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A fixed-point approach to the stability of a functional equation on quadratic forms

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Abstract

Using the fixed-point method, we prove the generalized Hyers-Ulam stability of the functional equation

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w).$$

The quadratic form $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2$ is a solution of the above functional equation.

Keywords: alternative of fixed point, functional equation, quadratic form, stability

1. Introduction

In 1940, S. M. Ulam [1] gave a wide-ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Thereafter, many authors investigated solutions or stability of various functional equations (see [3-11]).

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Throughout this paper, let X and Y be two real vector spaces and let $\phi : X \times X \times X \times X \rightarrow [0, \infty)$ be a function. For a mapping $f : X \times X \rightarrow Y$, consider the functional equation:

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w). \quad (1.1)$$

The quadratic form $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x, y) := ax^2 + bxy + cy^2$ is a solution of the Equation 1.1.

The authors [12] acquired the general solution and proved the stability of the functional Equation 1.1 for the case that X and Y are real vector spaces as follows.

Theorem A. *A mapping $f: X \times X \rightarrow Y$ satisfies the Equation 1.1 for all $x, y, z, w \in X$ if and only if there exist two symmetric bi-additive mappings $S, T: X \times X \rightarrow Y$ and a bi-additive mapping $B: X \times X \rightarrow Y$ such that*

$$f(x, y) = S(x, x) + B(x, y) + T(y, y)$$

for all $x, y \in X$.

From now on, let Y be a complete normed space.

Theorem B. *Assume that ϕ satisfies the condition*

$$\tilde{\phi}(x, y, z, w) := \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \phi(2^j x, 2^j y, 2^j z, 2^j w) < \infty$$

for all $x, y, z, w \in X$. Let $f: X \times X \rightarrow Y$ be a mapping such that

$$\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w)\| \leq \phi(x, y, z, w)$$

for all $x, y, z, w \in X$. Then, there exists a unique mapping $F: X \times X \rightarrow Y$ satisfying the Equation 1.1 such that

$$\|f(x, y) - F(x, y)\| \leq \tilde{\phi}(x, x, y, y) \tag{1.2}$$

for all $x, y \in X$. The mapping F is given by

$$F(x, y) := \lim_{j \rightarrow \infty} \frac{1}{4^j} f(2^j x, 2^j y)$$

for all $x, y \in X$.

In this paper, we prove the stability of the Equation 1.1 using the fixed-point method.

2. Stability using the alternative of fixed point

In this section, we investigate the stability of the functional Equation 1.1 using the alternative of fixed point. Before proceeding the proof, we will state the theorem, the alternative of fixed point.

Theorem 2.1. *(The alternative of fixed point [13,14]). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T: \Omega \rightarrow \Omega$ with Lipschitz constant L . Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or

there exists a positive integer n_0 such that

- $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
- the sequence $(T^n x)$ is convergent to a fixed point y^* of T ;
- y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega \mid d(T^{n_0} x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Lemma 2.2. Let $\psi : X \times X \rightarrow [0, \infty)$ be a function given by

$$\psi(x, y) := \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Consider the set $\Omega := \{g \mid g : X \times X \rightarrow Y, g(0, 0) = 0\}$ and the generalized metric d on Ω given by

$$d(g, h) = d_\psi(g, h) := \inf S_\psi(g, h),$$

where $S_\psi(g, h) := \{K \in [0, \infty] \mid \|g(x, y) - h(x, y)\| \leq K\psi(x, y) \text{ for all } x, y \in X\}$ for all $g, h \in \Omega$. Then, (Ω, d) is complete.

Proof. Let $\{g_n\}$ be a Cauchy sequence in (Ω, d) . Then, given $\varepsilon > 0$, there exists N such that $d(g_n, g_k) < \varepsilon$ if $n, k \geq N$. Let $n, k \geq N$. Since $d(g_n, g_k) = \inf S_\psi(g_n, g_k) < \varepsilon$, there exists $K \in [0, \varepsilon)$ such that

$$\|g_n(x, y) - g_k(x, y)\| \leq K\psi(x, y) \leq \varepsilon\psi(x, y) \tag{2.1}$$

for all $x, y \in X$. So, for each $x, y \in X$, $\{g_n(x, y)\}$ is a Cauchy sequence in Y . Since Y is complete, for each $x, y \in X$, there exists $g(x, y) \in Y$ such that $g_n(x, y) \rightarrow g(x, y)$ as $n \rightarrow \infty$ and $g(0, 0) = 0$. Thus, we have $g \in \Omega$. By (2.1), we obtain that

$$\begin{aligned} n \geq N &\Rightarrow \|g_n(x, y) - g(x, y)\| \leq \varepsilon\psi(x, y) \text{ for all } x, y \in X \\ &\Rightarrow \varepsilon \in S_\psi(g_n, g) \\ &\Rightarrow d(g_n, g) = \inf S_\psi(g_n, g) \leq \varepsilon. \end{aligned}$$

Hence, $g_n \rightarrow g \in \Omega$ as $n \rightarrow \infty$.

By using an idea of Cădariu and Radu (see [15]), we will prove the Hyers-Ulam stability of the functional equation related to quadratic forms.

Theorem 2.3. Assume that ϕ satisfies the condition

$$\lim_{n \rightarrow \infty} \frac{1}{4^n} \phi(2^n x, 2^n y, 2^n z, 2^n w) = 0$$

for all $x, y, z, w \in X$. Suppose that a mapping $f : X \times X \rightarrow Y$ satisfies the functional inequality

$$\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w)\| \leq \phi(x, y, z, w) \tag{2.2}$$

for all $x, y, z, w \in X$ and $f(0, 0) = 0$. If there exists $L < 1$ such that the function ψ given in Lemma 2.2 has the property

$$\psi(x, y) \leq 4L\psi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.3}$$

for all $x, y \in X$, then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (1.1) such that the inequality

$$\|f(x, y) - F(x, y)\| \leq \frac{L}{1-L}\psi(x, y) \tag{2.4}$$

holds for all $x, y \in X$.

Proof. Consider the complete generalized metric space (Ω, d) given in Lemma 2.2. Now we define a mapping $T : \Omega \rightarrow \Omega$ by

$$Tg(x, \gamma) := \frac{1}{4}g(2x, 2\gamma)$$

for all $g \in \Omega$ and all $x, \gamma \in X$. Observe that, for all $g, h \in \Omega$,

$$\begin{aligned} K' &\in S_\psi(g, h) \text{ and } K' < K \\ \Rightarrow \|g(x, \gamma) - h(x, \gamma)\| &\leq K'\psi(x, \gamma) \leq K\psi(x, \gamma) \text{ for all } x, \gamma \in X \\ \Rightarrow K &\in S_\psi(g, h). \end{aligned}$$

Let $g, h \in \Omega$ and $\varepsilon \in (0, \infty]$. Then, there is a $K' \in S_\psi(g, h)$ such that $K' < d(g, h) + \varepsilon$. By the above observation, we gain $d(g, h) + \varepsilon \in S_\psi(g, h)$. So we get $\|g(x, \gamma) - h(x, \gamma)\| \leq (d(g, h) + \varepsilon)\psi(x, \gamma)$ for all $x, \gamma \in X$. Thus, we have

$$\left\| \frac{1}{4}g(2x, 2\gamma) - \frac{1}{4}h(2x, 2\gamma) \right\| \leq \frac{1}{4}(d(g, h) + \varepsilon)\psi(2x, 2\gamma)$$

for all $x, \gamma \in X$. By (2.3), we obtain that

$$\left\| \frac{1}{4}g(2x, 2\gamma) - \frac{1}{4}h(2x, 2\gamma) \right\| \leq L(d(g, h) + \varepsilon)\psi(x, \gamma)$$

for all $x, \gamma \in X$. Hence, $d(Tg, Th) \leq L(d(g, h) + \varepsilon)$. Now we obtain that

$$d(Tg, Th) \leq L(d(g, h) + \varepsilon)$$

for all $\varepsilon \in (0, \infty]$. Taking the limit as $\varepsilon \rightarrow 0^+$ in the above inequality, we get

$$d(Tg, Th) \leq Ld(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with Lipschitz constant L .

Putting $y = x$ and $w = z$ in (2.2), by (2.3), we have the inequality

$$\left\| f(x, z) - \frac{1}{4}f(2x, 2z) \right\| \leq \frac{1}{4}\varphi(x, x, z, z) = \frac{1}{4}\psi(2x, 2z) \leq L\psi(x, z) \quad (2.5)$$

for all $x, z \in X$. Thus, we obtain that

$$d(f, Tf) \leq L < \infty. \quad (2.6)$$

Applying the alternative of fixed point, we see that there exists a fixed point F of T in Ω such that

$$F(x, \gamma) = \lim_{n \rightarrow \infty} \frac{1}{4^n}f(2^n x, 2^n \gamma)$$

for all $x, \gamma \in X$. Replacing x, γ, z, w by $2^n x, 2^n \gamma, 2^n z, 2^n w$ in (2.2), respectively, and dividing by 4^n , we have

$$\begin{aligned} &\|F(x + \gamma, z + w) + F(x - \gamma, z - w) - 2F(x, z) - 2F(\gamma, w)\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f(2^n(x + \gamma), 2^n(z + w)) + f(2^n(x - \gamma), 2^n(z - w)) \\ &\quad - 2f(2^n x, 2^n z) - 2f(2^n \gamma, 2^n w)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{4^n} \varphi(2^n x, 2^n \gamma, 2^n z, 2^n w) = 0 \end{aligned}$$

for all $x, y, z, w \in X$. Thus, the mapping F satisfies the Equation 1.1. By (2.3) and (2.5), we obtain that

$$\begin{aligned} \|T^n f(x, y) - T^{n+1} f(x, y)\| &= \frac{1}{4^n} \|f(2^n x, 2^n y) - \frac{1}{4} f(2^{n+1} x, 2^{n+1} y)\| \\ &\leq \frac{L}{4^n} \psi(2^n x, 2^n y) \leq \dots \leq \frac{L}{4^n} (4L)^n \psi(x, y) \\ &= L^{n+1} \psi(x, y) \end{aligned}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \leq L^{n+1} < \infty$ for all $n \in \mathbb{N}$. By the fixed-point alternative, there exists a natural number n_0 such that the mapping F is the unique fixed point of T in the set $\Delta = \{g \in \Omega \mid d(T^{n_0} f, g) < \infty\}$. So we have $d(T^{n_0} f, F) < \infty$. Since

$$d(f, T^{n_0} f) \leq d(f, Tf) + d(Tf, T^2 f) + \dots + d(T^{n_0-1} f, T^{n_0} f) < \infty,$$

we get $f \in \Delta$. Thus, we have $d(f, F) \leq d(f, T^{n_0} f) + d(T^{n_0} f, F) < \infty$. Hence, we obtain

$$\|f(x, y) - F(x, y)\| \leq K\psi(x, y)$$

for all $x \in X$ and some $K \in [0, \infty)$. Again using the fixed-point alternative, we have

$$d(f, F) \leq \frac{1}{1-L} d(f, Tf).$$

By (2.6), we may conclude that

$$d(f, F) \leq \frac{L}{1-L},$$

which implies the inequality (2.4).

Lemma 2.4. Let $\psi : X \times X \rightarrow [0, \infty)$ be a function given by

$$\psi(x, y) := \varphi(2x, 2x, 2y, 2y)$$

for all $x, y \in X$. Consider the set $\Omega := \{g \mid g : X \times X \rightarrow Y, g(0, 0) = 0\}$ and the generalized metric d on Ω given by

$$d(g, h) = d_\psi(g, h) := \inf S_\psi(g, h),$$

where $S_\psi(g, h) := \{K \in [0, \infty) \mid \|g(x, y) - h(x, y)\| \leq K\psi(x, y) \text{ for all } x, y \in X\}$ for all $g, h \in \Omega$. Then, (Ω, d) is complete.

Proof. The proof is similar to the proof of Lemma 2.2.

Theorem 2.5. Assume that ϕ satisfies the condition

$$\lim_{n \rightarrow \infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) = 0$$

for all $x, y, z, w \in X$. Suppose that a mapping $f : X \times X \rightarrow Y$ satisfies the functional inequality (2.2) for all $x, y, z, w \in X$ and $f(0, 0) = 0$. If there exists $L < 1$ such that the function ψ given in Lemma 2.4 has the property

$$\psi(x, y) \leq \frac{L}{4} \psi(2x, 2y) \tag{2.7}$$

for all $x, y \in X$, then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (1.1) such that the inequality

$$\|f(x, y) - F(x, y)\| \leq \frac{L^2}{16(1-L)}\psi(x, y) \tag{2.8}$$

holds for all $x, y \in X$.

Proof. Consider the complete generalized metric space (Ω, d) given in Lemma 2.4. Now we define a mapping $T : \Omega \rightarrow \Omega$ by

$$Tg(x, y) := 4g\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $g \in \Omega$ and all $x, y \in X$. By the same argument of the proof of Theorem 2.3, T is a strictly contractive mapping of Ω with Lipschitz constant L .

Replacing x, y, z, w by $\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}$ in (2.2), respectively, and using (2.7), we have the inequality

$$\|f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right)\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}\right) = \psi\left(\frac{x}{4}, \frac{z}{4}\right) \leq \frac{L}{4}\psi\left(\frac{x}{2}, \frac{z}{2}\right) \leq \frac{L^2}{16}\psi(x, y) \tag{2.9}$$

for all $x, z \in X$. Thus, we obtain that

$$d(f, Tf) \leq \frac{L^2}{16} < \infty. \tag{2.10}$$

Applying the alternative of fixed point, we see that there exists a fixed point F of T in Ω such that

$$F(x, y) = \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

for all $x, y \in X$. Replacing x, y, z, w by $\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}$ in (2.2), respectively, and multiplying by 4^n , we have

$$\begin{aligned} & \|F(x+y, z+w) + F(x-y, z-w) - 2F(x, z) - 2F(y, w)\| \\ &= \lim_{n \rightarrow \infty} 4^n \left\| f\left(\frac{x+y}{2^n}, \frac{z+w}{2^n}\right) + f\left(\frac{x-y}{2^n}, \frac{z-w}{2^n}\right) - 2f\left(\frac{x}{2^n}, \frac{z}{2^n}\right) - 2f\left(\frac{y}{2^n}, \frac{w}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) = 0 \end{aligned}$$

for all $x, y, z, w \in X$. Thus, the mapping F satisfies the Equation 1.1. By (2.7) and (2.9), we obtain that

$$\begin{aligned} \|T^n f(x, y) - T^{n+1} f(x, y)\| &= 4^n \left\| f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - 4f\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) \right\| \\ &\leq 4^{n-2} L^2 \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq 4^{n-3} L^3 \psi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) \\ &\leq \dots \leq \frac{L^{n+2}}{16} \psi(x, y) \end{aligned}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \leq \frac{L^{n+2}}{16} < \infty$ for all $n \in \mathbb{N}$. By the same reasoning of the proof of Theorem 2.3, we have

$$d(f, F) \leq \frac{1}{1-L} d(f, Tf).$$

By (2.10), we may conclude that

$$d(f, F) \leq \frac{L^2}{16(1-L)},$$

which implies the inequality (2.8).

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Authors' contributions

Both authors contributed equally to this work. The authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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References

1. Ulam, SM: A Collection of Mathematical Problems. pp. 63. Interscience Publishers, New York (1968)
2. Hyers, DH: On the stability of the linear functional equation. *Proc Natl Acad Sci USA*. **27**, 222–224 (1941). doi:10.1073/pnas.27.4.222
3. Bae, J-H, Park, W-G: On stability of a functional equation with n variables. *Nonlinear Anal TMA*. **64**, 856–868 (2006). doi:10.1016/j.na.2005.06.028
4. Bae, J-H, Park, W-G: On a cubic equation and a Jensen-quadratic equation. *Abstr Appl Anal* **2007** (2007). Article ID 45179
5. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J Math Anal Appl*. **184**, 431–436 (1994). doi:10.1006/jmaa.1994.1211
6. Jung, S-M: Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis. Springer, New York (2011)
7. Jung, S-M, Kim, T-S: A fixed point approach to the stability of the cubic functional equation. *Bol Soc Mat Mexicana* **12**(1), 51–57 (2006). (3)
8. Jung, S-M, Kim, T-S, Lee, K-S: A fixed point approach to the stability of quadratic functional equation. *Bull Korean Math Soc*. **43**, 531–541 (2006)
9. Jung, S-M, Lee, Z-H: A fixed point approach to the stability of quadratic functional equation with involution. *Fixed Point Theory Appl* **2008**, 11 (2008). Article ID 732086
10. Park, W-G, Bae, J-H: A multidimensional functional equation having quadratic forms as solutions. *J Inequal Appl* **2007** (2007). Article ID 24716
11. Rassias, TM: On the stability of linear mappings in Banach spaces. *Proc Amer Math Soc*. **72**, 297–300 (1978). doi:10.1090/S0002-9939-1978-0507327-1
12. Park, W-G, Bae, J-H: A functional equation originating from quadratic forms. *J Math Anal Appl*. **326**, 1142–1148 (2007). doi:10.1016/j.jmaa.2006.03.023
13. Margolis, B, Dias, JB: A fixed point theorem of the alternative for contractions on a generalized complete metric space. *Bull Amer Math Soc*. **74**, 305–309 (1968). doi:10.1090/S0002-9904-1968-11933-0
14. Rus, IA: Principles and Applications of Fixed Point Theory. (1979) in Romanian
15. Cădariu, L, Radu, V: Fixed points and the stability of Jensen's functional equation. *J Inequal Pure Appl Math* **4** (2003). Article 4

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