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Inequalities for Green's operator applied to the minimizers

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Abstract

In this paper, we prove both the local and global L^{ϕ}-norm inequalities for Green's operator applied to minimizers for functionals defined on differential forms in L^{ϕ}-averaging domains. Our results are extensions of L^p norm inequalities for Green's operator and can be used to estimate the norms of other operators applied to differential forms.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, B and σB with $\sigma > 0$ be the balls with the same center and $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$ throughout this paper. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is expressed by $|E|$. For any function u, we denote the average of u over B by $u_B = \frac{1}{|B|}\int_B u dx$. All integrals involved in this paper are the Lebesgue integrals.

A differential 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \cdots, x_n)dx_i$, where the coefficient functions $u_i(x_1, x_2,..., x_n)$, $i = 1, 2,..., n$, are differentiable. Similarly, a differential k-form $u(x)$ can be denoted as

$$u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $I = (i_1, i_2, ..., i_k)$, $1 \leq i_1 < i_2 < ... < i_k \leq n$. See [1-5] for more properties and some recent results about differential forms. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(\Omega, \wedge^l)$ be the space of all differential l-forms in Ω, and $L^p(\Omega, \wedge^l)$ be the Banach space of all l-forms $u(x) = \Sigma_I u_I(x)dx_I$ in Ω satisfying

$$\| u\|_{p,E} = \left(\int_E |u(x)|^p dx\right)^{1/p} = \left(\int_E \left(\sum_I |u_I(x)|^2\right)^{p/2} dx\right)^{1/p}$$

for all ordered l-tuples I, $l = 1, 2,..., n$. It is easy to see that the space $\wedge^l$ is of a basis

$$\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l},\ 1 \leq i_1 < i_2 < \cdots < i_l \leq n\},$$

and hence $dim(\wedge^l) = dim(\wedge^l(\mathbb{R}^n)) = \binom{n}{l}$ and

$$dim(\wedge) = \sum_{l=0}^{n} dim(\wedge^l(\mathbb{R}^n)) = \sum_{l=0}^{n} \binom{n}{l} = 2^n.$$

We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1,..., n - 1$. The exterior differential can be calculated as follows

$$d\omega(x) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_l \le n} \frac{\partial \omega_{i_1 i_2 \cdots i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}.$$

Its formal adjoint operator $d^\star$ which is called the Hodge codifferential is defined by $d^\star = (-1)^{nl+1} \star d\star$: $D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$, where $l = 0, 1,..., n - 1$, and $\star$ is the well known Hodge star operator. We say that $u \in L^1_{loc}(\wedge^l\Omega)$ has a generalized gradient if, for each coordinate system, the pullbacks of the coordinate function of u have generalized gradient in the familiar sense, see [6]. We write $\mathcal{W}(\wedge^l\Omega) = \{u \in L^1_{loc}(\wedge^l\Omega)$: u has generalized gradient$\}$. As usual, the harmonic l-fields are defined by $\mathcal{H}(\wedge^l\Omega) = \{u \in \mathcal{W}(\wedge^l\Omega) : du = d^\star u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$, The orthogonal complement of $\mathcal{H}$ in L^1 is defined by $\mathcal{H}^\perp = \{u \in L^1 : < u, h >= 0 \text{ for all } h \in \mathcal{H}\}$. Greens' operator G is defined as $G : C^\infty(\wedge^l\Omega) \to \mathcal{H}^\perp \cap C^\infty(\wedge^l\Omega)$ by assigning $G(u)$ be the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^l\Omega)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is either the harmonic projection or sometimes the harmonic part of u and Δ is the Laplace-Beltrami operator, see [2,7-11] for more properties of Green's operator. In this paper, we alway use G to denote Green's operator.

2. Local inequalities

The purpose of this paper is to establish the L^ϕ-norm inequalities for Green's operator applied to the following k-quasi-minimizer. We say a differential form $u \in W^{1,1}_{loc}(\Omega, \Lambda^\ell)$ is a k-quasi-minimizer for the functional

$$I(\Omega; v) = \int_\Omega (|dv|)dx \tag{2.1}$$

if and only if, for every $\varphi \in W^{1,1}_{loc}(\Omega, \Lambda^\ell)$ with compact support,

$$I(\text{supp } \varphi; u) \le k \cdot I(\text{supp } \varphi; u + \varphi),$$

where $k > 1$ is a constant. We say that ϕ satisfies the so called Δ_2-condition if there exists a constant $p > 1$ such that

$$\varphi(2t) \le p\varphi(t) \tag{2.2}$$

for all $t > 0$, from which it follows that $\phi(\lambda t) \le \lambda^p \phi(t)$ for any $t > 0$ and $\lambda \ge 1$, see [12].

We will need the following lemma which can be found in [13] or [12].

Lemma 2.1. *Let $f(t)$ be a nonnegative function defined on the interval $[a, b]$ with $a \ge 0$. Suppose that for $s, t \in [a, b]$ with $t < s$,*

$$f(t) \le \frac{M}{(s-t)^\alpha} + N + \theta f(s)$$

holds, where M, N, α *and* θ *are nonnegative constants with* $\theta < 1$. *Then, there exists a constant* $C = C(\alpha, \theta)$ *such that*

$$f(\rho) \le C\left(\frac{M}{(R-\rho)^\alpha} + N\right)$$

for any $\rho, R \in [a, b]$ with $\rho < R$.

A continuously increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$, is called an Orlicz function.

The Orlicz space $L^\phi(\Omega)$ consists of all measurable functions f on Ω such that $\int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx < \infty$ for some $\lambda = \lambda(f) > 0$. $L^\phi(\Omega)$ is equipped with the nonlinear Luxemburg functional

$$\| f \|_{\varphi(\Omega)} = inf\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx \le 1\}.$$

A convex Orlicz function ϕ is often called a Young function. A special useful Young function $\phi : [0, \infty) \to [0, \infty)$, termed an N-function, is a continuous Young function such that $\phi(x) = 0$ if and only if $x = 0$ and $lim_{x \to 0} \phi(x)/x = 0$, $lim_{x \to \infty} \phi(x)/x = +\infty$. If ϕ is a Young function, then $|| \cdot ||_\phi$ defines a norm in $L^\phi(\Omega)$, which is called the Luxemburg norm.

Definition 2.2[14]. We say a Young function ϕ lies in the class $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$, if (i) $1/C \le \phi(t^{1/p})/\Phi(t) \le C$ and (ii) $1/C \le \phi(t^{1/q})/\Psi(t) \le C$ for all $t > 0$, where Φ is a convex increasing function and Ψ is a concave increasing function on $[0, \infty)$.

From [14], each of ϕ, Φ and Ψ in above definition is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$C_1 t^q \le \Psi^{-1}(\varphi(t)) \le C_2 t^q,\ C_1 t^p \le \Phi^{-1}(\varphi(t)) \le C_2 t^p, \tag{2.3}$$

where C_1 and C_2 are constants. It is easy to see that $\phi \in G(p, q, C)$ satisfies the Δ_2-condition. Also, for all $1 \le p_1 < p < p_2$ and $\alpha \in \mathbb{R}$, the function $\varphi(t) = t^p log_+^\alpha t$ belongs to $G(p_1, p_2, C)$ for some constant $C = C(p, \alpha, p_1, p_2)$. Here $log_+(t)$ is defined by $log_+(t) = 1$ for $t \le e$; and $log_+(t) = log(t)$ for $t > e$. Particularly, if $\alpha = 0$, we see that $\phi(t) = t^p$ lies in $G(p_1, p_2, C)$, $1 \le p_1 < p < p_2$.

Theorem 2.3. *Let* $u \in W^{1,1}_{loc}(\Omega, \Lambda^\ell)$ *be a k-quasi-minimizer for the functional* (2.1), ϕ *be a Young function in the class* $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$ *and* $q(n - p) < np$, Ω *be a bounded domain and G be Green's operator. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi(|G(u) - (G(u))_B|)dx \le C \int_{2B} \varphi(|u - c|)dx \tag{2.4}$$

for all balls $B = B_r$ *with radius r and* $2B \subset \Omega$, *where c is any closed form.*

Proof. Using Jensen's inequality for Ψ^{-1}, (2.3), and noticing that ϕ and Ψ are doubling, for any ball $B = B_r \subset \Omega$, we obtain

$$\begin{aligned}\int_B \varphi\left(|G(u)-(G(u))_B|\right)dx &= \Psi\left(\Psi^{-1}\left(\int_B \varphi(|G(u)-(G(u))_B|)dx\right)\right)\\ &\le \Psi\left(\int_B \Psi^{-1}\left(\varphi(|G(u)-(G(u))_B|)\right)dx\right)\\ &\le \Psi\left(C_1\int_B |G(u)-(G(u))_B|^q dx\right)\\ &\le C_2\varphi\left(\left(C_1\int_B |G(u)-(G(u))_B|^q dx\right)^{1/q}\right)\\ &\le C_3\varphi\left(\left(\int_B |G(u)-(G(u))_B|^q dx\right)^{1/q}\right).\end{aligned} \tag{2.5}$$

Using the Poincaré-type inequality for differential forms $G(u)$ and noticing that

$$\| G(u) \|_{p,B} \le C_4||u||_{p,B}$$

holds for any differential form u, we obtain

$$\begin{aligned}&\left(\int_B |G(u)-(G(u))_B|^{np/(n-p)}dx\right)^{(n-p)/np}\\ &\le C_5\left(\int_B |d(G(u))|^p dx\right)^{1/p}\\ &\le C_5\left(\int_B |G(du)|^p dx\right)^{1/p}\\ &\le C_6\left(\int_B |du|^p dx\right)^{1/p}.\end{aligned} \tag{2.6}$$

If $1 < p < n$, by assumption, we have $q < \frac{np}{n-p}$. Then,

$$\left(\int_B |G(u)-(G(u))_B|^q dx\right)^{1/q} \le C_7\left(\int_B |du|^p dx\right)^{1/p}. \tag{2.7}$$

Note that the L^p-norm of $|G(u)$ - $(G(u))_B|$ increases with p and $\frac{np}{n-p} \to \infty$ as $p \to n$, it follows that (2.7) still holds when $p \ge n$. Since ϕ is increasing, from (2.5) and (2.7), we obtain

$$\int_B \varphi\left(|G(u)-(G(u))_B|\right)dx \le C_3\varphi\left(C_7\left(\int_B |du|^p dx\right)^{1/p}\right). \tag{2.8}$$

Applying (2.8), (i) in Definition 2.2, Jensen's inequality, and noticing that ϕ and Φ are doubling, we have

$$\begin{aligned}\int_B \varphi\left(|G(u)-(G(u))_B|\right)dx &\le C_3\varphi\left(C_7\left(\int_B |du|^p dx\right)^{1/p}\right)\\ &\le C_3\Phi\left(C_8\left(\int_B |du|^p dx\right)\right)\\ &\le C_9\int_B \Phi(|du|^p)dx.\end{aligned} \tag{2.9}$$

Using (i) in Definition 1.1 again yields

$$\int_B \Phi(|du|^p)dx \le C_{10}\int_B \varphi(|du|)dx. \tag{2.10}$$

Combining (2.9) and (2.10), we obtain

$$\int_B \varphi\left(|G(u)-(G(u))_B|\right)dx \le C_{11}\int_B \varphi(|du|)dx \tag{2.11}$$

for any ball $B \subset \Omega$. Next, let $B_{2r} = B(x_0, 2r)$ be a ball with radius $2r$ and center x_0, $r < t < s < 2r$. Set $\eta(x) = g(|x - x_0|)$, where

$$g(\tau) = \begin{cases} 1, & 0 \le \tau \le t \\ \text{affine}, & \tau < t < s \\ 0, & \tau \ge s. \end{cases}$$

Then, $\eta \in W_0^{1,\infty}(B_s)$, $\eta(x) = 1$ on B_t and

$$|d\eta(x)| = \begin{cases} (s-t)^{-1}, & t \le |x-x_0| \le s \\ 0, & \text{otherwise}. \end{cases} \tag{2.12}$$

Let $v(x) = u(x) + (\eta(x))^p(c - u(x))$, where c is any closed form. We find that

$$dv = (1-\eta^p)du + \eta^p p\frac{d\eta}{\eta}(c-u(x)). \tag{2.13}$$

Since ψ is an increasing convex function satisfying the Δ_2-condition, we obtain

$$\varphi(|dv|) \le (1-\eta^p)\varphi(|du|) + \eta^p\varphi(p\frac{|d\eta|}{\eta}|c-u(x)|). \tag{2.14}$$

Using the definition of the k-quasi-minimizer and (2.2), it follows that

$$\begin{aligned}\int_{B_s}\varphi(|du|)dx &\le k\int_{B_s}\varphi(|dv|)dx \\ &\le k\left(\int_{B_s\setminus B_t}(1-\eta^p)\varphi(|du|)dx + \int_{B_s}\eta^p\varphi\left(p\frac{|d\eta|}{\eta}|c-u(x)|\right)dx\right) \\ &\le k\left(\int_{B_s\setminus B_t}\varphi(|du|)dx + p^p\int_{B_s}\varphi\left(|d\eta||u-c|\right)dx\right).\end{aligned} \tag{2.15}$$

Applying (2.15), (2.12)) and (2.3), we have

$$\begin{aligned}\int_{B_t}\varphi(|du|)dx &\le \int_{B_s}\varphi(|du|)dx \\ &\le k\left(\int_{B_s\setminus B_t}\varphi(|du|)dx + p^p\int_{B_s}\varphi\left(4r\frac{|u-c|}{(s-t)2r}\right)dx\right) \\ &\le k\left(\int_{B_s\setminus B_t}\varphi(|du|)dx + \frac{(4pr)^p}{(s-t)^p}\int_{B_s}\varphi\left(\frac{|u-c|}{2r}\right)dx\right).\end{aligned} \tag{2.16}$$

Adding $k\int_{B_t}\varphi(|du|)dx$ to both sides of inequality (2.16) yields

$$\int_{B_t}\varphi(|du|)dx \le \frac{k}{k+1}\left(\int_{B_s}\varphi(|du|)dx + \frac{(4pr)^p}{(s-t)^p}\int_{B_s}\varphi\left(\frac{|u-c|}{2r}\right)dx\right). \tag{2.17}$$

In order to use Lemma 2.1, we write

$$f(t) = \int_{B_t} \varphi(|du|)dx,\ f(s) = \int_{B_s} \varphi(|du|)dx,\ M = (4pr)^p \int_{B_s} \varphi \left(\frac{|u-c|}{2r}\right) dx$$

and $N = 0$. From (2.17), we find that the conditions of Lemma 2.1 are satisfied. Hence, using Lemma 2.1 with $\rho = r$ and $\alpha = p$, we obtain

$$\int_{B_r} \varphi(|du|)dx \le C_{12} \int_{B_{2r}} \varphi \left(\frac{|u-c|}{2r}\right) dx, \tag{2.18}$$

Note that ϕ is doubling, $B = B_r$ and $2B = B_{2r}$. Then, (3.18) can be written as

$$\int_B \varphi(|du|)dx \le C_{13} \int_{2B} \varphi\ (|u-c|)dx. \tag{2.19}$$

Combining (2.11) and (2.19) yields

$$\int_B \varphi\left(|G(u) - (G(u))_B|\right) dx \le C_{14} \int_{2B} \varphi\ (|u-c|)dx. \tag{2.20}$$

The proof of Theorem 2.3 has been completed. □

Since each of ϕ, Φ and Ψ in Definition 2.2 is doubling, from the proof of Theorem 2.3 or directly from (2.3), we have

$$\int_B \varphi \left(\frac{|G(u) - (G(u))_B|}{\lambda}\right) dx \le C \int_{2B} \varphi \left(\frac{|u-c|}{\lambda}\right) dx \tag{2.21}$$

for all balls B with $2B \subset \Omega$ and any constant $\lambda > 0$. From definition of the Luxemburg norm and (2.21), the following inequality with the Luxemburg norm

$$\| G(u) - (G(u))_B \|_{\varphi(B)} \le C \| u - c\|_{\varphi(2B)} \tag{2.22}$$

holds under the conditions described in Theorem 2.3.

Note that in Theorem 2.3, c is any closed form. Hence, we may choose $c = 0$ in Theorem 2.3 and obtain the following version of ϕ-norm inequality which may be convenient to be used.

Corollary 2.4. *Let $u \in W^{1,1}_{loc}(\Omega, \Lambda^\ell)$be a k-quasi-minimizer for the functional* (2.1), ϕ *be a Young function in the class G(p, q, C), $1 \le p < q < \infty$, $C \ge 1$ and $q(n - p) < np$, Ω be a bounded domain and G be Green's operator. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi(|G(u) - (G(u))_B|)dx \le C \int_{2B} \varphi\ (|u|)dx \tag{2.23}$$

for all balls $B = B_r$ with radius r and $2B \subset \Omega$.

3. Global inequalities

In this section, we extend the local Poincaré type inequalities into the global cases in the following L^ϕ-averaging domains, which are extension of John domains and L^s-averaging domain, see [15,16].

Definition 3.1[16]. Letϕ be an increasing convex function on $[0, \infty)$ with $\phi(0) = 0$. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an L^ϕ-averaging domain, if $|\Omega| < \infty$ and there exists a constant C such that

$$\int_{\Omega} \varphi(\tau|u-u_{B_0}|)dx \le C \sup_{B\subset\Omega} \int_B \varphi(\sigma|u-u_B|)dx \tag{3.1}$$

for some ball $B_0 \subset \Omega$ and all u such that $\varphi(|u|) \in L^1_{loc}(\Omega)$, where τ, σ are constants with $0 < \tau < \infty$, $0 < \sigma < \infty$ and the supremum is over all balls $B \subset \Omega$.

From above definition we see that L^s-averaging domains and $L^s(\mu)$-averaging domains are special L^{ϕ}-averaging domains when $\phi(t) = t^s$ in Definition 3.1. Also, uniform domains and John domains are very special L^{ϕ}-averaging domains, see [1,15,16] for more results about domains.

Theorem 3.2. *Let $u \in W^{1,1}_{loc}(\Omega, \Lambda^0)$ be a k-quasi-minimizer for the functional* (2.1), *ϕ be a Young function in the class $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$ and $q(n - p) < np$, Ω be any bounded L^{ϕ}-averaging domain and G be Green's operator. Then, there exists a constant C, independent of u, such that*

$$\int_{\Omega} \varphi(|G(u)-(G(u))_{B_0}|)dx \le C \int_{\Omega} \varphi\,(|u-c|)dx, \tag{3.2}$$

where $B_0 \subset \Omega$ is some fixed ball and c is any closed form.

Proof. From Definition 3.1, (2.4) and noticing that ϕ is doubling, we have

$$\begin{aligned}\int_{\Omega} \varphi(|G(u)-(G(u))_{B_0}|)dx &\le C_1 \sup_{B\subset\Omega} \int_B \varphi(|G(u)-(G(u))_B|)dx\\ &\le C_1 \sup_{B\subset\Omega} \left(C_2 \int_{2B} \varphi(|u-c|)dx \right)\\ &\le C_1 \sup_{B\subset\Omega} \left(C_2 \int_{\Omega} \varphi(|u-c|)dx \right)\\ &\le C_3 \int_{\Omega} \varphi(|u-c|)dx.\end{aligned}$$

We have completed the proof of Theorem 3.2. □

Similar to the local inequality, the following global inequality with the Orlicz norm

$$\| G(u)-(G(u))_{B_0} \|_{\varphi(\Omega)} \le C \| u \|_{\varphi(\Omega)} \tag{3.3}$$

holds if all conditions in Theorem 3.2 are satisfied.

We know that any John domain is a special L^{ϕ}-averaging domain. Hence, we have the following inequality in John domain.

Theorem 3.3. *Let $u \in W^{1,1}_{loc}(\Omega, \Lambda^0)$be a k-quasi-minimizer for the functional* (2.1), *ϕ be a Young function in the class $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$ and $q(n - p) < np$, Ω be any bounded John domain and G be Green's operator. Then, there exists a constant C, independent of u, such that*

$$\int_{\Omega} \varphi(|G(u)-(G(u))_{B_0}|)dx \le C \int_{\Omega} \varphi\,(|u-c|)dx, \tag{3.4}$$

where $B_0 \subset \Omega$ is some fixed ball and c is any closed form.

Choosing $\varphi(t) = t^p \log_+^{\alpha} t$ in Theorems 3.2, we obtain the following inequalities with the $L^p(\log_+^{\alpha} L)$-norms.

Corollary 3.4. *Let $u \in W^{1,1}_{loc}(\Omega, \Lambda^0)$be a k-quasi-minimizer for the functional* (2.1), *$\varphi(t) = t^p \log_+^{\alpha} t$, $\alpha \in \mathbb{R}$, $q(n - p) < np$ for $1 \le p < q < \infty$ and G be Green's operator. Then,* *there exists a constant C, independent of u, such that*

$$\int_\Omega |G(u)-(G(u))_{B_0}|^p log_+^\alpha(|G(u)-(G(u))_{B_0}|)dx \le C\int_\Omega |u-c|^p log_+^\alpha(|u-c|)dx \quad (3.5)$$

for any bounded L^ϕ-averaging domain Ω, where $B_0 \subset \Omega$ is some fixed ball and c is any closed form.

We can also write (3.5) as the following inequality with the Luxemburg norm

$$\| G(u)-(G(u))_{B_0}\|_{L^p(log_+^\alpha L)(\Omega)} \le C \parallel u-c\|_{L^p(log_+^\alpha L)(\Omega)} \quad (3.6)$$

provided the conditions in Corollary 3.5 are satisfied.

Similar to the local case, we may choose $c = 0$ in Theorem 3.2 and obtain he following version of L^ϕ-norm inequality.

Corollary 3.5. *Let $u \in W_{loc}^{1,1}(\Omega, \Lambda^0)$be a k-quasi-minimizer for the functional* (2.1), ϕ *be a Young function in the class G(p, q, C), $1 \le p < q < \infty$, $C \ge 1$ and $q(n - p) < np$, Ω be any bounded L^ϕ -averaging domain and G be Green's operator. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi(|G(u)-(G(u))_{B_0}|)dx \le C\int_\Omega \varphi(|u|)dx, \quad (3.2a)$$

where $B_0 \subset \Omega$ is some fixed ball.

4. Applications

It should be noticed that both of the local and global norm inequalities for Green's operator proved in this paper can be used to estimate other operators applied to a *k*-quasi-minimizer. Here, we give an example using Theorem 2.3 to estimate the projection operator *H*. Using the basic Poincaré inequality to $\Delta G(u)$ and noticing that *d* commute with Δ and *G*, we can prove the following Lemma 4.1

Lemma 4.1. *Let $u \in D'(\Omega, \wedge^l)$, $l = 0, 1,..., n - 1$, be an A-harmonic tensor on Ω. Assume that $\rho > 1$ and $1 < s < \infty$. Then, there exists a constant C, independent of u, such that*

$$\| \Delta G(u)-(\Delta G(u))_B\|_{s,B} \le C diam(B) \parallel du\|_{s,\rho B} \quad (4.1)$$

for any ball B with $\rho B \subset \Omega$.

Using Lemma 4.1 and the method developed in the proof of Theorem 2.3, we can prove the following inequality for the composition of Δ and *G*.

Theorem 4.2. *Let $u \in W_{loc}^{1,1}(\Omega, \Lambda^\ell)$be a k-quasi-minimizer for the functional* (2.1), ϕ *be a Young function in the class G(p, q, C), $1 \le p < q < \infty$, $C \ge 1$ and $q(n - p) < np$, Ω be a bounded domain and G be Green's operator. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi(|\Delta G(u)-(\Delta G(u))_B|)dx \le C\int_{2B} \varphi(|u-c|)dx \quad (4.2)$$

for all balls $B = B_r$ with radius r and $2B \subset \Omega$, where c is any closed form.

Now, we are ready to develop the estimate for the projection operator applied to a *k*-quasi-minimizer for the functional defined by (2.1).

Theorem 4.3. *Let $u \in W^{1,1}_{loc}(\Omega, \Lambda^\ell)$be a k-quasi-minimizer for the functional* (2.1), ϕ *be a Young function in the class $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$ and $q(n - p) < np$, Ω be a bounded domain and H be projection operator. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi(|H(u) - (H(u))_B|)dx \le C \int_{2B} \varphi\ (|u - c|)dx \tag{4.3}$$

for all balls $B = B_r$ with radius r and $2B \subset \Omega$, where c is any closed form.

Proof. Using the Poisson's equation $\Delta G(u) = u - H(u)$ and the fact that ϕ is convex and doubling as well as Theorem 4.2, we have

$$\begin{aligned}
\int_B \varphi(|H(u) - (H(u))_B|)dx &\le \int_B \varphi\ (|u - u_B| + |\Delta G(u) - (\Delta G(u))_B|)dx \\
&= \int_B \varphi\ ((1/2)2|u - u_B|dx + (1/2)2|\Delta G(u) - (\Delta G(u))_B|)dx \\
&\le \frac{1}{2}\int_B \varphi\ (2|u - u_B|)dx + \frac{1}{2}\int_B \varphi\ (2|\Delta G(u) - (\Delta G(u))_B|)dx \\
&\le \frac{C_1}{2}\int_B \varphi\ (|u - u_B|)dx + \frac{C_2}{2}\int_B \phi\ (|\Delta G(u) - (\Delta G(u))_B|)dx \qquad (4.4) \\
&\le \frac{C_3}{2}\left(\int_B \varphi\ (|u - u_B|)dx + \int_B \varphi\ (|\Delta G(u) - (\Delta G(u))_B|)dx\right) \\
&\le \frac{C_3}{2}\left(C_4 \int_{\sigma B} \varphi\ (|u - c|)dx + C_5 \int_{\sigma B} \varphi\ (|u - c|)dx\right) \\
&\le C_6 \int_{\sigma B} \varphi\ (|u - c|)dx,
\end{aligned}$$

that is

$$\int_B \varphi(|H(u) - (H(u))_B|)dm \le C \int_{\sigma B} \varphi(|u - c|)dm.$$

We have completed the proof of Theorem 4.3. □

Remark. (i) We know that the L^s-averaging domains uniform domains are the special L^ϕ-averaging domains. Thus, Theorems 3.2 also holds if Ω is tan L^s-averaging domain or uniform domain. (ii) Theorem 4.3 can also be extended into the global case in L^ϕ (m)-averaging domain.

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