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Convolution estimates related to space curves

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Abstract

Based on a uniform estimate of convolution operators with measures on a family of plane curves, we obtain optimal L^p-L^q boundedness of convolution operators with affine arclength measures supported on space curves satisfying a suitable condition. The result generalizes the previously known estimates.

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1 Introduction

Let $I \subset \mathbb{R}$ be an open interval and $\psi : I \to \mathbb{R}$ be a C^3 function. Let $\gamma : I \to \mathbb{R}^3$ be the curve given by $\gamma(t) = (t, t^2/2, \psi(t))$, $t \in I$. Associated to γ is the affine arclength measure $d\sigma_\gamma$ on $\mathbb{R}^3$ determined by

$$\int_{\mathbb{R}^3} f \, d\sigma_\gamma = \int_I f(\gamma(t))\lambda(t)\, dt, \qquad f \in C_0^\infty(\mathbb{R}^3)$$

with

$$\lambda(t) = \left|\psi^{(3)}(t)\right|^{\frac{1}{6}}, \quad t \in I.$$

The L^p - L^q mapping properties of the corresponding convolution operator T_{σ_γ} given by

$$T_{\sigma_\gamma} f(x) = f * \sigma_\gamma(x) = \int_I f(x - \gamma(t))\, \lambda(t)\, dt \tag{1.1}$$

have been studied by many authors [1-8]. The use of the affine arclength measure was suggested by Drury [2] to mitigate the effect of degeneracy and has been helpful to obtain uniform estimates.

We denote by Δ the closed convex hull of $\{(0, 0), (1, 1), (p_0^{-1}, q_0^{-1})\ (p_1^{-1}, q_1^{-1})\}$ in the plane, where $p_0 = 3/2$, $q_0 = 2$, $p_1 = 2$ and $q_1 = 3$. The line segment joining (p_0^{-1}, q_0^{-1}) and (p_1^{-1}, q_1^{-1}) is denoted by $\mathfrak{S}$. It is well known that the typeset of T_{σ_γ} is contained in Δ and that under suitable conditions T_{σ_γ} is bounded from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ with uniform bounds whenever $(p^{-1}, q^{-1}) \in \mathfrak{S}$. The most general result currently available was obtained by Oberlin [5]. In this article, we establish uniform endpoint estimates on T_{σ_γ} for a wider class of curves γ.

Before we state our main result, we introduce certain conditions on functions defined on intervals. For an interval J_1 in $\mathbb{R}$, a locally integrable function $\Phi : J_1 \to \mathbb{R}^+$, and a positive real number A, we let

$$\mathfrak{G}(\Phi, A) := \left\{\omega : J_1 \to \mathbb{R}^+ | \sqrt{\omega(s_1)\omega(s_2)} \le \frac{A}{s_2 - s_1} \int_{s_1}^{s_2} \Phi(s) ds \quad \text{whenever } s_1 < s_2 \text{ and } [s, s_2] \subset J_1 \right\}$$

and

$$\mathcal{E}_1(A) := \{\Phi : J \to \mathbb{R}^+ | \Phi \in \mathfrak{G}(\Phi, A)\}.$$

An interesting subclass of $\mathcal{E}_1(2A)$ is the collection $\mathcal{E}_2(A)$, introduced in [9], of functions $\Phi : J \to \mathbb{R}^+$ such that

1. Φ is monotone; and
2. whenever $s_1 < s_2$ and $[s_1, s_2] \subset J$,

$$\sqrt{\Phi(s_1)\Phi(s_2)} \le A\Phi((s_1 + s_2)/2)$$

Our main theorem is the following:

Theorem 1.1. *Let $I = (a, b) \subset \mathbb{R}$ be an open interval and let $\psi : I \to \mathbb{R}$ be a C^3 function such that*

1. *$\psi^{(3)}(t) \ge 0$, whenever $t \in I$;*
2. *there exists $A \in (0, \infty)$ such that, for each $u \in (0, b - a)$, $\mathfrak{F}_u : (a, b - u) \to \mathbb{R}^+$ given by $\mathfrak{F}_u(s) := \sqrt{\psi^{(3)}(s + u)\psi^{(3)}(s)}$ satisfies*

$$\mathfrak{F}_u \in \mathcal{E}_1(A). \tag{1.2}$$

Then, the operator T_{σ_γ} defined by (1.1) *is a bounded operator from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ whenever $(p^{-1}, q^{-1}) \in \mathfrak{S}$, and the operator norm $\|T_{\sigma_\gamma}\|_{L^p \to L^q}$ is dominated by a constant that depends only on A.*

The case when $\psi^{(3)} \in \mathcal{E}_2(A)$ was considered by Oberlin [5]. One can easily see that $\psi^{(3)} \in \mathcal{E}_2(A/2)$ implies (1.2) uniformly in $u \in (0, b - a)$. The theorem generalizes many results previously known for convolution estimates related to space curves, namely [1-6].

This article is organized as follows: in the following section, a uniform estimate for convolution operators with measures supported on plane curves. The proof of Theorem 1.1 based on a T^*T method is given in Section 3.

2 Uniform estimates on the plane

The following theorem motivated by Oberlin [10] which is interesting in itself will be useful:

Theorem 2.1. *Let J be an open interval in $\mathbb{R}$, and $\varphi : J \to \mathbb{R}$ be a C^2 function such that $\varphi'' \ge 0$. Let $\omega : J \to \mathbb{R}$ be a nonnegative measurable function. Suppose that there exists a positive constant A such that $\omega \in \mathfrak{G}(\phi'', A)$, i.e.*

$$\omega(s_1)^{1/2}\omega(s_2)^{1/2} \le \frac{A}{s_2 - s_1} \int_{s_1}^{s_2} \phi''(v) dv$$

holds whenever $s_1 < s_2$ and $[s_1, s_2] \subset J$. Let S be the operator given by

$$\mathcal{S}g(x_2, x_3) = \int_J g(x_2 - s, x_3 - \phi(s))\omega^{1/3}(s) ds$$

for $g \in C_0^\infty(\mathbb{R}^2)$. Then, there exists a constant C that depends only on A such that

$$||\mathcal{S}g||_{L^3(\mathbb{R}^2)} \leq C||g||_{L^{3/2}(\mathbb{R}^2)}$$

holds uniformly in $g \in C_0^\infty(\mathbb{R}^2)$.

Proof of Theorem 2.1. Our proof is based on the method introduced by Drury and Guo [11], which was later refined by Oberlin [10].

We have

$$\begin{aligned} ||\mathcal{S}g||_3^3 &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_J\int_J\int_J\prod_{j=1}^{3}\left(g\left(x_2 - s_j, x_3 - \phi\left(s_j\right)\right)\omega^{1/3}\left(s_j\right)\right)ds_1ds_2ds_3dx_2dx_3 \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\mathcal{G}\left(g\left(z_1,\cdot\right), g\left(z_2,\cdot\right), g\left(z_3,\cdot\right)\right)\right]\left(z_1, z_2, z_3\right)dz_1dz_2dz_3, \end{aligned}$$

where for z_1, z_2, $z_3 \in \mathbb{R}$ and suitable functions h_1, h_2, h_3 defined on $\mathbb{R}$,

$$[\mathcal{G}(h_1, h_2, h_3)(z_1, z_2, z_3) := \int_{\mathbb{R}}\int_{J(z_1,z_2,z_3)}\prod_{j=1}^{3}[h_j(x_3 - \phi(x_2 - z_j))\omega^{1/3}(x_2 - z_j)] \\ dx_2dx_3,$$

and

$$J(z_1, z_2, z_3) := (J + z_1) \cap (J + z_2) \cap (J + z_3).$$

We will prove that the estimate

$$|[\mathcal{G}(h_1, h_2, h_3)](z_1, z_2, z_3)| \leq \frac{C||h_1||_{L^{3/2}(\mathbb{R})}||h_2||_{L^{3/2}(\mathbb{R})}||h_3||_{L^{3/2}(\mathbb{R})}}{|(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)|^{1/3}} \tag{2.1}$$

holds uniformly in h_1, h_2, h_3, z_1, z_2, and z_3.

To establish (2.1) we let

$$[\mathcal{G}_k(h_1, h_2, h_3)](z_1, z_2, z_3) := \int_{\mathbb{R}}\int_{J(z_1,z_2,z_3)} h_k(x_3 - \phi(x_2 - z_k)) \\ \prod_{\substack{1\leq j\leq 3 \\ j\neq k}}[h_j(x_3 - \phi(x_2 - z_j))\omega^{1/2}(x_2 - z_j)]dx_2dx_3$$

for k = 1, 2, 3. Then, we have

$$|[\mathcal{G}_1(h_1, h_2, h_3)](z_1, z_2, z_3)| \leq ||h_1||_\infty \int_{\mathbb{R}}\int_{J(z_1,z_2,z_3)}\prod_{j=2}^{3}\left(|h_j(x_3 - \phi(x_2 - z_j))|\omega^{1/2}(x_2 - z_j)\right) \\ dx_2dx_3.$$

For z_2, $z_3 \in \mathbb{R}$ and $x_2 \in J(z_1, z_2, z_3)$, we have

$$\begin{aligned} \left|\phi')(x_2 - z_2) - \phi'(x_2 - z_3)\right| &= \left|\int_{x_2-z_2}^{x_2-z_3}\phi''(s)ds\right| \\ &\geq A^{-1}|z_2 - z_3|\omega^{1/2}(x_2 - z_2)\omega^{1/2}(x_2 - z_3) \end{aligned}$$

by hypothesis. Hence,

$$|[\mathcal{G}_1(h_1, h_2, h_3)](z_1, z_2, z_3)| \leq \frac{A||h_1||_\infty}{|z_2 - z_3|}\int_{\mathbb{R}}\int_{J(z_1,z_2,z_3)}\prod_{j=2}^{3}|h_j(x_3 - \phi(x_2 - z_j))| \\ |\phi'(x_2 - z_2) - \phi'(x_2 - z_3)|dx_2dx_3.$$

A change of variables gives

$$|[\mathcal{G}_1(h_1, h_2, h_3)](z_1, z_2, z_3)| \le \frac{A||h_1||_\infty}{|z_2 - z_3|} \int_{\mathbb{R}} \int_{\mathbb{R}} |h_2(z_2)|\,|h_3(z_3)| dz_2 dz_3.$$

Thus, we obtain

$$|[\mathcal{G}_1(h_1, h_2, h_3)](z_1, z_2, z_3)| \le \frac{A||h_1||_\infty ||h_2||_1 ||h_3||_1}{|z_2 - z_3|}. \tag{2.2}$$

Similarly, we get

$$|[\mathcal{G}_2(h_1, h_2, h_3)](z_1, z_2, z_3)| \le \frac{A||h_1||_1 ||h_2||_\infty ||h_3||_1}{|z_1 - z_3|} \tag{2.3}$$

and

$$|[\mathcal{G}_2(h_1, h_2, h_3)](z_1, z_2, z_3)| \le \frac{A||h_1||_1 ||h_2||_1 ||h_3||_\infty}{|z_1 - z_2|}. \tag{2.4}$$

Interpolating (2.2), (2.3) and (2.4) provides (2.1). Combining this with Proposition 2.2 in Christ [12] finishes the proof.

The special case in which $\omega = \varphi''$ provides a uniform estimate for the convolution operators with affine arclength measure on plane curves.

Corollary 2.2. *Let J be an open interval in $\mathbb{R}$, and $\varphi : J \to \mathbb{R}$ be a C^2 function such that $\varphi'' \ge 0$. Suppose that there exists a constant A such that $\phi'' \in \mathcal{E}_1(A)$, i.e.*

$$\phi''(s_1)^{1/2} \phi''(s_2)^{1/2} \le \frac{A}{s_2 - s_1} \int_{s_1}^{s_2} \phi''(v) dv$$

holds whenever $s_1 < s_2$ and $[s_1, s_2] \subset J$. Let $\mathcal{S}$be the operator given by

$$\mathcal{S}g(x_2, x_3) = \int_J g(x_2 - s, x_3 - \phi(s)) \phi''(s)^{1/3} ds$$

for $g \in C_0^\infty(\mathbb{R}^2)$. Then, there exists a constant C that depends only on A such that

$$||\mathcal{S}g||_{L^3(\mathbb{R}^2)} \le C||g||_{L^{3/2}(\mathbb{R}^2)}$$

holds uniformly in $g \in C_0^\infty(\mathbb{R}^2)$.

3 Proof of the main theorem

Before we proceed the proof of Theorem 1.1, we note that the uniform estimate (1.2) in $u \in (0, b - a)$ implies

$$\psi^{(3)} \in \mathcal{E}_1(A) \tag{3.1}$$

by continuity of $\psi^{(3)}$.

By duality and interpolation, it suffices to show that

$$||T_{\sigma_\gamma} f||_{L^2(\mathbb{R}^3)} \le C||f||_{L^{3/2}(\mathbb{R}^3)} \tag{3.2}$$

holds uniformly for $f \in L^{3/2}(\mathbb{R}^3)$.

Recall the following lemma observed by Oberlin [3]:

Lemma 3.1. *Suppose there exists a constant C_1 such that*

$$||T^*_{\sigma_\gamma} T_{\sigma_\gamma} f||_{L^3(\mathbb{R}^3)} \le C_1||f||_{L^{3/2}(\mathbb{R}^3)} \tag{3.3}$$

holds uniformly in $f \in L^{3/2}$ (R^3). Then, (3.2) holds for each $f \in L^{3/2}$ ($\mathbb{R}^3$).

To establish (3.3), we write

$$T^*_{\sigma_\gamma} T_{\sigma_\gamma} f(x) = \int_I \int_I f(x - \gamma(t) + \gamma(s))\lambda(t)\lambda(s)\, dtds$$
$$equiv \mathcal{T}^{(1)} f(x) + \mathcal{T}^{(2)} f(x),$$

where

$$\mathcal{T}^{(1)} f(x) = \iint_{\substack{t,s\in I\\ t>s}} f(x - \gamma(t) + \gamma(s))\lambda(t)\lambda(s)\, dtds,$$
$$\mathcal{T}^{(2)} f(x) = \iint_{\substack{t,s\in I\\ t<s}} f(x - \gamma(t) + \gamma(s))\lambda(t)\lambda(s)\, dtds.$$

By symmetry, it suffices to prove

$$||\mathcal{T}^{(1)} f||_{L^3(\mathbb{R}^3)} \le C_1 ||f||_{L^{3/2}(\mathbb{R}^3)}.$$

Next we make a change of variables, $u = t - s$ and write for $u \in (0, b - a)$

$$I_u = \{s \in \mathbb{R} : a < s < b - u\},$$
$$\Psi_u(s) = \psi(s + u) - \psi(s).$$

Then, we obtain:

$$\mathcal{T}^{(1)} f(x) = \int_I \int_0^{b-s} f(x_1 - u, x_2 - u(s + u/2), x_3 - \Psi_u(s))\lambda(s + u)\lambda(s)\, duds$$
$$= \int_0^{b-a} \int_{I_u} f(x_1 - u, x_2 - u(s + u/2), x_3 - \Psi_u(s))\lambda(s + u)\lambda(s)\, dsdu,$$

and so

$$\mathcal{T}^{(1)} f(x_1, x_2, x_3) = \int_0^{b-a} \mathcal{T}_u[f_u(x_1 - u, \cdot, \cdot\cdot)]((x_2 - u^2/2)/u, x_3)\, \frac{du}{u^{2/3}},$$

where

$$f_u(x_1, x_2, x_3) := u^{1/3} f(x_1, ux_2, x_3)$$
$$\mathcal{T}_u g(x_2, x_3) := \int_{I_u} g(x_2 - s, x_3 - \Psi_u(s))\Lambda_u^{1/3}(s)\, ds$$
$$\Lambda_u(s) := u\lambda^3(s + u)\lambda^3(s)$$
$$= u\sqrt{\psi^{(3)}(s + u)\psi^{(3)}(s)}$$

for $x_1, x_2, x_3 \in \mathbb{R}$, $u \in (0, b - a)$, $s \in I_u$.

Notice that for $u \in (0, b - a)$ and $[s_1, s_2] \subset I_u$, we have

$$\Lambda_u^{1/2}(s_1)\Lambda_u^{1/2}(s_2) \le \frac{Au}{s_2 - s_1} \int_{s_1}^{s_2} \sqrt{\psi^{(3)}(s + u)\psi^{(3)}(s)} ds$$
$$\le \frac{A^2 u}{s_2 - s_1} \int_{s_1}^{s_2} \frac{1}{u} \int_s^{s+u} \psi^{(3)}(v) dvds$$
$$= \frac{A^2}{s_2 - s_1} \int_{s_1}^{s_2} (\psi''(s + u) - \psi''(s)) ds$$
$$= \frac{A^2}{s_2 - s_1} \int_{s_1}^{s_2} \Psi''{}_u(s) ds$$

by (1.2) and (3.1). By Theorem 2.1, $||\mathcal{T}_u||_{L^{3/2}(\mathbb{R}^2)\to L^3(\mathbb{R}^2)}$ is uniformly bounded. Hence, we obtain

$$
\begin{aligned}
||\mathcal{T}^{(1)}f||_3 &\le \left(\int_{\mathbb{R}}\left[\iint_{\mathbb{R}^2}\left(\int_0^{b-a}\left|\mathcal{T}_u f_u(x_1-u,\cdot,\cdot\cdot)\left(\frac{x_2-u^2/2}{u},x_3\right)\right|\frac{du}{u^{2/3}}\right)^3 dx_2dx_3\right]^{\frac{1}{3}\cdot 3} dx_1\right)^{\frac{1}{3}} \\
&\le \left(\int_{\mathbb{R}}\left[\int_0^{b-a}\left(\iint_{\mathbb{R}^2}\left|\mathcal{T}_u f_u(x_1-u,\cdot,\cdot\cdot)\left(\frac{x_2-u^2/2}{u},x_3\right)\right|^3 dx_2dx_3\right)^{\frac{1}{3}}\frac{du}{u^{2/3}}\right]^3 dx_1\right)^{\frac{1}{3}} \\
&\le C(A)\left(\int_{\mathbb{R}}\left[\int_0^{b-a} u^{\frac{1}{3}}\,||f_u\,(x_1-u,\cdot,\cdot\cdot)\,||_{L^{3/2}(\mathbb{R}^2)}\frac{du}{u^{2/3}}\right]^3 dx_1\right)^{\frac{1}{3}} \\
&\le C(A)\left(\int_{\mathbb{R}}\left[\int_0^{b-a}||f(x_1-u,\cdot,\cdot\cdot)||_{L^{3/2}(\mathbb{R}^2)}\frac{du}{u^{2/3}}\right]^3 dx_1\right)^{\frac{1}{3}}.
\end{aligned}
$$

By Hardy-Littlewood-Sobolev theorem on fractional integration, we obtain

$$||\mathcal{T}^{(1)}f||_3 \le C_1(A)||f||_{3/2}$$

This finishes the proof of Theorem 1.1.

Competing interests

The author declares that they have no competing interests.

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References

1. Choi, Y: The L^p-L^q mapping properties of convolution operators with the affine arclength measure on space curves. J Aust Math Soc. **75**, 247–261 (2003). doi:10.1017/S144678870000375X
2. Drury, SW: Degenerate curves and harmonic analysis. Math Proc Cambridge Philos Soc. **108**, 89–96 (1990). doi:10.1017/S0305004100068973
3. Oberlin, DM: Convolution estimates for some measures on curves. Proc Am Math Soc. **99**, 56–60 (1987). doi:10.1090/S0002-9939-1987-0866429-6
4. Oberlin, DM: Convolutions with measures on curves in $\mathbb{R}^3$. Can Math Bull. **41**, 478–480 (1998). doi:10.4153/CMB-1998-062-7
5. Oberlin, DM: Convolution with measures on flat curves in low dimensions. J Funct Anal. **259**, 1799–1815 (2010). doi:10.1016/j.jfa.2010.05.008
6. Pan, Y: L^p-improving properties for some measures supported on curves. Math Scand. **73**, 215–228 (1993)
7. Pan, Y: Convolution estimates for some degenerate curves. Math Proc Cambridge Philos Soc. **116**, 143–146 (1994). doi:10.1017/S0305004100072431
8. Secco, S: Fractional integration along homogeneous curves in $\mathbb{R}^3$. Math Scand. **85**, 259–270 (1999)
9. Bak, JG, Oberlin, D, Seeger, A: Restriction of Fourier transforms to curves. II. Some classes with vanishing torsion. J Aust Math Soc. **85**, 1–28 (2008). doi:10.1017/S1446788708000578
10. Oberlin, DM: Two estimates for curves in the plane. Proc Am Math Soc. **132**, 3195–3201 (2004). doi:10.1090/S0002-9939-04-07610-5
11. Drury, SW, Guo, K: Convolution estimates related to surfaces of half the ambien dimension. Math Proc Cambridge Philos Soc. **110**, 151–159 (1991). doi:10.1017/S0305004100070201
12. Christ, M: On the restriction of the Fourier transform to curves: endpoint results and the degenerated case. Trans Am Math Soc. **287**, 223–238 (1985). doi:10.1090/S0002-9947-1985-0766216-6

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