n - z, \gamma f(z) - Az \rangle \\
&= [1 - 2(\bar\gamma - \gamma\eta)\varepsilon_n]\|x_n - z\|^2 + \varepsilon_n \left\{ 2\langle e_n - z, \gamma f(z) - Az \rangle + \varepsilon_n K \right\}.
\end{aligned}
$$

where K is an appropriate constant such that

$$K \ge \max \left\{ \sup_{n\ge 1} \left\{ \bar\gamma^2 \|x_n - z\|^2 + \|\gamma f(u_n) - Az\|^2 + 2\|A(e_n - z)\| \|\gamma f(u_n) - Az\| \right\} \right\},$$

Set $b_n = 2(\bar\gamma - \gamma\eta)\varepsilon_n$ and $c_n = \epsilon_n \{ 2\langle e_n - z, \gamma f(z) - Az \rangle + \epsilon_n K \}$. Then we have

$$\|x_{n+1} - z\|^2 \le (1 - b_n)\|x_n - z\|^2 + c_n, \quad \forall n \ge 0. \tag{3.43}$$

From the conditions (C1) and (3.42), we see that

$$\lim_{n\to\infty} b_n = 0, \quad \sum_{n=0}^{\infty} b_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} c_n \le 0.$$

Therefore, applying Lemma 2.8 to (3.43), we get that $\{x_n\}$ converges strongly to $z \in \Theta$. This completes the proof.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3,..., M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let B be ξ-inverse strongly monotone such that*

$$\Theta := \left(\cap_{k=1}^M SEP(F_k)\right) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\eta \in (0,1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ y_n = (1 - \delta_n)x_n + \delta_n P_C(x_n - \alpha_n Bx_n), \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} y_n, \\ x_{n+1} = \varepsilon_n f(u_n) + \beta_n x_n + (1 - \beta_n - \varepsilon_n) P_C(u_n - \lambda_n Bu_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in (0,1) and $\{r_{k,n}\}$, $k \in \{1,2,3,...,M\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

(C1) $\lim_{n \to \infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \varepsilon_n = \infty$,

(C2) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$,

(C3) $\{\alpha_n\}, \{\lambda_n\} \subset [e, g] \subset (0, 2\xi)$, $\lim_{n \to \infty} \alpha_n = 0$ *and* $\lim_{n \to \infty} \lambda_n = 0$,

(C4) $\{\delta_n\} \subset [0, b]$, *for some* $b \in (0,1)$ *and* $\lim_{n \to \infty} |\delta_{n+1} - \delta_n| = 0$,

(C5) $\liminf_{n \to \infty} r_{k,n} > 0$ *and* $\lim_{n \to \infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1, 2, 3,..., M\}$,

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (f(z) - z, x - z \rangle \geq 0, \quad \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta f(z)$.

Proof. Put $T_n \equiv I$ for all $n \geq 1$ and for all $x \in C$. Then $W_n = I$, $A = I$ and $\gamma = 1$. The conclusion follows from Theorem 3.2. This completes the proof.

If $\delta_n = 0$ and $M = 1$, in Theorem 3.2, then we can obtain the following result immediately.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3,..., M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be an infinite family of nonexpansive mappings of C into itself and let B be ξ-inverse strongly monotone such that*

$$\Theta := \cap_{n=1}^{\infty} F(T_n) \cap EP(F) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\eta \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ F(u_n, \gamma) + \frac{1}{r_n} \langle \gamma - u_n, u_n - x_n \rangle \geq 0, \quad \forall \gamma \in C, \\ x_{n+1} = \varepsilon_n \gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \varepsilon_n A) P_C(W_n u_n - \lambda_n BW_n u_n), \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequence generated by (3.1) and $\{\epsilon_n\}$, $\{\beta_n\}$ are two sequences in (0,1) and $\{r_n\}$ are a real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \varepsilon_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\{\lambda_n\} \subset [e, g] \subset (0, 2\xi)$ *and* $\lim_{n\to\infty} \lambda_n = 0$,

(C4) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Theta$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in \Theta.$$

Equivalently, we have $z = P_\Theta(I - A + \gamma f)(z)$.

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Authors' contributions
All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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