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Fuzzy Stability of Generalized Mixed Type Cubic, Quadratic, and Additive Functional Equation

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Abstract

In this paper, we prove the generalized Hyers-Ulam stability of generalized mixed type cubic, quadratic, and additive functional equation, in fuzzy Banach spaces.

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [4] has provided a lot of influence in the development of what we now call *generalized Hyers-Ulam stability* of functional equations. In 1994, a generalization of the Rassias' theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping [6,7]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive mapping B such that $f(x) = B(x, x)$ for all x (see [6,7]). The bi-additive mapping B is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)) \tag{1.2}$$

A generalized Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : A \to B$, where A is normed space and B is a Banach space [8] (see [9-12]).

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Jun and Kim [13] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers-Ulam stability for the functional equation (1.3). They proved that a mapping f between two real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique mapping $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, moreover, C is symmetric for each fixed one variable and is additive for fixed two variables. The mapping C is given by

$$C(x,y,z)=\frac{1}{24}(f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)) \tag{1.4}$$

for all $x, y, z \in X$. During the last decades, several stability problems for various functional equations have been investigated by many mathematicians; [14-21].

Eshaghi and Khodaei [22] have established the general solution and investigated the generalized Hyers-Ulam stability for a mixed type of cubic, quadratic, and additive functional equation with $f(0) = 0$,

$$f(x+ky)+f(x-ky)=k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(x) \tag{1.5}$$

in quasi-Banach spaces, where k is nonzero integer numbers with $k \neq \pm 1$. Obviously, the function $f(x) = ax + bx^2 + cx^3$ is a solution of the functional equation (1.5). Interesting new results concerning mixed functional equations has recently been obtained by Najati et. al. [23,24], Jun and Kim [25,26] as well as for the fuzzy stability of a mixed type of additive and quadratic functional equation by Park [27] (see also [28-43]).

This paper is organized as follows: In Section 3, we prove the generalized Hyers-Ulam stability of the functional equation (1.5) in fuzzy Banach spaces for an even case. In Section 4, we prove the generalized Hyers-Ulam stability of the functional equation (1.5) in fuzzy Banach spaces for an odd case. In Section 5, we prove the generalized Hyers-Ulam stability of generalized mixed cubic, quadratic, and additive functional equation (1.5) in fuzzy Banach spaces.

2. Preliminaries

We use the definition of fuzzy normed spaces given in [44] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the functional equation (1.5) in the fuzzy normed space setting.

Definition 2.1. (Bag and Samanta [44], Mirmostafaee [45]). Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$;

(N_1) $N(x, t) = 0$ for all $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N_5) $N(x, \cdot)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) $N(x, \cdot)$ is left continuous on $\mathbb{R}$ for every $x \neq 0$.

The pair (X, N) is called a fuzzy normed linear space.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in ([3,45-47]).

Definition 2.2. (Bag and Samanta [44], Mirmostafaee [45]). Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence (x_n) and we write $N - lim_{n\to\infty} x_n = x$.

Definition 2.3. (Bag and Samanta [44], Mirmostafaee [45]). Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $N(x_m - x_n, \delta) > 1 - \epsilon$ $(m, n \geq n_0)$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a function $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_k\}$ converging to x_0 in X, then the sequence $\{f(x_k)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [48]).

In the rest of this paper, unless otherwise explicitly stated, we will assume that X is a vector space, (Z, N') is a fuzzy normed space, and (Y, N) is a fuzzy Banach space. For convenience, we use the following abbreviation for a given function $f : X \to Y$,

$$D_f(x, y) = f(x + ky) + f(x - ky) - k^2 f(x + y) - k^2 f(x - y) - 2(1 - k^2)f(x)$$

for all $x, y \in X$, where k is nonzero integer numbers with $k \neq \pm 1$.

3. Fuzzy stability of the functional equation (1.5): an even case

In this section, we prove the generalized Hyers-Ulam stability of the functional equation (1.5) in fuzzy Banach spaces, for an even case. From now on, V_1 and V_2 will be real vector spaces.

Lemma 3.1. [22]. *If an even mapping $f : V_1 \to V_2$ satisfies (1.5), then $f(x)$ is quadratic.*

Theorem 3.2. *Let $\ell \in \{-1, 1\}$ be fixed and let $\phi_q : X \times X \to Y$ be a mapping such that*

$$\varphi_q(kx, ky) = \alpha \varphi_q(x, y) \tag{3.1}$$

for all $x, y \in X$ and for some positive real number α with $\alpha\ell < k^2\ell$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_f(x, y), t) \geq N'(\varphi_q(x, y), t) \tag{3.2}$$

for all $x, y \in X$ and all $t > 0$. Then, the limit

$$Q(x) = N - \lim_{n\to\infty} \frac{f(k^{\ell n}x)}{k^{2\ell n}})$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic mapping satisfying

$$N(f(x) - Q(x), t) \geq N'\left(\varphi_q(0, x), \frac{\ell(k^2 - \alpha)t}{2}\right) \tag{3.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Case (1): $\ell = 1$. By putting $x = 0$ in (3.2) and then using evenness of f and $f(0) = 0$, we obtain

$$N(2f(ky) - 2k^2 f(y), t) \geq N'(\varphi_q(0, y), t) \tag{3.4}$$

for all $x, y \in X$ and all $t > 0$. If we replace y in (3.4) by x, we get

$$N\left(f(kx) - k^2 f(x), \frac{t}{2}\right) \geq N'(\varphi_q(0, x), t) \tag{3.5}$$

for all $x \in X$. So

$$N\left(\frac{f(kx)}{k^2} - f(x), \frac{t}{2k^2}\right) \geq N'(\varphi_q(0, x), t) \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Then by our assumption

$$N'(\varphi_q(0, kx), t) = N'\left(\varphi_q(0, x), \frac{t}{\alpha}\right) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. Replacing x by $k^n x$ in (3.6) and using (3.7), we obtain

$$N\left(\frac{f(k^{n+1}x)}{k^{2(n+1)}} - \frac{f(k^n x)}{k^{2n}}, \frac{t}{k^2(k^{2n})}\right) \geq N'(\varphi_q(0, k^n x), t) = N'\left(\varphi_q(0, x), \frac{t}{\alpha^n}\right) \tag{3.8}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Replacing t by $\alpha^n t$ in (3.8), we see that

$$N\left(\frac{f(k^{n+1}x)}{k^{2(n+1)}} - \frac{f(k^n x)}{k^{2n}}, \frac{\alpha^n t}{k^2(k^{2n})}\right) \geq N'(\varphi_q(0, x), t) \tag{3.9}$$

for all $x \in X$, $t > 0$ and $n > 0$. It follows from $\frac{f(k^n x)}{k^{2n}} - f(x) = \sum_{j=0}^{n-1}\left(\frac{f(k^{j+1}x)}{k^{2(j+1)}} - \frac{f(k^j x)}{k^{2j}}\right)$ and (3.9) that

$$\begin{aligned} N\left(\frac{f(k^n x)}{k^{2n}} - f(x), \sum_{j=0}^{n-1} \frac{\alpha^j t}{k^2(k^2)^j}\right) &\geq \min \bigcup_{j=0}^{n-1} \left\{N\left(\frac{f(k^{j+1}x)}{k^{2(j+1)}} - \frac{f(k^j x)}{k^{2j}}, \frac{\alpha^j t}{k^2(k^2)^j}\right)\right\} \\ &\geq N'(\varphi_q(0, x), t) \end{aligned} \tag{3.10}$$

for all $x \in X$, $t > 0$ and $n > 0$. Replacing x by $k^m x$ in (3.10), we observe that

$$N\left(\frac{f(k^{n+m}x)}{k^{2(n+m)}} - \frac{f(k^m x)}{k^{2m}}, \sum_{j=0}^{n-1} \frac{\alpha^j t}{k^2(k^2)^{j+m}}\right) \geq N'(\varphi_q(0, k^m x), t) = N'\left(\varphi(0, x), \frac{t}{\alpha^m}\right)$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. Hence

$$N\left(\frac{f(k^{n+m}x)}{k^{2(n+m)}} - \frac{f(k^m x)}{k^{2m}}, \sum_{j=m}^{n+m-1} \frac{\alpha^j t}{k^2(k^2)^j}\right) \geq N'(\varphi_q(0, x), t)$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. By last inequality, we obtain

$$N\left(\frac{f(k^{n+m}x)}{k^{2(n+m)}} - \frac{f(k^m x)}{k^{2m}}, t\right) \geq N'\left(\varphi_q(0, x), \frac{t}{\sum_{j=m}^{n+m-1} \frac{\alpha^j}{k^2(k^2)^j}}\right) \tag{3.11}$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. Since $0 < \alpha < k^2$ and $\sum_{j=0}^{\infty}\left(\frac{\alpha}{k^2}\right)^j < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\left\{\frac{f(k^n x)}{k^{2n}}\right\}$ is a Cauchy sequence in Y. Since Y is a fuzzy Banach space, this sequence converges to some point $Q(x) \in Y$. So one can define the function $Q : X \to Y$ by

$$Q(x) = N - \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}}) \tag{3.12}$$

for all $x \in X$. Fix $x \in X$ and put $m = 0$ in (3.11) to obtain

$$N\left(\frac{f(k^n x)}{k^{2n}} - f(x), t\right) \geq N'\left(\varphi_q(0, x), \frac{t}{\sum_{j=0}^{n-1} \frac{\alpha^j}{k^2 (k^2)^j}}\right)$$

for all $x \in X$, all $t > 0$ and all $n > 0$. From which we obtain

$$\begin{aligned} N(Q(x) - f(x), t) &\geq \min\left\{N(Q(x) - \frac{f(k^n x)}{k^{2n}}, \frac{t}{2}), N\left(\frac{f(k^n x)}{k^{2n}} - f(x), \frac{t}{2}\right)\right\} \\ &\geq N'\left(\varphi_q(0, x), \frac{t}{\sum_{j=0}^{n-1} \frac{2\alpha^j}{k^2 (k^2)^j}}\right) \end{aligned} \tag{3.13}$$

for n large enough. Taking the limit as $n \to \infty$ in (3.13), we obtain

$$N(Q(x) - f(x), t) \geq N'\left(\varphi_q(0, x), \frac{(k^2 - \alpha)t}{2}\right) \tag{3.14}$$

for all $x \in X$ and all $t > 0$. It follows from (3.8) and (3.12) that

$$\begin{aligned} N\left(\frac{Q(kx)}{k^2} - Q(x), t\right) \geq \min\Big\{&N\left(\frac{Q(kx)}{k^2} - \frac{f(k^{n+1}x)}{k^{2(n+1)}}, \frac{t}{3}\right), N(\frac{f(k^n x)}{k^{2n}} - Q(x), \frac{t}{3}), \\ &N\left(\frac{f(k^{n+1}x)}{k^{2(n+1)}} - \frac{f(k^n x)}{k^{2n}}, \frac{t}{3}\right)\Big\} = N'\left(\varphi_q(0, x), \frac{k^2 (k^{2n})t}{3\alpha^n}\right) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Therefore,

$$Q(kx) = k^2 Q(x) \tag{3.15}$$

for all $x \in X$. Replacing x, y by $k^n x, k^n y$ in (3.2), respectively, we obtain

$$N\left(\frac{1}{k^{2n}} D_f(k^n x, k^n y), t\right) \geq N'(\varphi_q(k^n x, k^n y), k^{2n} t) = N'\left(\varphi_q(x, y), \frac{k^{2n} t}{\alpha^n}\right)$$

which tends to 1 as $n \to \infty$ for all $x, y \in X$ and all $t > 0$. So, we see that Q satisfies (1.5). Thus, by Lemma 3.1, the function $x \rightsquigarrow f(x)$ is quadratic. Therefore, (3.15) implies that the function Q is quadratic.

Now, to prove the uniqueness property of Q, let Q': $X \to Y$ be another quadratic function satisfying (3.3). It follows from (3.3), (3.7) and (3.15) that

$$\begin{aligned}N(Q(x)-Q'(x),t) &= N\left(\frac{Q(k^n x)}{k^{2n}}-\frac{Q'(k^n x)}{k^{2n}},t\right)\\ &\geq \min\left\{N\left(\frac{Q(k^n x)}{k^{2n}}-\frac{f(k^n x)}{k^{2n}},\frac{t}{2}\right),N\left(\frac{f(k^n x)}{k^{2n}}-\frac{Q'(k^n x)}{k^{2n}},\frac{t}{2}\right)\right\}\\ &\geq N'\left(\varphi_q(0,k^n x),\frac{k^{2n}(k^2-\alpha)t}{4}\right)=N'\left(\varphi_q(0,x),\frac{k^{2n}(k^2-\alpha)t}{4\alpha^n}\right)\end{aligned}$$

for all $x \in X$ and all $t > 0$. Since $\alpha < k^2$, we obtain $\lim_{n\to\infty} N'\left(\varphi_q(0,x),\frac{k^{2n}(k^2-\alpha)t}{4\alpha^n}\right)=1$. Thus, $Q(x) = Q'(x)$.

Case (2): $\ell = -1$. We can state the proof in the same pattern as we did in the first case.

Replacing x by $\frac{x}{k}$ in (3.5), we obtain

$$N\left(f(x)-k^2 f\left(\frac{x}{k}\right),\frac{t}{2}\right)\geq N'\left(\varphi_q\left(0,\frac{x}{k}\right),t\right) \tag{3.16}$$

for all $x \in X$ and all $t > 0$. Replacing x and t by $\frac{x}{k^n}$ and $\frac{t}{k^{2n}}$ in (3.16), respectively, we obtain

$$N\left(k^{2n}f\left(\frac{x}{k^n}\right)-k^{2(n+1)}f\left(\frac{x}{k^{n+1}}\right),\frac{t}{2}\right)\geq N'\left(\varphi_q\left(0,\frac{x}{k^{n+1}}\right),\frac{t}{k^{2n}}\right)=N'\left(\varphi_q(0,x),\left(\frac{\alpha}{k^2}\right)^n\alpha t\right)$$

for all $x \in X$, all $t > 0$ and all $n > 0$. One can deduce

$$N\left(k^{2(n+m)}f\left(\frac{x}{k^{n+m}}\right)-k^{2m}f\left(\frac{x}{k^m}\right),t\right)\geq N'\left(\varphi_q(0,x),\frac{t}{\sum_{j=m+1}^{n+m}\frac{2k^{2j}}{k^2\alpha^j}}\right) \tag{3.17}$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n \geq 0$. From which we conclude that $\left\{k^{2n}f\left(\frac{x}{k^n}\right)\right\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N). Therefore, there is a mapping $Q: X \to Y$ defined by $Q(x) := N-\lim_{n\to\infty} k^{2n}f\left(\frac{x}{k^n}\right)$. Employing (3.17) with $m = 0$, we obtain

$$N(Q(x)-f(x),t)\geq N'\left(\varphi_q(0,x),\frac{(\alpha-k^2)t}{2}\right)$$

for all $x \in X$ and all $t > 0$. The proof for uniqueness of Q for this case proceeds similarly to that in the previous case, hence it is omitted. □

Remark 3.3. Let $0 < \alpha < k^2$. Suppose that the function $t \mapsto N(f(x) - Q(x), .)$ from $(0, \infty)$ into $[0, 1]$ is right continuous. Then, we obtain a better fuzzy (3.14) as follows.

We obtain

$$\begin{aligned}N(Q(x)-f(x),t+s) &\geq \min\left\{N\left(Q(x)-\frac{f(k^n x)}{k^{2n}},s\right),N\left(\frac{f(k^n x)}{k^{2n}}-f(x),t\right)\right\}\\ &\geq N'\left(\varphi_q(0,x),\frac{t}{\sum_{j=0}^{n-1}\frac{\alpha^j}{k^2(k)^{2j}})}\right)\\ &\geq N'\left(\varphi_q(0,x),(k^2-\alpha)t\right).\end{aligned}$$

Tending s to zero we infer that

$$N(Q(x) - f(x), t) \geq N'(\varphi_q(0, x), (k^2 - \alpha)t)$$

for all $x \in X$ and all $t > 0$.

From Theorem 3.2, we obtain the following corollary concerning the generalized Hyers-Ulam stability [4] of quadratic mappings satisfying (1.5), in normed spaces.

Corollary 3.4. *Let X be a normed space and Y be a Banach space. Let ε, λ be non-negative real numbers such that λ ≠ 2. Suppose that an even mapping f : X → Y with f (0) = 0 satisfies*

$$\| D_f(x, y) \| \leq \varepsilon(\| x\|^{\lambda} + \| y\|^{\lambda}) \tag{3.18}$$

for all x, y ∈ X. Then, the limit

$$Q(x) = N - \lim_{n\to\infty} \frac{f(k^{\ell n}x)}{k^{2\ell n}})$$

exists for all x ∈ X and Q : X → Y is a unique quadratic mapping satisfying

$$||f(x) - Q(x)|| \leq \frac{2\varepsilon \| x \|^{\lambda}}{\ell(k^2 - k^{\lambda})} \tag{3.19}$$

for all x ∈ X, where λℓ < 2ℓ.

Proof. Define the function N by

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

It is easy to see that (X, N) is a fuzzy normed space and (Y, N) is a fuzzy Banach space. Denote $\phi_q : X \times X \to \mathbb{R}$, the function sending each (x, y) to $\varepsilon(||x||^{\lambda} + ||y||^{\lambda})$. By assumption

$$N(D_f(x, y), t) \geq N'(\varphi_q(x, y), t)$$

note that N': $\mathbb{R} \times \mathbb{R} \to [0, 1]$ given by

$$N'(x, t) = \begin{cases} \frac{t}{t+|x|}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

is a fuzzy norm on $\mathbb{R}$. By Theorem 3.2, there exists a unique quadratic mapping $Q : X \to Y$ satisfying the equation (1.5) and

$$\begin{aligned} \frac{t}{t+ \| f(x) - Q(x) \|} &= N(f(x) - Q(x), t) \\ &\geq N'\left(\varphi_q(0, x), \frac{\ell(k^2 - k^{\lambda})t}{2}\right) \\ &= N'\left(\varepsilon \| x\|^{\lambda}, \frac{\ell(k^2 - k^{\lambda})t}{2}\right) = \frac{\ell(k^2 - k^{\lambda})t}{\ell(k^2 - k^{\lambda})t + 2\varepsilon \| x\|^{\lambda}} \end{aligned}$$

and thus

$$\frac{t}{t+ \| f(x) - Q(x) \|} \geq \frac{\ell(k^2 - k^{\lambda})t}{\ell(k^2 - k^{\lambda})t + 2\varepsilon \| x\|^{\lambda}}$$

which implies that, $\ell(k^2 - k^{\lambda})||f(x) - Q(x)|| \leq 2\varepsilon||x||^{\lambda}$ for all $x \in X$. □

In the following theorem, we will show that under some extra conditions on Theorem 3.2, the quadratic function $r \mapsto Q(rx)$ is fuzzy continuous. It follows that in such a case, $Q(rx) = r^2Q(x)$ for all $x \in X$ and $r \in \mathbb{R}$.

In the following result, we will assume that all conditions of the theorem 3.2 hold.

Theorem 3.5. *Denote N_1 the fuzzy norm obtained as Corollary 3.4 on $\mathbb{R}$. Let for all $x \in X$, the functions $r \mapsto f(rx)$ (from $(\mathbb{R}, N_1)$ into (Y, N)) and $r \mapsto \phi_q(0, rx)$ (from $(\mathbb{R}, N_1)$ into (Z, N')) be fuzzy continuous. Then, for all $x \in X$, the function $r \mapsto Q(rx)$ is fuzzy continuous and $Q(rx) = r^2Q(x)$ for all $r \in \mathbb{R}$.*

Proof. Case (1): $\ell = 1$. Let $\{r_k\}$ be a sequence in $\mathbb{R}$ that converge to some $r \in \mathbb{R}$, and let $t > 0$. Let $\varepsilon > 0$ be given, since $0 < \alpha < k^2$, so $\lim_{n\to\infty} \frac{(k^2-\alpha)k^{2n}t}{12\alpha^n} = \infty$, there is $m \in \mathbb{N}$ such that

$$N'\left(\varphi_q(0, rx), \frac{(k^2-\alpha)k^{2m}t}{12\alpha^m}\right) > 1 - \varepsilon \tag{3.20}$$

It follows from (3.14) and (3.20) that

$$N\left(\frac{f(k^m rx)}{k^{2m}} - \frac{Q(k^m rx)}{k^{2m}}, \frac{t}{3}\right) > 1 - \varepsilon \tag{3.21}$$

By the fuzzy continuity of functions $r \mapsto f(rx)$ and $r \mapsto \phi_q(0, rx)$, we can find some $\mathcal{J} \in \mathbb{N}$ such that for any $n \geq j$,

$$N\left(\frac{f(k^m r_k x)}{k^{2m}} - \frac{f(k^m rx)}{k^{2m}}, \frac{t}{3}\right) > 1 - \varepsilon \tag{3.22}$$

and

$$N'(\varphi_q(0, r_k x) - \varphi_q(0, rx), \frac{(k^2-\alpha)k^{2m}t}{12\alpha^m}) > 1 - \varepsilon \tag{3.23}$$

It follows from (3.20) and (3.23) that

$$N'(\varphi_q(0, r_k x), \frac{(k^2-\alpha)k^{2m}t}{6\alpha^m}) > 1 - \varepsilon \tag{3.24}$$

On the other hand,

$$\begin{aligned} N\left(Q(r_k x) - \frac{f(k^m r_k x)}{k^{2m}}, \frac{t}{k^{2m}}\right) &= N\left(\frac{Q(k^m r_k x)}{k^{2m}} - \frac{f(k^m r_k x)}{k^{2m}}, \frac{t}{k^{2m}}\right) \\ &\geq N'\left(\varphi_q(0, r_k x), \frac{(k^2-\alpha)t}{2\alpha^m}\right) \end{aligned} \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$N\left(Q(r_k x) - \frac{f(k^m r_k x)}{k^{2m}}, \frac{t}{3}\right) > 1 - \varepsilon \tag{3.26}$$

So, it follows from (3.21), (3.22) and (3.26) that for any $n \geq j$,

$$N(Q(r_k x) - Q(rx), t) > 1 - \varepsilon$$

Therefore, for every choice $x \in X$, $t > 0$, and $\varepsilon > 0$, we can find some $\mathcal{J} \in \mathbb{N}$ such that $N(Q(r_k x) - Q(rx), t) > 1 - \varepsilon$ for every $n \geq \mathcal{J}$. This shows that $Q(r_k x) \to Q(rx)$. The proof for $\ell = -1$ proceeds similarly to that in the previous case.

It is not hard to see that $Q(rx) = r^2Q(x)$ for each rational number r. Since Q is a fuzzy continuous function, by the same reasoning as in the proof of [45], the quadratic mapping $Q : X \to Y$ satisfies $Q(rx) = r^2Q(x)$ for each $r \in \mathbb{R}$. □

4. Fuzzy stability of the functional equation (1.5): an odd case

In this section, we prove the generalized Hyers-Ulam stability of the functional equation (1.5) in fuzzy Banach spaces for an odd case.

Lemma 4.1. [22,24]. *If an odd mapping* $f : V_1 \to V_2$ *satisfies* (1.5), *then the mapping* $g : V_1 \to V_2$, *defined by* $g(x) = f(2x) - 8f(x)$, *is additive.*

Theorem 4.2. *Let* $\ell \in \{-1, 1\}$ *be fixed and let* $\phi_q : X \times X \to Z$ *be a function such that*

$$\varphi_a(2x, 2y) = \alpha\varphi_a(x, y) \tag{4.1}$$

for all $x, y \in X$ *and for some positive real number* α *with* $\alpha\ell < 2\ell$. *Suppose that an odd mapping* $f : X \to Y$ *satisfies the inequality*

$$N(D_f(x, y), t) \geq N'(\varphi_a(x, y), t) \tag{4.2}$$

for all $x, y \in X$ *and all* $t > 0$. *Then, the limit*

$$A(x) = N - \lim_{n\to\infty} \frac{1}{2^{\ell n}}(f(2^{\ell n+1}x) - 8f(2^{\ell n}x))$$

exists for all $x \in X$ *and* $A : X \to Y$ *is a unique additive mapping satisfying*

$$N(f(2x) - 8f(x) - A(x), t) \geq M_a\left(x, \frac{\ell(2-\alpha)}{2}t\right) \tag{4.3}$$

for all $x \in X$ *and all* $t > 0$, *where*

$$\begin{aligned} M_a(x, t) = \min\Bigg\{ & N'\left(\varphi_a(x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_a(2x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a(x, 2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_a((k+1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a((k-1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right)), N'\left(\varphi_a(2x, 2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a(x, 3x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_a((2k+1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a((2k-1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right)\Bigg\}. \end{aligned}$$

Proof. Case (1): $\ell = 1$. It follows from (4.2) and using oddness of f that

$$\begin{aligned} & N(f(ky+x) - f(ky-x) - k^2f(x+y) - k^2f(x-y) + 2(k^2-1)f(x), t) \\ & \quad \geq N'(\varphi_a(x, y), t) \end{aligned} \tag{4.4}$$

for all $x, y \in X$ and all $t > 0$. Putting $y = x$ in (4.4), we have

$$N(f((k+1)x) - f((k-1)x) - k^2f(2x) + 2(k^2-1)f(x), t) \geq N'(\varphi_a(x, x), t) \tag{4.5}$$

for all $x \in X$ and all $t > 0$. It follows from (4.5) that

$$N(f(2(k+1)x) - f(2(k-1)x) - k^2f(4x) + 2(k^2-1)f(2x), t) \geq N'(\varphi_a(2x, 2x), t) \tag{4.6}$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $2x$ and x in (4.4), respectively, we get

$$\begin{aligned} &N(f((k+2)x) - f((k-2)x) - k^2 f(3x) - k^2 f(x) + 2(k^2-1)f(2x), t) \\ &\quad \geq N'(\varphi_a(2x, x), t) \end{aligned} \tag{4.7}$$

for all $x \in X$. Setting $y = 2x$ in (4.4), we have

$$\begin{aligned} &N(f((2k+1)x) - f((2k-1)x) - k^2 f(3x) - k^2 f(-x) + 2(k^2-1)f(x), t) \\ &\quad \geq N'(\varphi_a(x, 2x), t) \end{aligned} \tag{4.8}$$

for all $x \in X$ and all $t > 0$. Putting $y = 3x$ in (4.4), we obtain

$$\begin{aligned} &N(f((3k+1)x) - f((3k-1)x) - k^2 f(4x) - k^2 f(-2x) + 2(k^2-1)f(x), t) \\ &\quad \geq N'(\varphi_a(x, 3x), t) \end{aligned} \tag{4.9}$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $(k + 1)x$ and x in (4.4), respectively, we get

$$\begin{aligned} &N(f((2k+1)x) - f(-x) - k^2 f((k+2)x) - k^2 f(kx) + 2(k^2-1)f((k+1)x), t) \\ &\quad \geq N'(\varphi_a((k+1)x, x), t) \end{aligned} \tag{4.10}$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $(k - 1)x$ and x in (4.4), respectively, one gets

$$\begin{aligned} &N(f((2k-1)x) - f(x) - k^2 f((k-2)x) - k^2 f(kx) + 2(k^2-1)f((k-1)x), t) \\ &\quad \geq N'(\varphi_a((k-1)x, x), t) \end{aligned} \tag{4.11}$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $(2k + 1)x$ and x in (4.4), respectively, we obtain

$$\begin{aligned} &N(f((3k+1)x) - f(-(k+1)x) - k^2 f(2(k+1)x) - k^2 f(2kx) \\ &\quad +2(k^2-1)f((2k+1)x), t) \geq N'(\varphi_a((2k+1)x, x), t) \end{aligned} \tag{4.12}$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $(2k - 1)x$ and x in (4.4), respectively, we have

$$\begin{aligned} &N(f((3k-1)x) - f(-(k-1)x) - k^2 f(2(k-1)x) - k^2 f(2kx) \\ &\quad +2(k^2-1)f((2k-1)x), t) \geq N'(\varphi_a((2k-1)x, x), t) \end{aligned} \tag{4.13}$$

for all $x \in X$ and all $t > 0$. It follows from (4.5), (4.7), (4.8), (4.10) and (4.11) that

$$\begin{aligned} &N\left(2f(3x) - 8f(2x) + 10f(x), \frac{2}{k^2(k^2-1)}(2(k^2-1) + k^2 + 3)t\right) \\ &\quad \geq \min\left\{N'(\varphi_a(x, x), t), N'(\varphi_a(2x, x), t), N'(\varphi_a(x, 2x), t), \right. \\ &\qquad \left. N'(\varphi_a((k+1)x, x), t), N'(\varphi_a((k-1)x, x), t)\right\} \end{aligned} \tag{4.14}$$

for all $x \in X$ and all $t > 0$. And, from (4.5), (4.6), (4.8), (4.9), (4.12) and (4.14), we conclude that

$$\begin{aligned} &N\left(f(4x) - 2f(3x) - 2f(2x) + 6f(x), \frac{1}{k^2(k^2-1)}(2(k^2-1) + k^2 + 4)t\right) \\ &\quad \geq \min\{N'(\varphi_a(x, x), t), N'(\varphi_a(2x, 2x), t), N'(\varphi_a(x, 2x), t), N'(\varphi_a(x, 3x), t), \\ &\qquad N'(\varphi_a((2k+1)x, x), t), N'(\varphi_a((2k-1)x, x), t)\} \end{aligned} \tag{4.15}$$

for all $x \in X$ and all $t > 0$. Finally, by using (4.14) and (4.15), we obtain that Similar to the proof Theorem 3.2, we have

$$N\left(f(4x) - 10f(2x) + 16f(x), \frac{9k^2+4}{k^2(k^2-1)}t\right) \geq \min\{N'(\varphi_a(x,x),t), N'(\varphi_a(2x,x),t), N'(\varphi_a(x,2x),t), N'(\varphi_a((k+1)x,x),t), N'(\varphi_a((k-1)x,x),t)), N'(\varphi_a(2x,2x),t), N'(\varphi_a(x,3x),t), N'(\varphi_a((2k+1)x,x),t), N'(\varphi_a((2k-1)x,x),t)\} \tag{4.16}$$

for all $x \in X$ and all $t > 0$, where

$$\begin{aligned} M_a(x,t) = \min\Bigg\{ & N'\left(\varphi_a(x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_a(2x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a(x,2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_a((k+1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a((k-1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right)), N'\left(\varphi_a(2x,2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a(x,3x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_a((2k+1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_a((2k-1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right)\Bigg\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus, (4.16) means that

$$N(f(4x) - 10f(2x) + 16f(x), t) \geq M_a(x,t) \tag{4.17}$$

for all $x \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 8f(x)$ for all $x \in X$. From (4.17), we conclude that

$$N(g(2x) - 2g(x), t) \geq M_a(x,t) \tag{4.18}$$

for all $x \in X$ and all $t > 0$. So

$$N\left(\frac{g(2x)}{2} - g(x), \frac{t}{2}\right) \geq M_a(x,t) \tag{4.19}$$

for all $x \in X$ and all $t > 0$. Then, by our assumption

$$M_a(2x,t) = M_a\left(x, \frac{t}{\alpha}\right) \tag{4.20}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^n x$ in (4.19) and using (4.20), we obtain

$$N\left(\frac{g(2^{n+1}x)}{2^{n+1}} - \frac{g(2^n x)}{2^n}, \frac{t}{2(2^n)}\right) \geq M_a(2^n x, t) = M_a\left(x, \frac{t}{\alpha^n}\right) \tag{4.21}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Replacing t by $\alpha^n t$ in (4.21), we see that

$$N\left(\frac{g(2^{n+1}x)}{2^{n+1}} - \frac{g(2^n x)}{2^n}, \frac{t\alpha^n}{2(2^n)}\right) \geq M_a(x,t) \tag{4.22}$$

for all $x \in X$, $t > 0$ and $n > 0$. It follows from $\frac{g(2^n x)}{2^n} - g(x) = \sum_{j=0}^{n-1}\left(\frac{g(2^{j+1}x)}{2^{j+1}} - \frac{g(2^j x)}{2^j}\right)$ and (4.22) that

$$N\left(\frac{g(2^n x)}{2^n} - g(x), \sum_{j=0}^{n-1} \frac{\alpha^j t}{2(2)^j}\right) \geq \min \bigcup_{j=0}^{n-1} \left\{ N\left(\frac{g(2^{j+1}x)}{2^{j+1}} - \frac{g(2^j x)}{2^j}, \frac{\alpha^j t}{2(2)^j}\right)\right\} \geq M_a(x,t) \tag{4.23}$$

for all $x \in X$, $t > 0$ and $n > 0$. Replacing x by $2^m x$ in (4.23), we observe that

$$N\left(\frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^m x)}{2^m}, \sum_{j=0}^{n-1} \frac{\alpha^j t}{2(2)^{j+m}}\right) \geq M_a(2^m x, t) = M_a\left(x, \frac{t}{\alpha^m}\right)$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. So

$$N\left(\frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^m x)}{2^m}, \sum_{j=m}^{n+m-1} \frac{\alpha^j t}{2(2)^j}\right) \geq M_a(x,t)$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. Hence

$$N\left(\frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^m x)}{2^m}, t\right) \geq M_a\left(x, \frac{t}{\sum_{j=m}^{n+m-1} \frac{\alpha^j}{2(2)^j}}\right) \tag{4.24}$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. Since $0 < \alpha < 2$ and $\sum_{n=0}^{\infty} \left(\frac{\alpha}{2}\right)^n < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\left\{\frac{g(2^n x)}{2^n}\right\}$ is a Cauchy sequence in (Y, N) to some point $A(x) \in Y$. So one can define the mapping $A : X \to Y$ by

$$A(x) = N - \lim_{n\to\infty} \frac{g(2^n x)}{2^n}) \tag{4.25}$$

for all $x \in X$. Fix $x \in X$ and put $m = 0$ in (4.24) to obtain

$$N\left(\frac{g(2^n x)}{2^n} - g(x), t\right) \geq M_a\left(x, \frac{t}{\sum_{j=0}^{n-1} \frac{\alpha^j}{2(2)^j}}\right)$$

for all $x \in X$, $t > 0$ and $n > 0$. From which we obtain

$$N(A(x) - g(x), t) \geq \min\left\{N\left(A(x) - \frac{g(2^n x)}{2^n}, \frac{t}{2}\right), N\left(\frac{g(2^n x)}{2^n} - g(x), \frac{t}{2}\right)\right\} \geq M_a\left(x, \frac{t}{\sum_{j=0}^{n-1} \frac{\alpha^j}{2^j}}\right) \tag{4.26}$$

for n large enough. Taking the limit as $n \to \infty$ in (4.26), we obtain

$$N(A(x) - g(x), t) \geq M_a\left(x, \frac{t(2-\alpha)}{2}\right) \tag{4.27}$$

for all $x \in X$ and all $t > 0$. It follows from (4.21) and (4.25) that

$$N\left(\frac{A(2x)}{2} - A(x), t\right) \geq \min\left\{N\left(\frac{A(2x)}{2} - \frac{g(2^{n+1}x)}{2^{n+1}}\right), \frac{t}{3}), N\left(\frac{g(2^n x)}{2^n} - A(x), \frac{t}{3}\right)\right.$$
$$\left., N\left(\frac{g(2^{n+1}x)}{2^{n+1}} - \frac{g(2^n x)}{2^n}\right), \frac{t}{3})\right\} = M_a\left(x, \frac{2(2)^n t}{3\alpha^n}\right)$$

for all $x \in X$ and all $t > 0$. Therefore,

$$A(2x) = 2A(x) \tag{4.28}$$

for all $x \in X$. Replacing x, y by $2^n x$, $2^n y$ in (4.2), respectively, we obtain

$$\begin{aligned} N\left(\frac{1}{2^n} D_g(2^n x, 2^n y), t\right) &= N(D_f(2^{n+1}x, 2^{n+1}y) - 8D_f(2^n x, 2^n y), 2^n t) \\ &= \min\left\{N(D_f(2^{n+1}x, 2^{n+1}y), \frac{2^n t}{2}), N(D_f(2^n x, 2^n y), \frac{2^n t}{16})\right\} \\ &\geq \min\left\{N'\left(\varphi_a(2^{n+1}x, 2^{n+1}y), \frac{2^n t}{2}\right), N'\left(\varphi_a(2^n x, 2^n y), \frac{2^n t}{16}\right)\right\} \\ &= \min\left\{N'\left(\varphi_a(x, y), \frac{2^n t}{2\alpha^{n+1}}\right), N'\left(\varphi_a(x, y), \frac{2^n t}{16\alpha^n}\right)\right\} \end{aligned}$$

which tends to 1 as $n \to \infty$ for all x, $y \in X$ and all $t > 0$. So we see that A satisfies (1.5). Thus, by Lemma 4.1, the mapping $x \rightsquigarrow A(2x) - 8A(x)$ is additive. So (4.28) implies that the mapping A is additive.

The rest of the proof is similar to the proof of Theorem 3.2 and we omit the details. □

Remark 4.3. Let $0 < \alpha < 2$. Suppose that the function $t \mapsto N(f(2x) - 8f(x) - A(x), .)$ from $(0, \infty)$ into $[0, 1]$ is right continuous. Then, we obtain a better fuzzy approximation than (4.27).

Corollary 4.4. *Let X be a normed space and Y be a Banach space. Let ε, λ be non-negative real numbers such that λ ≠ 1. Suppose that an odd mapping $f : X \to Y$ satisfies the inequality* (3.18) *for all x, y ∈ X. Then, the limit*

$$A(x) = \lim_{n\to\infty} \frac{1}{2^{\ell n}} (f(2^{\ell n+1}x) - 8f(2^{\ell n}x))$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping satisfying

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{(|2k+\ell|^\lambda + 1)(9k^2+4)\varepsilon \, \| x\|^\lambda}{\ell k^2(k^2-1)(1-2^{\lambda-1})} \tag{4.29}$$

for all $x \in X$, where $\lambda\ell < \ell$.

Proof. The proof is similar to the proof of Corollary 3.4 and the result follows from Theorem 4.2. □

Theorem 4.5. *Denote N_1 the fuzzy norm obtained as Corollary* 3.4 *on R. Let for all x ∈ X, the functions $r \mapsto f(2rx) - 8f(rx)$ (from (R, N_1) into (Y, N)) and $r \mapsto \phi_a(\iota_1 rx, \iota_2 ry)$ (from (R, N_1) into (Z, N')) be fuzzy continuous, where $\iota_1 \in \{1, 2, (k+1), (k-1), (2k+1), (2k-1)\}$ and $\iota_2 \in \{1, 2, 3\}$. Then, for all $x \in X$, the function $r \mapsto A(rx)$ is fuzzy continuous and $A(rx) = rA(x)$ for all $r \in R$.*

Proof. The proof is similar to the proof of Theorem 3.5 and the result follows from Theorem 4.2. □

Lemma 4.6. [22,24]. *If an odd mapping* $f : V_1 \to V_2$ *satisfies* (1.5), *then the mapping* $h : V_1 \to V_2$ *defined by* $h(x) = f(2x) - 2f(x)$ *is cubic.*

Theorem 4.7. *Let* $\ell \in \{-1, 1\}$ *be fixed and let* $\phi_c : X \times X \to Z$ *be a mapping such that*

$$\varphi_c(2x, 2y) = \alpha\varphi_c(x, y) \tag{4.30}$$

for all $x, y \in X$ *and for some positive real number* α *with* $\alpha\ell < 8\ell$. *Suppose that an odd mapping* $f : X \to Y$ *satisfies the inequality*

$$N(D_f(x, y), t) \geq N'(\varphi_c(x, y), t) \tag{4.31}$$

for all $x, y \in X$ *and all* $t > 0$. *Then, the limit*

$$C(x) = N - \lim_{n\to\infty} \frac{1}{8^{\ell n}}(f(2^{\ell n+1}x) - 2f(2^{\ell n}x))$$

exists for all $x \in X$ *and* $C : X \to Y$ *is a unique cubic mapping satisfying*

$$N(f(2x) - 2f(x) - C(x), t) \geq M_c\left(x, \frac{\ell(8 - \alpha)}{2}t\right) \tag{4.32}$$

for all $x \in X$ *and all* $t > 0$, *where*

$$\begin{aligned} M_c(x, t) = \min\Bigg\{ & N'\left(\varphi_c(x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_c(2x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_c(x, 2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_c((k+1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_c((k-1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right)), N'\left(\varphi_c(2x, 2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_c(x, 3x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi_c((2k+1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi_c((2k-1)x, x), \frac{k^2(k^2-1)}{9k^2+4}t\right)\Bigg\}. \end{aligned}$$

Proof. Case (1): $\ell = 1$. Similar to the proof of Theorem 4.2, we have

$$N(f(4x) - 10f(2x) + 16f(x), t) \geq M_c(x, t)$$

for for all $x \in X$ and all $t > 0$, where $M_c(x, t)$ is defined as in above. Letting $h : X \to Y$ be a mapping defined by $h(x) := f(2x) - 2f(x)$. Then, we conclude that

$$N(h(2x) - 8h(x), t) \geq M_c(x, t) \tag{4.33}$$

for all $x \in X$ and all $t > 0$. So

$$N\left(\frac{h(2x)}{8} - h(x), \frac{t}{8}\right) \geq M_c(x, t) \tag{4.34}$$

for all $x \in X$ and all $t > 0$. Then, by our assumption

$$M_c(2x, t) = M_c\left(x, \frac{t}{\alpha}\right) \tag{4.35}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^n x$ in (4.34) and using (4.35), we obtain

$$N\left(\frac{h(2^{n+1}x)}{8^{n+1}} - \frac{h(2^n x)}{8^n}, \frac{t}{8(8^n)}\right) \geq M_c(2^n x, t) = M_c\left(x, \frac{t}{\alpha^n}\right) \tag{4.36}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Replacing t by $\alpha^n t$ in (4.36), we see that

$$N\left(\frac{h(2^{n+1}x)}{8^{n+1}} - \frac{h(2^n x)}{8^n}, \frac{t\alpha^n}{8(8^n)}\right) \geq M_c(x, t) \tag{4.37}$$

for all $x \in X$, $t > 0$ and $n > 0$. It follows from $\frac{h(2^n x)}{8^n} - h(x) = \sum_{j=0}^{n-1}\left(\frac{h(2^{j+1}x)}{8^{j+1}} - \frac{h(2^j x)}{8^j}\right)$ and (4.37) that

$$\begin{aligned} N\left(\frac{h(2^n x)}{8^n} - h(x), \sum_{j=0}^{n-1}\frac{\alpha^j t}{8(8)^j}\right) &\geq \min \bigcup_{j=0}^{n-1}\left\{N\left(\frac{h(2^{j+1}x)}{8^{j+1}} - \frac{h(2^j x)}{8^j}, \frac{\alpha^j t}{8(8)^j}\right)\right\} \\ &\geq M_c(x, t) \end{aligned} \tag{4.38}$$

for all $x \in X$, $t > 0$ and $n > 0$. Replacing x by $2^m x$ in (4.38), we observe that

$$N\left(\frac{h(2^{n+m}x)}{8^{n+m}} - \frac{h(2^m x)}{8^m}, \sum_{j=0}^{n-1}\frac{\alpha^j t}{8(8)^{j+m}}\right) \geq M_c(2^m x, t) = M_c\left(x, \frac{t}{\alpha^m}\right)$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. So

$$N\left(\frac{h(2^{n+m}x)}{8^{n+m}} - \frac{h(2^m x)}{8^m}, \sum_{j=m}^{n+m-1}\frac{\alpha^j t}{8(8)^j}\right) \geq M_c(x, t)$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. Hence

$$N\left(\frac{h(2^{n+m}x)}{8^{n+m}} - \frac{h(2^m x)}{8^m}, t\right) \geq M_c\left(x, \frac{t}{\sum_{j=m}^{n+m-1}\frac{\alpha^j}{8(8)^j})}\right) \tag{4.39}$$

for all $x \in X$, all $t > 0$ and all $m \geq 0$, $n > 0$. Since $0 < \alpha < 8$ and $\sum_{n=0}^{\infty}\left(\frac{\alpha}{8}\right)^n < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\left\{\frac{h(2^n x)}{8^n}\right\}$ is a Cauchy sequence in (Y, N) to some point $C(x) \in Y$. So one can define the mapping $C : X \to Y$ by

$$C(x) = N - \lim_{n\to\infty}\frac{h(2^n x)}{8^n}) \tag{4.40}$$

for all $x \in X$. Fix $x \in X$ and put $m = 0$ in (4.39) to obtain

$$N\left(\frac{h(2^n x)}{8^n} - h(x), t\right) \geq M_c\left(x, \frac{t}{\sum_{j=0}^{n-1}\frac{\alpha^j}{8(8)^j})}\right)$$

for all $x \in X$, $t > 0$ and $n > 0$. From which we obtain

$$\begin{aligned} N(C(x) - h(x), t) &\geq \min\left\{N(C(x) - \frac{h(2^n x)}{8^n}, \frac{t}{2}), N(\frac{h(2^n x)}{8^n} - h(x), \frac{t}{2})\right\} \\ &\geq M_c\left(x, \frac{t}{\sum_{j=0}^{n-1} \frac{\alpha^j}{4(8^j)})}\right) \end{aligned} \tag{4.41}$$

for n large enough. Taking the limit as $n \to \infty$ in (4.41), we obtain

$$N(C(x) - h(x), t) \geq M_a\left(x, \frac{t(8-\alpha)}{2}\right) \tag{4.42}$$

for all $x \in X$ and all $t > 0$. It follows from (4.36) and (4.40) that

$$\begin{aligned} N\left(\frac{C(2x)}{8} - C(x), t\right) \geq \min\Bigg\{&N\left(\frac{C(2x)}{8} - \frac{h(2^{n+1}x)}{8^{n+1}}\right), \frac{t}{3}), N\left(\frac{h(2^n x)}{8^n} - C(x), \frac{t}{3}\right), \\ &N\left(\frac{h(2^{n+1}x)}{8^{n+1}} - \frac{h(2^n x)}{8^n}\right), \frac{t}{3}) = M_c\left(x, \frac{8(8)^n t}{3\alpha^n}\right) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Therefore,

$$C(2x) = 8C(x) \tag{4.43}$$

for all $x \in X$. Replacing x, y by $2^n x$, $2^n y$ in (4.31), respectively, we obtain

$$\begin{aligned} N\left(\frac{1}{8^n} D_h(2^n x, 2^n y), t\right) &= N(D_f(2^{n+1}x, 2^{n+1}y) - 2D_f(2^n x, 2^n y), 8^n t) \\ &= \min\left\{N\left(D_f(2^{n+1}x, 2^{n+1}y), \frac{8^n t}{2}\right), N\left(D_f(2^n x, 2^n y), \frac{8^n t}{4}\right)\right\} \\ &\geq \min\left\{N'\left(\varphi_c(2^{n+1}x, 2^{n+1}y), \frac{8^n t}{2}\right), N'\left(\varphi_c(2^n x, 2^n y), \frac{8^n t}{4}\right)\right\} \\ &= \min\left\{N'\left(\varphi_c(x, y), \frac{8^n t}{2\alpha^{n+1}}\right), N'(\varphi_c(x, y), \frac{8^n t}{4\alpha^n})\right\} \end{aligned}$$

which tends to 1 as n $\to \infty$ for all x, $y \in X$ and all $t > 0$. So we see that C, satisfies (1.5). Thus, by Lemma 4.6, the mapping $x \rightsquigarrow C(2x) - 2C(x)$ is cubic. So (4.43) implies that the function C is cubic. The rest of the proof is similar to the proof of Theorem 3.2 and we omit the details. □

Remark 4.8. Let $0 < \alpha < 8$. Suppose that the function $t \mapsto N(f(2x) - 2f(x) - C(x), .)$ from $(0, \infty)$ into $[0, 1]$ is right continuous. Then, we obtain a better fuzzy approximation than(4.42).

Corollary 4.9. *Let X be a normed space and Y be a Banach space. Let* ε, λ *be non-negative real numbers such that* $\lambda \neq 3$. *Suppose that an odd mapping* $f : X \to Y$ *satisfies the inequality* (3.18) *for all* x, $y \in X$. *Then, the limit*

$$C(x) = \lim_{n\to\infty} \frac{1}{8^{\ell n}}(f(2^{\ell n+1}x) - 2f(2^{\ell n}x))$$

exists for all $x \in X$ *and* $C : X \to Y$ *is a unique cubic mapping satisfying*

$$\| f(2x) - 2f(x) - C(x)\| \leq \frac{(|2k+\ell|^{\lambda}+1)(9k^2+4)\varepsilon \|x\|^{\lambda}}{\ell k^2(k^2-1)(4-2^{\lambda-1})} \tag{4.44}$$

for all $x \in X$, *where* $\lambda\ell < 3\ell$.

Theorem 4.10. *Denote* N_1 *the fuzzy norm obtained as Corollary* 3.4 *on R. Let for all* $x \in X$, *the functions* $r \mapsto f(2rx) - 2f(rx)$ *(from* (R, N_1) *into* (Y, N)*) and* $r \mapsto \phi_q(\iota_1 rx, \iota_2 ry)$ *(from* (R, N_1) *into* (Z, N')*) be fuzzy continuous, where* $\iota_1 \in \{1, 2, (k+1), (k-1), (2k+1), (2k-1)\}$ *and* $\iota_2 \in \{1, 2, 3\}$. *Then, for all* $x \in X$, *the function* $r \mapsto C(rx)$ *is fuzzy continuous and* $C(rx) = r^3C(x)$ *for all* $r \in R$.

Proof. The proof is similar to the proof of Theorem 3.5 and the result follows from Theorem 4.7. □

Theorem 4.11. *Let* $\phi : X \times X \to Z$ *be a mapping such that*

$$\varphi(2x, 2y) = \alpha\varphi(x, y) \tag{4.45}$$

for all $x, y \in X$ *and for some positive real number* α. *Suppose that an odd mapping* $f : X \to Y$ *satisfies the inequality*

$$N(D_f(x, y), t) \geq N'(\varphi(x, y), t) \tag{4.46}$$

for all $x, y \in X$ *and all* $t > 0$. *Then, there exist a unique cubic mapping* $C : X \to Y$ *and a unique additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x) - C(x), t) \geq \begin{cases} \min\left\{M(x, \frac{3t(2-\alpha)}{2}), M(x, \frac{3t(8-\alpha)}{2})\right\}, & 0 < \alpha < 2 \\ \min\left\{M(x, \frac{3t(\alpha-2)}{2}), M(x, \frac{3t(8-\alpha)}{2})\right\}, & 2 < \alpha < 8 \\ \min\left\{M(x, \frac{3t(\alpha-2)}{2}), M(x, \frac{3t(\alpha-8)}{2})\right\}, & \alpha > 8 \end{cases} \tag{4.47}$$

for all $x \in X$ *and all* $t > 0$, *where*

$$\begin{aligned} M(x,t) = \min\Big\{ & N'\left(\varphi(x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi(2x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi(x,2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi((k+1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi((k-1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t)\right), N'\left(\varphi(2x,2x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi(x,3x), \frac{k^2(k^2-1)}{9k^2+4}t\right), N'\left(\varphi((2k+1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right), \\ & N'\left(\varphi((2k-1)x,x), \frac{k^2(k^2-1)}{9k^2+4}t\right)\Big\}. \end{aligned}$$

Proof. Case (1): $0 < \alpha < 2$. By Theorems 4.2 and 4.7, there exist an additive mapping $A_0 : X \to Y$ and a cubic mapping $C_0 : X \to Y$ such that

$$N(f(2x) - 8f(x) - A_0(x), t) \geq M\left(x, \frac{t(2-\alpha)}{2}\right) \tag{4.48}$$

and

$$N(f(2x) - 2f(x) - C_0(x), t) \geq M\left(x, \frac{t(8-\alpha)}{2}\right) \tag{4.49}$$

for all $x \in X$ and all $t > 0$. It follows from (4.48) and (4.49) that

$$N\left(f(x) + \frac{1}{6}A_0(x) - \frac{1}{6}C_0(x), t\right) \geq \min\left\{M\left(x, \frac{3t(2-\alpha)}{2}\right), M\left(x, \frac{3t(8-\alpha)}{2}\right)\right\} \quad (4.50)$$

for all $x \in X$ and all $t > 0$. Letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$ in (4.50), we obtain

$$N(f(x) - A(x) - C(x), t) \geq \min\left\{M\left(x, \frac{3t(2-\alpha)}{2}\right), M\left(x, \frac{3t(8-\alpha)}{2}\right)\right\} \quad (4.51)$$

for all $x \in X$ and all $t > 0$. To prove the uniqueness of A and C, let A', C': $X \to Y$ be another additive and cubic mappings satisfying (4.51). Let $\bar{A} = A - A'$ and $\bar{C} = C - C'$. So

$$\begin{aligned} N(\bar{A}(x) + \bar{C}(x), t) &\geq \min\left\{N\left(f(x) - A(x) - C(x), \frac{t}{2}\right), N\left(f(x) - A'(x) - C'(x), \frac{t}{2}\right)\right\} \\ &\geq \min\left\{M\left(x, \frac{3t(2-\alpha)}{4}\right), M\left(x, \frac{3t(8-\alpha)}{4}\right)\right\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Therefore, it follows from the last inequalities that

$$\begin{aligned} N(\bar{A}(2^n x) + \bar{C}(2^n x), 8^n t) &\geq \min\left\{M\left(2^n x, \frac{3(8^n)t(2-\alpha)}{4}\right), M\left(2^n x, \frac{3(8^n)t(8-\alpha)}{4}\right)\right\} \\ &= \min\left\{M\left(x, \frac{3(8^n)t(2-\alpha)}{4\alpha^n}\right), M\left(x, \frac{3(8^n)t(8-\alpha)}{4\alpha^n}\right)\right\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So, $\lim_{n\to\infty} N\left(\frac{1}{8^n}(\bar{A}(2^n x) + \bar{C}(2^n x)), t\right) = 1$, hence $\bar{C} = 0$ and then $\bar{A} = 0$. The rest of the proof, proceeds similarly to that in the previous case. □

Remark 4.12. Let $0 < \alpha < 2$. Suppose that the function $t \mapsto N(f(x) - A(x) - C(x), .)$ from $(0, \infty)$ into $[0, 1]$ is right continuous. Then, we obtain a better fuzzy approximation than (4.51).

Corollary 4.13. *Let X be a normed space and Y be a Banach space. Let ε, λ be non-negative real numbers. Suppose that an odd mapping $f : X \to Y$ satisfies the inequality* (3.18) *for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\begin{aligned} &\| f(x) - A(x) - C(x) \| \\ &\leq \begin{cases} \frac{1}{6}\frac{(|2k+\ell|^\lambda+1)(9k^2+4)\varepsilon}{k^2(k^2-1)}\left(\frac{1}{(1-2^{\lambda-1})} + \frac{1}{(4-2^{\lambda-1})}\right) \| x\|^\lambda, \ \lambda < 1 \\ \frac{1}{6}\frac{(|2k+\ell|^\lambda+1)(9k^2+4)\varepsilon}{k^2(k^2-1)}\left(\frac{1}{(2^{\lambda-1}-1)} + \frac{1}{(4-2^{\lambda-1})}\right) \| x\|^\lambda, \ 1 < \lambda < 3 \\ \frac{1}{6}\frac{(|2k+\ell|^\lambda+1)(9k^2+4)\varepsilon}{k^2(k^2-1)}\left(\frac{1}{(2^{\lambda-1}-1)} + \frac{1}{(2^{\lambda-1}-4)}\right) \| x\|^\lambda, \ \lambda > 3 \end{cases} \end{aligned} \quad (4.52)$$

for all $x \in X$.

Proof. The result follows by Corollaries 4.4 and 4.9. □

Theorem 4.14. *Denote N_1 the fuzzy norm obtained as Corollary* 3.4 *on R. Let for all $x \in X$, the functions $r \mapsto f(rx)$ (from (R, N_1) into (Y, N)) and $r \mapsto \phi(\iota_1 rx, \iota_2 ry)$ (from (R, N_1) into (Z, N')) be fuzzy continuous, where $\iota_1 \in$* {1, 2, (k + 1), (k - 1), (2k + 1), (2k - 1)} *and $\iota_2 \in$* {1, 2, 3}. *Then, for all $x \in X$, the function $r \mapsto A(rx) + C(rx)$ is fuzzy continuous and $A(rx) + C(rx) = rA(x) + r^3C(x)$ for all $r \in R$.*

Proof. The result follows by Theorems 4.5 and 4.10. □

5. Fuzzy stability of the functional equation (1.5)

In this section, we prove the generalized Hyers-Ulam stability of a mixed cubic, quadratic, and additive functional equation (1.5) in fuzzy Banach spaces.

Theorem 5.1. *Let $\phi : X \times X \to Z$ be a function which satisfies* (3.1) *and* (4.45) *for all $x, y \in X$ and for some positive real number α. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$N(D_f(x, y), t) \geq N'(\varphi(x, y), t) \tag{5.1}$$

for all $x, y \in X$ and all $t > 0$. Furthermore, assume that $f(0) = 0$ in (5.1) *for the case f is even. If $|k| = 2$, then there exist a unique cubic mapping $C : X \to Y$, a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that*

$$N(f(x) - C(x) - Q(x) - A(x), t) \geq \begin{cases} \min\{\tilde{M}_1(x, t), \tilde{M}_1(-x, t)\}, \ 0 < \alpha < 2 \\ \min\{\tilde{M}_2(x, t), \tilde{M}_2(-x, t)\}, \ 2 < \alpha < k^2 \\ \min\{\tilde{M}_3(x, t), \tilde{M}_3(-x, t)\}, \ k^2 < \alpha < 8 \\ \min\{\tilde{M}_4(x, t), \tilde{M}_4(-x, t)\}, \ \alpha > 8 \end{cases} \tag{5.2}$$

for all $x \in X$ and all $t > 0$, otherwise

$$N(f(x) - C(x) - Q(x) - A(x), t) \geq \begin{cases} \min\{\tilde{M}_1(x, t), \tilde{M}_1(-x, t)\}, \ 0 < \alpha < 2 \\ \min\{\tilde{M}_2(x, t), \tilde{M}_2(-x, t)\}, \ 2 < \alpha < 8 \\ \min\{\tilde{M}_5(x, t), \tilde{M}_5(-x, t)\}, \ 8 < \alpha < k^2 \\ \min\{\tilde{M}_4(x, t), \tilde{M}_4(-x, t)\}, \ \alpha > k^2 \end{cases} \tag{5.3}$$

for all $x \in X$ and all $t > 0$, where

$$\begin{aligned}
\tilde{M}_1(x, t) &= \min\left\{N'\left(\varphi(0, x), \frac{(k^2 - \alpha)}{4}t\right), M\left(x, \frac{3t(2 - \alpha)}{4}\right), M(x, \frac{3t(8 - \alpha)}{4}\right\}, \\
\tilde{M}_2(x, t) &= \min\left\{N'\left(\varphi(0, x), \frac{(k^2 - \alpha)}{4}t\right), M\left(x, \frac{3t(\alpha - 2)}{4}\right), M(x, \frac{3t(8 - \alpha)}{4}\}\right., \\
\tilde{M}_3(x, t) &= \min\left\{N'\left(\varphi(0, x), \frac{(\alpha - k^2)}{4}t\right), M\left(x, \frac{3t(\alpha - 2)}{4}\right), M(x, \frac{3t(8 - \alpha)}{4}\right\}, \\
\tilde{M}_4(x, t) &= \min\left\{N'\left(\varphi(0, x), \frac{(\alpha - k^2)}{4}t\right), M\left(x, \frac{3t(\alpha - 2)}{4}\right), M(x, \frac{3t(\alpha - 8)}{4}\right\}, \\
\tilde{M}_5(x, t) &= \min\left\{N'\left(\varphi(0, x), \frac{(k^2 - \alpha)}{4}t\right), M\left(x, \frac{3t(\alpha - 2)}{4}\right), M(x, \frac{3t(\alpha - 8)}{4}\right\}
\end{aligned}$$

and

$$\begin{aligned}
M(x, t) = \min\Bigg\{ & N'\left(\varphi(x, x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right), N'\left(\varphi(2x, x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right), \\
& N'\left(\varphi(x, 2x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right), N'\left(\varphi((k + 1)x, x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right), \\
& N'\left(\varphi((k - 1)x, x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t)\right), N'\left(\varphi(2x, 2x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right), \\
& N'\left(\varphi(x, 3x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right), N'\left(\varphi((2k + 1)x, x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right), \\
& N'\left(\varphi((2k - 1)x, x), \frac{k^2(k^2 - 1)}{9k^2 + 4}t\right)\Bigg\}.
\end{aligned}$$

Proof. Case (1): $0 < \alpha < 2$. Assume that $\phi : X \times X \to Z$ satisfies (1.6) for all $x, y \in X$. Let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$, then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$N(D_{f_e}(x,y),t) \geq \min\{N'(\varphi(x,y),t), N'(\varphi(-x,-y),t)\}$$

for all $x, y \in X$ and all $t > 0$. By Theorem 3.2 for all $x, y \in X$, there exist a unique quadratic mapping $Q : X \to Y$ such that

$$N(f_e(x) - Q(x), t) \geq \min\left\{N'\left(\varphi(0,x), \frac{(k^2-\alpha)}{2}t\right), N'\left(\varphi(0,-x), \frac{(k^2-\alpha)}{2}t\right)\right\} \tag{5.4}$$

for all $x \in X$ and all $t > 0$. Now, if $\phi : X \times X \to Z$ satisfies (4.45) for all $x, y \in X$, and let $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$, then

$$N(D_{f_o}(x,y),t) \geq \min\{N'(\varphi(x,y),t), N'(\varphi(-x,-y),t)\}$$

for all $x, y \in X$ and all $t > 0$. By Theorem 4.11, it follows that there exist a unique cubic mapping $C : X \to Y$ and a unique additive mapping A ; $X \to Y$ such that

$$\begin{aligned} N(f_o(x) - C(x) - A(x), t) \geq \min&\left\{M\left(x, \frac{3t(2-\alpha)}{2}\right), M\left(x, \frac{3t(8-\alpha)}{2}\right)\right. \\ &\left., M\left(-x, \frac{3t(2-\alpha)}{2}\right), M\left(-x, \frac{3t(8-\alpha)}{2}\right)\right\} \end{aligned} \tag{5.5}$$

for all $x \in X$ and all $t > 0$. It follows from (5.4) and (5.5) that

$$\begin{aligned} &N(f(x) - C(x) - Q(x) - A(x), t) \\ &\geq \min\left\{N'\left(\varphi(0,x), \frac{(k^2-\alpha)}{4}t\right), M\left(x, \frac{3t(2-\alpha)}{4}\right), M\left(x, \frac{3t(8-\alpha)}{4}\right),\right. \\ &\quad \left. N'\left(\varphi(0,-x), \frac{(k^2-\alpha)}{4}t\right), M\left(-x, \frac{3t(2-\alpha)}{4}\right), M(-x, \frac{3t(8-\alpha)}{4}\right\} \\ &= \min\left\{\tilde{M}_1(x,t), \tilde{M}_1(-x,t)\right\} \end{aligned} \tag{5.6}$$

The rest of the proof proceeds similarly to that in the previous case. □

Remark 5.2. Let $0 < \alpha < 2$. Suppose that the function $t \mapsto N(f(x) - C(x) - Q(x) - A(x), .)$ from $(0, \infty)$ into $[0, 1]$ is right continuous. Then, we obtain a better fuzzy approximation than (5.2) or (5.3).

Corollary 5.3. *Let X be a normed space and Y be a Banach space. Let ε, λ be non-negative real numbers. Suppose that $f(0) = 0$ in (3.18) for the case $f : X \to Y$ is even. Then, there exist a unique cubic mapping $C : X \to Y$, a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned} &\| f(x) - C(x) - Q(x) - A(x) \| \\ &\leq \begin{cases} \frac{1}{6}\frac{(|2k+\ell|^\lambda+1)(9k^2+4)\varepsilon}{k^2(k^2-1)}\left(\frac{1}{(1-2^{\lambda-1})} + \frac{1}{(4-2^{\lambda-1})}\right)\| x\|^\lambda + \frac{2\varepsilon}{k^2-k^\lambda}\| x\|^\lambda, & \lambda < 1 \\ \frac{1}{6}\frac{(|2k+\ell|^\lambda+1)(9k^2+4)\varepsilon}{k^2(k^2-1)}\left(\frac{1}{(2^{\lambda-1}-1)} + \frac{1}{(4-2^{\lambda-1})}\right)\| x\|^\lambda + \frac{2\varepsilon}{k^2-k^\lambda}\| x\|^\lambda, & 1 < \lambda < 2 \\ \frac{1}{6}\frac{(|2k+\ell|^\lambda+1)(9k^2+4)\varepsilon}{k^2(k^2-1)}\left(\frac{1}{(2^{\lambda-1}-1)} + \frac{1}{(4-2^{\lambda-1})}\right)\| x\|^\lambda + \frac{2\varepsilon}{k^\lambda-k^2}\| x\|^\lambda, & 2 < \lambda < 3 \\ \frac{1}{6}\frac{(|2k+\ell|^\lambda+1)(9k^2+4)\varepsilon}{k^2(k^2-1)}\left(\frac{1}{(2^{\lambda-1}-1)} + \frac{1}{(2^{\lambda-1}-4)}\right)\| x\|^\lambda + \frac{2\varepsilon}{k^\lambda-k^2}\| x\|^\lambda, & \lambda > 3 \end{cases} \end{aligned} \tag{5.7}$$

for all $x \in X$.

Proof. The result follows by Corollaries 3.4 and 4.13. □

Theorem 5.4. *Denote N_1 the fuzzy norm obtained as Corollary 3.4 on R. Let for all x ∈ X, the functions $r \mapsto f(rx)$ (from (R, N_1) into (Y, N)) and $r \mapsto \phi(\iota_1 rx, \iota_2 ry)$ (from (R, N_1) into (Z, N')) be fuzzy continuous, where $\iota_1 \in \{0, \pm 1, \pm 2, \pm (k+1), \pm (k-1), \pm (2k+1), \pm (2k-1)\}$ and $\iota_2 \in \{\pm 1, \pm 2, \pm 3\}$. Then, for all $x \in X$, the function $r \mapsto C(rx) + Q(rx) + A(rx)$ is fuzzy continuous and $C(rx) + Q(rx) + A(rx) = r^3C(x) + r^2Q(x) + rA(x)$ for all $r \in R$.*

Proof. The result follows by Theorems 3.5 and 4.14. □

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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