Bi and Ding *Journal of Inequalities and Applications* 2011, **2011**:69
http://www.journalofinequalitiesandapplications.com/content/2011/1/69

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Orlicz norm inequalities for the composite operator and applications

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Abstract

In this article, we first prove Orlicz norm inequalities for the composition of the homotopy operator and the projection operator acting on solutions of the nonhomogeneous A-harmonic equation. Then we develop these estimates to $L^{\phi}(\mu)$-averaging domains. Finally, we give some specific examples of Young functions and apply them to the norm inequality for the composite operator.

2000 Mathematics Subject Classification: Primary 26B10; Secondary 30C65, 31B10, 46E35.

Keywords: Orlicz norm, the projection operator, the homotopy operator, $L^{\phi?\varphi?}(?\mu?)$-averaging domains

1. Introduction

Differential forms as the extensions of functions have been rapidly developed. In recent years, some important results have been widely used in PDEs, potential theory, nonlinear elasticity theory, and so forth; see [1-7] for details. However, the study on operator theory of differential forms just began in these several years and hence attracts the attention of many people. Therefore, it is necessary for further research to establish some norm inequalities for operators. The purpose of this article is to establish Orlicz norm inequalities for the composition of the homotopy operator T and the projection operator H.

Throughout this article, we always let E be an open subset of $\mathbb{R}^n$, $n \geq 2$. The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$. Assume that $B \subset \mathbb{R}^n$ is a ball, and σB is the ball with the same center as B and with $diam(\sigma B) = \sigma diam(B)$. Let $\wedge^k = \wedge^k(\mathbb{R}^n)$, $k = 0, 1,..., n$, be the linear space of all k-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1,i_2,...,i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$, where $I = (i_1, i_2,...,i_k)$, $1 \leq i_1 < i_2 <... < i_k \leq n$. We use $\mathcal{D}'(E, \wedge^k)$ to denote the space of all differential k-forms in E. In fact, a differential k-form $\omega(x)$ is a Schwarz distribution in E with value in $\wedge^k(\mathbb{R}^n)$. As usual, we still use $\star$ to denote the Hodge star operator, and use $d^\star : \mathcal{D}'(E, \wedge^{k+1}) \to \mathcal{D}'(E, \wedge^k)$ to denote the Hodge codifferential operator defined by $d^\star = (-1)^{nk+1} \star d \star$ on $\mathcal{D}'(E, \wedge^{k+1})$, $k = 0, 1, \ldots, n-1$. Here $d : \mathcal{D}'(E, \wedge^k) \to \mathcal{D}'(E, \wedge^{k+1})$ denotes the differential operator.

A weight $w(x)$ is a nonnegative locally integrable function on $\mathbb{R}^n$. $L^p(E, \wedge^k)$ is a Banach space equipped with norm $||\omega||_{p,E} = (\int_E |\omega(x)|^p dx)^{1/p} = \left(\int_E \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}$. Let D

be a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$, and $C^\infty(\wedge^k D)$ be the space of smooth k-forms on D, where $\wedge^k D$ is the kth exterior power of the cotangent bundle. The harmonic k-field is defined by $\mathcal{H}(\wedge^k D) = \{u \in \mathcal{W}(\wedge^k D) : d\omega = d^\star\omega = 0,\ \omega \in L^p \text{ for some } 1 < p < \infty\}$, where $\mathcal{W}(\wedge^k D) = \{\omega \in L^1_{loc}(\wedge^k D) : \omega \text{ has generalized gradient}\}$. If we use $\mathcal{H}^\perp$ to denote the orthogonal complement of $\mathcal{H}$ in L^1, then the Green's operator G is defined by $G : C^\infty(\wedge^k D) \to \mathcal{H}^\perp \cap C^\infty(\wedge^k D)$ by assigning $G(\omega)$ as the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^k D)$ satisfying $\Delta G(\omega) = \omega - H(\omega)$, where H is the projection operator that maps $C^\infty(\wedge^k D)$ onto $\mathcal{H}$ such that $H(\omega)$ is the harmonic part of ω; see [8] for more properties on the projection operator and Green's operator. The definition of the homotopy operator for differential forms was first introduced in [9]. Assume that $D \subset \mathbb{R}^n$ is a bounded convex domain. To each $y \in D$, there corresponds a linear operator $K_y : C^\infty(\wedge^k D) \to C^\infty(\wedge^{k-1} D)$ satisfying that $(K_y\omega)(x; \xi_1, \xi_2, \ldots, \xi_{k-1}) = \int_0^1 t^{k-1}\omega(tx + y - ty; x - y, \xi_1, \xi_2, \ldots, \xi_{k-1})dt$. Then by averaging K_y over all points y in D, The homotopy operator $T : C^\infty(\wedge^k D) \to C^\infty(\wedge^{k-1} D)$ is defined by $T\omega = \int_D \varphi(y) K_y \omega dy$, where $\varphi \in C_0^\infty(D)$ is normalized so that $\int \phi(y)dy = 1$. In [9], those authors proved that there exists an operator $T : L^1_{loc}(D, \wedge^k) \to L^1_{loc}(D, \wedge^{k-1})$, $k = 1, 2, \ldots, n$, such that

$$T(d\omega) + dT\omega = \omega; \tag{1.1}$$

$$|T\omega(x)| \leq C \int_D \frac{|\omega(y)|}{|y - x|^{n-1}}\, dy \tag{1.2}$$

for all differential forms $\omega \in L^p(D, \wedge^k)$ such that $d\omega \in L^p(D, \wedge^k)$. Furthermore, we can define the k-form $\omega_D \in \mathcal{D}'(D, \wedge^k)$ by the homotopy operator as

$$\omega_D = |D|^{-1} \int_D \omega(y)dy,\ k = 0; \omega_D = d(T\omega),\ k = 1, 2, \ldots, n \tag{1.3}$$

for all $\omega \in L^p(D, \wedge^k)$, $1 \leq p < \infty$.

Consider the nonhomogeneous A-harmonic equation for differential forms

$$d^\star A(x, d\omega) = B(x, d\omega), \tag{1.4}$$

where $A : E \times \wedge^k(\mathbb{R}^n) \to \wedge^k(\mathbb{R}^n)$ and $B : E \times \wedge^k(\mathbb{R}^n) \to \wedge^{k-1}(\mathbb{R}^n)$ are two operators satisfying the conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \tag{1.5}$$

$$A(x, \xi) \cdot \xi \geq |\xi|^p, \tag{1.6}$$

$$|B(x, \xi)| \leq b|\xi|^{p-1} \tag{1.7}$$

for almost every $x \in E$ and all $\xi \in \wedge^k(\mathbb{R}^n)$. Here, $a, b > 0$ are some constants and $1 < p < \infty$ is a fixed exponent associated with (1.4). A solution to (1.4) is an element of the Sobolev space $W^{1,p}_{loc}(E, \wedge^{k-1})$ such that

$$\int_E A(x, d\omega) \cdot d\varphi + B(x, d\omega) \cdot \varphi = 0 \tag{1.8}$$

for all $\varphi \in W^{1,p}_{loc}(E, \wedge^{k-1})$ with compact support.

2. Orlicz norm inequalities for the composite operator

In this section, we establish the weighted inequalities for the composite operator $T \circ H$ in terms of Orlicz norms. To state our results, we need some definitions and lemmas.

We call a continuously increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ an Orlicz function. If the Orlicz function Φ is convex, then Φ is often called a Young function. The Orlicz space $L^{\Phi}(E)$ consists of all measurable functions f on E such that $\int_E \Phi(|f|/\lambda)dx < \infty$ for some $\lambda = \lambda(f) > 0$ with the nonlinear Luxemburg functional

$$||f||_{\Phi,E} = inf\{\lambda > 0 : \int_E \Phi\left(\frac{|f|}{\lambda}\right) dx \le 1\}. \tag{2.1}$$

Moreover, if Φ is a restrictively increasing Young function, then $L^{\Phi}(E)$ is a Banach space and the corresponding norm $|| \cdot ||_{\Phi,E}$ is called Luxemburg norm or Orlicz Norm. The following definition appears in [10].

Definition 2.1. *We say that an Orlicz function* Φ *lies in the class* $G(p, q, C)$, $1 \le p < q < \infty$ *and* $C \ge 1$, *if (1)* $1/C \le \Phi(t^{1/p})/g(t) \le C$ *and (2)* $1/C \le \Phi(t^{1/q})/h(t) \le C$ *for all* $t > 0$, *where* $g(t)$ *is a convex increasing function and* $h(t)$ *is a concave increasing function on* $[0, \infty)$.

We note from [10] that each of Φ, g, and h mentioned in Definition 2.1 is doubling, from which it is easy to know that

$$C_1 t^q \le h^{-1}(\Phi(t)) \le C_2 t^q,\ C_1 t^p \le g^{-1}(\Phi(t)) \le C_2 t^p \tag{2.2}$$

for all $t > 0$, where C_1 and C_2 are constants.

We also need the following lemma which appears in [1].

Lemma 2.2. *Let* $u \in L^s_{loc}(D, \wedge^k)$, $k = 1, 2, \ldots, n$, $1 < s < \infty$, *be a smooth solution of the nonhomogeneous A-harmonic equation in a bounded convex domain D, H be the projection operator and* $T : C^{\infty}(\wedge^k D) \to C^{\infty}(\wedge^{k-1} D)$ *be the homotopy operator. Then there exists a constant C, independent of u, such that*

$$||T(H(u)) - (T(H(u)))_B||_{s,B} \le Cdiam(B)||u||_{s,\rho B}$$

for all balls B with $\rho B \subset D$, *where* $\rho > 1$ *is a constant.*

The A_r weights, $r > 1$, were first introduced by Muckenhoupt [11] and play a crucial role in weighted norm inequalities for many operators. As an extension of A_r weights, the following class was introduced in [2].

Definition 2.3. *We call that a measurable function* $w(x)$ *defined on a subset* $E \subset \mathbb{R}^n$ *satisfies the* $A(\alpha, \beta, \gamma; E)$*-condition for some positive constants* α, β, γ; *write* $w(x) \in A(\alpha, \beta, \gamma; E)$, *if* $w(x) > 0$ *a.e. and*

$$\sup_B \left(\frac{1}{|B|}\int_B w^{\alpha} dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{\beta} dx\right)^{\gamma/\beta} = c_{\alpha,\beta,\gamma} < \infty,$$

where the supremum is over all balls $B \subset E$.

We also need the following reverse Hölder inequality for the solutions of the nonhomogeneous A-harmonic equation, which appears in [3].

Lemma 2.4. *Let u be a solution of the nonhomogeneous A-harmonic equation,* $\sigma > 1$ *and* $0 < s, t < \infty$. *Then there exists a constant C, independent of u and B, such that*

$$||u||_{s,B} \le C|B|^{(t-s)/st}||u||_{t,\sigma B}$$

for all balls B with $\sigma B \subset E$.

Theorem 2.5. *Assume that u is a smooth solution of the nonhomogeneous A-harmonic equation in a bounded convex domain D, $1 < p, q < \infty$ and $w(x) \in A(\alpha, \beta, \frac{\alpha q}{p}; D)$for some $\alpha > 1$ and $\beta > 0$. Let H be the projection operator and $T : C^\infty(\wedge^k D) \to C^\infty(\wedge^{k-1} D)$, $k = 1, 2, \ldots, n$, be the homotopy operator. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |T(H(u)) - (T(H(u)))_B|^q w(x)dx\right)^{1/q} \le Cdiam(B)|B|^{(p-q)/pq}\left(\int_{\sigma B} |u|^p w(x)dx\right)^{1/p}$$

for all balls with $\sigma B \subset D$ for some $\sigma > 1$.

Proof. Set $s = \alpha q$ and $m = \beta p/(\beta + 1)$. From Lemma 2.2 and the reverse Hölder inequality, we have

$$\begin{aligned} &\left(\int_B |T(H(u)) - (T(H(u)))_B|^q w(x)dx\right)^{1/q} \\ &\le \left(\int_B |T(H(u)) - (T(H(u)))_B|^{\frac{qs}{s-q}} dx\right)^{\frac{s-q}{sq}} \left(\int_B (w(x))^\alpha dx\right)^{\frac{1}{\alpha q}} \\ &\le C_1 diam(B)|B|^{\frac{1}{q}-\frac{1}{s}-\frac{1}{m}} \left(\int_{\sigma B} |u|^m dx\right)^{1/m} \left(\int_B (w(x))^\alpha dx\right)^{1/\alpha q}. \end{aligned} \tag{2.3}$$

Let $n = \frac{pm}{p-m}$, then $\frac{1}{p} + \frac{1}{n} = \frac{1}{m}$. Thus, using the Hölder inequality, we obtain

$$\begin{aligned} &\left(\int_{\sigma B} |u|^m dx\right)^{1/m} \\ &= \left(\int_{\sigma B} |u|^m \left(w^{\frac{1}{p}} \cdot w^{\frac{-1}{p}}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\sigma B} |u|^p w(x)dx\right)^{1/p} \left(\int_{\sigma B} w^{\frac{-n}{p}} dx\right)^{\frac{1}{n}}. \end{aligned} \tag{2.4}$$

Note that $w(x) \in A(\alpha, \beta, \frac{\alpha q}{p}; D)$. It is easy to find that

$$\begin{aligned} &\left(\int_B (w(x))^\alpha dx\right)^{1/\alpha q} \left(\int_{\sigma B} w^{\frac{-n}{p}} dx\right)^{\frac{1}{n}} \\ &= \left(\int_B (w(x))^\alpha dx\right)^{1/\alpha q} \left(\int_{\sigma B} w^{-\beta} dx\right)^{\frac{1}{\beta p}} \\ &\le |\sigma B|^{\frac{1}{s}+\frac{1}{n}} \left[\left(\frac{1}{|\sigma B|}\int_{\sigma B} (w(x))^\alpha dx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B} w^{-\beta} dx\right)^{\frac{\alpha q}{\beta p}}\right]^{1/\alpha q} \\ &\le C^{1/\alpha q}_{\alpha,\beta,\frac{\alpha q}{p}} |\sigma B|^{\frac{1}{s}+\frac{1}{n}}. \end{aligned} \tag{2.5}$$

Combining (2.3)-(2.5) immediately yields that

$$\begin{aligned}&\left(\int_B |T(H(u)) - (T(H(u)))_B|^q w(x)dx\right)^{1/q}\\ &\le C_2 diam(B)|B|^{\frac{1}{q}-\frac{1}{s}-\frac{1}{m}}|\sigma B|^{\frac{1}{s}+\frac{1}{n}}\left(\int_{\sigma B} |u|^p w(x)dx\right)^{1/p}\\ &\le C_3 diam(B)|B|^{(p-q)/pq}\left(\int_{\sigma B} |u|^p w(x)dx\right)^{1/p}.\end{aligned}$$

This ends the proof of Theorem 2.5.

If we choose $p = q$ in Theorem 2.5, we have the following corollary.

Corollary 2.6. *Assume that u is a solution of the nonhomogeneous A-harmonic equation in a bounded convex domain D,* $1 < q < \infty$ *and* $w(x) \in A(\alpha, \beta, \alpha; D)$ *for some* $\alpha > 1$ *and* $\beta > 0$. *Let H be the projection operator and* $T : C^\infty(\wedge^k D) \to C^\infty(\wedge^{k-1} D)$, $k = 1, 2,..., n$, *be the homotopy operator. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |T(H(u)) - (T(H(u)))_B|^q w(x)dx\right)^{1/q} \le C diam(B)\left(\int_{\sigma B} |u|^q w(x)dx\right)^{1/q}$$

for all balls with $\sigma B \subset D$ *for some* $\sigma > 1$.

Next, we prove the following inequality, which is a generalized version of the one given in Lemma 2.2. More precisely, the inequality in Lemma 2.2 is a special case of the following result when $\phi(t) = t^p$.

Theorem 2.7. *Assume that* ϕ *is a Young function in the class* $G(p, q, C_0)$, $1 < p < q < \infty$, $C_0 \ge 1$ *and D is a bounded convex domain. If* $u \in C^\infty(\wedge^k D)$, $k = 1, 2,..., n$, *is a solution of the nonhomogeneous A-harmonic equation in D,* $\varphi(|u|) \in L^1_{loc}(D, dx)$ *and* $1/p - 1/q \le 1/n$, *then there exists a constant C, independent of u, such that*

$$\int_B \varphi(|T(H(u)) - (T(H(u)))_B|)dx \le C\int_{\sigma B} \varphi(|u|)dx$$

for all balls B with $\sigma B \subset D$, *where* $\sigma > 1$ *is a constant.*

Proof. From Lemma 2.2, we know that

$$||T(H(u)) - (T(H(u)))_B||_{s,B} \le C_1 diam(B)||u||_{s,\sigma B}$$

for $1 < s < \infty$. Note that u is a solution of the nonhomogeneous A-harmonic equation. Hence, by the reverse Hölder inequality, we have

$$\begin{aligned}&\left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{1/q}\\ &\le C_1 diam(B)\left(\int_{\sigma_1 B} |u|^q dx\right)^{1/q} \qquad (2.6)\\ &\le C_2 diam(B)|\sigma_1 B|^{(p-q)/pq}\left(\int_{\sigma_2 B} |u|^p dx\right)^{1/p},\end{aligned}$$

where $\sigma_2 > \sigma_1 > 1$ are some constants. Thus, using that ϕ and g are increasing functions as well as Jensen's inequality for g, we deduce that

$$\begin{aligned}
&\varphi\left(\left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{1/q}\right)\\
&\le \varphi\left(C_2 diam(B)|\sigma_1 B|^{(p-q)/pq}\left(\int_{\sigma_2 B}|u|^p dx\right)^{1/p}\right)\\
&\le \varphi\left(\left(C_2^p (diam(B))^p |\sigma_1 B|^{(p-q)/q}\int_{\sigma_2 B}|u|^p dx\right)^{1/p}\right)\\
&\le C_3 g\left(C_2^p (diam(B))^p |\sigma_1 B|^{(p-q)/q}\int_{\sigma_2 B}|u|^p dx\right)\\
&= C_3 g\left(\int_{\sigma_2 B} C_2^p (diam(B))^p |\sigma_1 B|^{(p-q)/q}|u|^p dx\right)\\
&\le C_3 \int_{\sigma_2 B} g(C_2^p (diam(B))^p |\sigma_1 B|^{(p-q)/q}|u|^p) dx.
\end{aligned} \tag{2.7}$$

Since $1/p - 1/q \le 1/n$, we have

$$diam(B)|\sigma_1 B|^{\frac{p-q}{pq}} \le C_4 |D|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} \le C_5. \tag{2.8}$$

Applying (2.7) and (2.8) and noting that $g(t) \le C_0 \phi(t^{1/p})$, we have

$$\begin{aligned}
&\int_{\sigma_2 B} g(C_2^p (diam(B))^p |\sigma_1 B|^{(p-q)/q}|u|^p) dx\\
&\le C_0 \int_{\sigma_2 B} \varphi(C_2 diam(B)|\sigma_1 B|^{(p-q)/pq}|u|) dx\\
&\le C_0 \int_{\sigma_2 B} \varphi(C_6 |u|) dx.
\end{aligned} \tag{2.9}$$

It follows from (2.7) and (2.9) that

$$\begin{aligned}
&\varphi\left(\left(\int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{1/q}\right)\\
&\le C_7 \int_{\sigma_2 B} \varphi(C_6 |u|) dx.
\end{aligned} \tag{2.10}$$

Applying Jensen's inequality once again to h^{-1} and considering that ϕ and h are doubling, we have

$$\begin{aligned}
&\int_B \varphi(|T(H(u)) - (T(H(u)))_B|) dx\\
&= h\left(h^{-1}\left(\int_B \varphi(|T(H(u)) - (T(H(u)))_B|) dx\right)\right)\\
&\le h\left(\int_B h^{-1}(\varphi(|T(H(u)) - (T(H(u)))_B|) dx)\right)\\
&\le h\left(C_8 \int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)\\
&\le C_0 \varphi\left(\left(C_8 \int_B |T(H(u)) - (T(H(u)))_B|^q dx\right)^{1/q}\right)\\
&\le C_9 \int_{\sigma_2 B} \varphi\ (C_6 |u|) dx\\
&\le C_{10} \int_{\sigma_2 B} \varphi\ (|u|) dx.
\end{aligned}$$

This ends the proof of Theorem 2.7.

To establish the weighted version of the inequality obtained in the above Theorem 2.7, we need the following lemma which appears in [4].

Lemma 2.8. *Let u be a solution of the nonhomogeneous A-harmonic equation in a domain E and* $0 < p, q < \infty$. *Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |u|^q d\mu\right)^{1/q} \le C(\mu(B))^{\frac{p-q}{pq}} \left(\int_{\sigma B} |u|^p d\mu\right)^{1/p}$$

for all balls B with $\sigma B \subset E$ *for some* $\sigma > 1$, *where the Radon measure* μ *is defined by* $d\mu = w(x)dx$ *and* $w \in A(\alpha, \beta, \alpha; E)$, $\alpha > 1$, $\beta > 0$.

Theorem 2.9. *Assume that* ϕ *is a Young function in the class* $G(p, q, C_0)$, $1 < p < q < \infty$, $C_0 \ge 1$ *and D is a bounded convex domain. Let* $d\mu = w(x)dx$, *where* $w(x) \in A(\alpha, \beta, \alpha; D)$ *for* $\alpha > 1$ *and* $\beta > 0$. *If* $u \in C^\infty(\wedge^k D)$, $k = 1, 2,..., n$, *is a solution of the nonhomogeneous A-harmonic equation in D*, $\varphi(|u|) \in L^1_{loc}(D, d\mu)$, *then there exists a constant C, independent of u, such that*

$$\int_B \varphi(|T(H(u)) - (T(H(u)))_B|)d\mu \le C \int_{\sigma B} \varphi(|u|)d\mu$$

for all balls B with $\sigma B \subset D$ *and* $|B| \ge d_0 > 0$, *where* $\sigma > 1$ *is a constant.*

Proof. From Corollary 2.6 and Lemma 2.8, we have

$$\begin{aligned} &\left(\int_B |T(H(u)) - (T(H(u)))_B|^q d\mu\right)^{1/q} \\ &\le C_1 diam(B)\left(\int_{\sigma_1 B} |u|^q d\mu\right)^{1/q} \\ &\le C_2 diam(B)(\mu(B))^{(p-q)/pq}\left(\int_{\sigma_2 B} |u|^p d\mu\right)^{1/p}, \end{aligned} \tag{2.11}$$

where $\sigma_2 > \sigma_1 > 1$ is some constant. Note that ϕ and g are increasing functions and g is convex in D. Hence by Jensen's inequality for g, we deduce that

$$\begin{aligned} &\varphi\left(\left(\int_B |T(H(u)) - (T(H(u)))_B|^q d\mu\right)^{1/q}\right) \\ &\le \varphi\left(C_2 diam(B)(\mu(B))^{(p-q)/pq}\left(\int_{\sigma_2 B} |u|^p d\mu\right)^{1/p}\right) \\ &= \varphi\left(\left(C_2^p (diam(B))^p (\mu(B))^{(p-q)/q} \int_{\sigma_2 B} |u|^p d\mu\right)^{1/p}\right) \\ &\le C_3 g\left(C_2^p (diam(B))^p (\mu(B))^{(p-q)/q} \int_{\sigma_2 B} |u|^p d\mu\right) \\ &= C_3 g\left(\int_{\sigma_2 B} C_2^p (diam(B))^p (\mu(B))^{(p-q)/q} |u|^p d\mu\right) \\ &\le C_3 \int_{\sigma_2 B} g\left(C_2^p (diam(B))^p (\mu(B))^{(p-q)/q} |u|^p\right) d\mu. \end{aligned} \tag{2.12}$$

Set $D_1 = \{x \in D : 0 < w(x) < 1\}$ and $D_2 = \{x \in D : w(x) \ge 1\}$. Then $D = D_1 \cup D_2$. We let $\tilde{w}(x) = 1$, if $x \in D_1$ and $\tilde{w}(x) = w(x)$, if $x \in D_2$. It is easy to check that $w(x) \in A(\alpha, \beta, \alpha; D)$ if and only if $\tilde{w}(x) \in A(\alpha, \beta, \alpha; D)$. Thus, we may always assume that $w(x) \ge 1$ a.e. in D. Hence, we have $\mu(B) = \int_B w(x)dx \ge |B|$ for all balls $B \subset D$. Since $p < q$ and |

$B| = d_0 > 0$, it is easy to find that

$$diam(B)\mu(B)^{(p-q)/pq} \leq diam(D)d_0^{(p-q)/pq} \leq C_3. \tag{2.13}$$

It follows from (2.13) and $g(t) \leq C_0\phi(t^{1/p})$ that

$$\begin{aligned} &\int_{\sigma_2 B} g(C_2^p(diam(B))^p(\mu(B))^{(p-q)/q}|u|^p)d\mu \\ &\leq C_0 \int_{\sigma_2 B} \varphi(C_2 diam(B)(\mu(B))^{(p-q)/pq}|u|)d\mu \\ &\leq C_0 \int_{\sigma_2 B} \varphi(C_4|u|)d\mu. \end{aligned} \tag{2.14}$$

Applying Jensen's inequality to h^{-1} and considering that ϕ and h are doubling, we have

$$\begin{aligned} &\int_B \varphi(|T(H(u)) - (T(H(u)))_B|)d\mu \\ &= h\left(h^{-1}\left(\int_B \varphi(|T(H(u)) - (T(H(u)))_B|)d\mu\right)\right) \\ &\leq h\left(\int_B h^{-1}(\varphi(|T(H(u)) - (T(H(u)))_B|)d\mu)\right) \\ &\leq h\left(C_8 \int_B |T(H(u)) - (T(H(u)))_B|^q d\mu\right) \\ &\leq C_0\varphi\left(\left(C_8 \int_B |T(H(u)) - (T(H(u)))_B|^q d\mu\right)^{1/q}\right) \\ &\leq C_9 \int_{\sigma_2 B} \varphi(C_6|u|)d\mu \\ &\leq C_{10} \int_{\sigma_2 B} \varphi(|u|)d\mu. \end{aligned}$$

This ends the proof of Theorem 2.9.

Note that if we remove the restriction on balls B, then we can obtain a weighted inequality in the class $A(\alpha, \beta, \frac{\alpha q}{p}; D)$, for which the method of proof is analogous to the one in Theorem 2.9. We now give the statement as follows.

Theorem 2.10. *Assume that ϕ is a Young function in the class $G(p, q, C_0)$, $1 < p < q < \infty$, $C_0 \geq 1$ and D is a bounded convex domain. Let $d\mu = w(x)dx$, where $w(x) \in A(\alpha, \beta, \frac{\alpha q}{p}; D)$ for $\alpha > 1$ and $\beta > 0$. If $u \in C^\infty(\wedge^k D)$, $k = 1, 2,..., n$, is a solution of the nonhomogeneous A-harmonic equation in D, $\varphi(|u|) \in L^1_{loc}(D, d\mu)$ and $1/p - 1/q \leq 1/n$, then there exists a constant C, independent of u, such that*

$$\int_B \varphi(|T(H(u)) - (T(H(u)))_B|)d\mu \leq C \int_{\sigma B} \varphi(|u|)d\mu$$

for all balls B with $\sigma B \subset D$, where $\sigma > 1$ is a constant.

Directly from the proof of Theorem 2.7, if we replace $|T(H(u))-(T(H(u)))_B|$ by $\frac{1}{\lambda}|T(H(u)) - (T(H(u)))_B|$, then we immediately have

$$\int_B \varphi\left(\frac{|T(H(u)) - (T(H(u)))_B|}{\lambda}\right) dx \leq C \int_{\sigma B} \varphi\left(\frac{|u|}{\lambda}\right) dx \tag{2.15}$$

for all balls B with $\sigma B \subset D$ and $\lambda > 0$. Furthermore, from the definition of the Orlicz norm and (2.15), the following Orlicz norm inequality holds.

Corollary 2.11. *Assume that ϕ is a Young function in the class $G(p, q, C_0)$, $1 < p < q < \infty$, $C_0 \geq 1$ and D is a bounded convex domain. If $u \in C^\infty(\wedge^k D)$, $k = 1, 2, \ldots, n$, is a solution of the nonhomogeneous A-harmonic equation in D, $\varphi(|u|) \in L^1_{loc}(D, dx)$and $1/p - 1/q \leq 1/n$, then there exists a constant C, independent of u, such that*

$$||T(H(u)) - (T(H(u)))_B||_{\varphi,B} \leq C||u||_{\varphi,\sigma B} \tag{2.16}$$

for all balls B with $\sigma B \subset D$, where $\sigma > 1$ is a constant.

Next, we extend the local Orlicz norm inequality for the composite operator to the global version in the $L^\phi(\mu)$-averaging domains.

In [12], Staples introduced L^s-averaging domains in terms of Lebesgue measure. Then, Ding and Nolder [6] developed L^s-averaging domains to weighted versions and obtained a similar characterization. At the same time, they also established a global norm inequality for conjugate A-harmonic tensors in $L^s(\mu)$-averaging domains. In the following year, Ding [5] further generalized L^s-averaging domains to $L^\phi(\mu)$-averaging domains, for which $L^s(\mu)$-averaging domains are special cases when $\phi(t) = t^s$. The following definition appears.

Definition 2.12. *Let ϕ be an increasing convex function defined on $[0, \infty)$ with $\phi(0) = 0$. We say a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L^\phi(\mu)$-averaging domain, if $\mu(\Omega) < \infty$ and there exists a constant C such that*

$$\int_\Omega \varphi(\tau|u - u_{B_0}|)d\mu \leq C \sup_B \int_B \varphi(\sigma|u - u_B|)d\mu$$

for some balls $B_0 \subset \Omega$ and all u such that $\varphi(|u|) \in L^1_{loc}(\Omega, d\mu)$, where $0 < \tau, \sigma < \infty$ are constants and the supremum is over all balls $B \subset \Omega$.

Theorem 2.13. *Let ϕ be a Young function in the class $G(p, q, C_0)$, $1 < p < q < \infty$, $C_0 \geq 1$ and D is a bounded convex $L^\phi(dx)$-averaging domain. Suppose that $\phi(|u|) \in L^1(D, dx)$, $u \in C^\infty(\wedge^1 D)$ is a solution of the nonhomogeneous A-harmonic equation in D and $1/p - 1/q \leq 1/n$. Then there exists a constant C, independent of u, such that*

$$\int_D \varphi(|T(H(u)) - (T(H(u)))_{B_0}|)dx \leq C\int_D \varphi(|u|)dx, \tag{2.17}$$

where $B_0 \subset D$ is a fixed ball.

Proof. Since D is an $L^\phi(dx)$-averaging domain and ϕ is doubling, from Theorem 2.7, we have

$$\begin{aligned}
&\int_D \varphi(|T(H(u)) - (T(H(u)))_{B_0}|)dx \\
&\leq C_1 \sup_{B\subset D} \int_B \varphi(|T(H(u)) - (T(H(u)))_B|)dx \\
&\leq C_1 \sup_{B\subset D} \left(C_2 \int_{\sigma B} \varphi(|u|)dx\right) \\
&\leq C_3 \int_D \varphi(|u|)dx.
\end{aligned}$$

We have completed the proof of Theorem 2.13.

Clearly, (2.17) implies that

$$||T(H(u)) - (T(H(u)))_{B_0}||_{\varphi,D} \leq C||u||_{\varphi,D}. \tag{2.18}$$

Similarly, we also can develop the inequalities established in Theorems 2.9 and 2.10 to $L^{\phi}(\mu)$-averaging domains, for which $d\mu = w(x)dx$ and $w(x) \in A(\alpha, \beta, \alpha; D)$ and $A(\alpha, \beta, \frac{\alpha q}{p}; D)$, respectively.

3. Applications

The homotopy operator provides a decomposition to differential forms $\omega \in L^p(D, \wedge^k)$ such that $d\omega \in L^p(D, \wedge^{k+1})$. Sometimes, however, the expression of $T(H(u))$ or $(TH(u))_B$ may be quite complicated. However, using the estimates in the previous section, we can obtain the upper bound for the Orlicz norms of $T(H(u))$ or $(TH(u))_B$. In this section, we give some specific estimates for the solutions of the nonhomogeneous A-harmonic equation. Meantime, we also give several Young functions that lie in the class $G(p, q, C)$ and then establish some corresponding norm inequalities for the composite operator.

In fact, the nonhomogeneous A-harmonic equation is an extension of many familiar equations. Let $B = 0$ and u be a 0-form in the nonhomogeneous A-harmonic equation (1.4). Thus, (1.4) reduces to the usual A-harmonic equation:

$$divA(x, \nabla u) = 0. \tag{3.1}$$

In particular, if we take the operator $A(x, \xi) = \xi|\xi|^{p-2}$, then Equation 3.1 further reduces to the p-harmonic equation

$$div(\nabla u|\nabla u|^{p-2}) = 0. \tag{3.2}$$

It is easy to verify that the famous Laplace equation $\Delta u = 0$ is a special case of $p = 2$ to the p-harmonic equation.

In $\mathbb{R}^3$, consider that

$$\omega = \frac{1}{r^3}(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2), \tag{3.3}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It is easy to check that $d\omega = 0$ and $|\omega| = \frac{1}{r^2}|$. Hence, ω is a solution of the nonhomogeneous A-harmonic equation. Let B be a ball with the origin $O \notin \sigma B$, where $\sigma > 1$ is a constant. Usually the term $\int_B \varphi(|T(H(\omega)) - (T(H(\omega)))_B|)dx$ is not easy to estimate due to the complexity of the operators T and H as well as the function ϕ. However, by Theorem 2.7, we can give an upper bound of Orlicz norm. Specially, if the Young function ϕ is not very complicated, sometimes it is possible to obtain a specific upper bound. For instance, take $\phi(t) = t^p log_+ t$, where $log_+ t = 1$ if $t \leq e$ and $log_+ t = log\, t$ if $t > e$. It is easy to verify that $\phi(t) = t^p log_+ t$ is a Young function and belongs to $G(p_1, p_2, C)$ for some constant $C = C(p_1, p_2, p)$. Let $0 < M < \infty$ be the upper bound of $|\omega|$ in σB. Thus, we have

$$\begin{aligned} &\int_B |T(H(\omega)) - (T(H(\omega)))_B|^p log_+ |T(H(\omega)) - (T(H(\omega)))_B| dx \\ &\leq \int_{\sigma B} |\omega|^p log_+(|\omega|)dx \leq \int_{\sigma B} M^p log_+ M dx = M^p log_+ M |\sigma B|, \end{aligned}$$

where $\sigma > 1$ is some constant. Also, if we let $\phi(t) = t^p log_+ t$ in Theorem 2.13, we can obtain a global estimate in a bounded convex $L^{\phi}(dx)$-averaging domain D without the origin. That is

$$\int_D |T(H(\omega)) - (T(H(\omega)))_{B_0}|^p log_+ |T(H(\omega)) - (T(H(\omega)))_{B_0}| dx$$
$$\leq \int_D |\omega|^p log_+(|\omega|) dx \leq \int_D N^p log_+ N dx = N^p log_+ N |D|,$$

where $B_0 \subset D$ is a fixed ball and N is the upper bound of $|\omega|$ in D.

Next we give some examples of Young functions that lie in $G(p, q, C)$ and then apply them to Theorem 2.9.

Consider the function $\Psi(t) = t^p log_+^{\alpha} t$, $1 < p < \infty$, $\alpha \in \mathbb{R}$. Obviously, if we take $\alpha = 1$, then $\Psi(t)$ reduces to $\phi(t) = t^p log_+ t$ mentioned above. It is easy to check that for all $1 \leq p_1 < p < p_2 < \infty$ and $\alpha \in \mathbb{R}$, the function $\Psi(t) \in G(p_1, p_2, C)$, where C is dependent on p, p_1, p_2 and α. However, $\Psi(t)$ is not always a Young function. More precisely, $\Psi(t)$ cannot guarantee to be both increasing and convex. However, note that for $\Psi(t)$, we can always find $K > 1$ depending on p and α such that the function $\Psi(t)$ is increasing and convex on both $[0, 1]$ and $[K, \infty)$. Furthermore, if let $\Psi_K(t) = \Psi(t)$ on $[0, 1] \cup [K, \infty)$ and $\Psi_K(t) = \Psi(1) + \frac{\Psi(K)-\Psi(1)}{K-1}(t-1)$ in $(1, K)$, then $\Psi_K(t)$ still lies in $G(p_1, p_2, C)$ for some $C = C(p, \alpha, p_1, p_2)$. It is worth noting that after such modification $\Psi_K(t)$ is convex in the entire interval $[0, \infty)$, in the sense that $\Psi_K(t)$ is a Young function that lies in the class $G(p, q, C)$; see [10] for more details on $\Psi_K(t)$. Thus, we have the following result.

Corollary 3.1. *Assume that $u \in C^{\infty}(\wedge^k D)$, $k = 1, 2, \ldots, n$, is a solution of the nonhomogeneous A-harmonic equation in D, where D is a bounded convex domain. Let $d\mu = w(x)dx$ and $\Psi_K(|u|) \in L^1_{loc}(D, d\mu)$, where $w(x) \in A(\alpha, \beta, \alpha; D)$ for $\alpha > 1$ and $\beta > 0$. Then, for the composition of the homotopy operator T and the projection operator H, we have*

$$\int_B \Psi_K(|T(H(u)) - (T(H(u)))_B|) d\mu \leq C \int_{\sigma B} \Psi_K(|u|) d\mu$$

for all balls B with $\sigma B \subset D$ and $|B| \geq d_0 > 0$. Here σ and C are constants and C is independent of u.

For the other example consider the function $\Phi(t) = t^p \sin t$, on $[0, \frac{\pi}{2}]$ and $\Phi(t) = t^p$, in $(\frac{\pi}{2}, \infty)$, $3 < p < \infty$. It is easy to check that $\Phi(t)$ is a Young function and for all $0 < p_1 < p + 1 < p_2 < \infty$, $\Phi(t) \in G(p_1, p_2, C)$, where $C = C(p, p_1, p_2) \geq 1$ is some constant. Thus, Theorem 2.9 holds for $\Phi(t)$ and we have the following corollary.

Corollary 3.2. *Assume that $u \in C^{\infty}(\wedge^k D)$, $k = 1, 2, \ldots, n$, is a solution of the nonhomogeneous A-harmonic equation in D, where D is a bounded convex domain. Let $d\mu = w(x)dx$ and $\Phi(|u|) \in L^1_{loc}(D, d\mu)$, where $w(x) \in A(\alpha, \beta, \alpha; D)$ for $\alpha > 1$ and $\beta > 0$. Then, for the composition of the homotopy operator T and the projection operator H, we have*

$$\int_B \Phi(|T(H(u)) - (T(H(u)))_B|) d\mu \leq C \int_{\sigma B} \Phi(|u|) d\mu$$

for all balls B with $\sigma B \subset D$ and $|B| \geq d_0 > 0$. Here σ and C are constants and C is independent of u.

Acknowledgements
The authors express their sincere thanks to the referee for his/her thoughtful suggestions. H.B. was supported by the Foundation of Education Department of Heilongjiang Province in 2011 (#12511111) and by the Youth Foundation at the Harbin University of Science and Technology (# 2009YF033).

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Authors' contributions
HB and SD jointly contributed to the main results and HB wrote the paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 8 March 2011 Accepted: 24 September 2011 Published: 24 September 2011

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doi:10.1186/1029-242X-2011-69
Cite this article as: Bi and Ding: Orlicz norm inequalities for the composite operator and applications. *Journal of Inequalities and Applications* 2011 **2011**:69.