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Weighted differentiation composition operators from weighted bergman space to *n*th weighted space on the unit disk

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Abstract

This paper characterizes the boundedness and compactness of the weighted differentiation composition operator from weighted Bergman space to *n*th weighted space on the unit disk of ≤.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex space ≤, dA the Lebesegue measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$.

Let α >-1, p >0. f is said to belong to the weighted Bergman space, denoted by $A^p_\alpha (= A^p_\alpha(\mathbb{D}))$, if $f \in H(\mathbb{D})$ and

$$\| f \|^p = (\alpha + 1) \int_{\mathbb{D}} | f(z) |^p (1 - |z|^2)^\alpha dA < \infty.$$

When $0 < p < 1$, it is complete metric space; when $p \geq 1$, it is a Banach space.

Let $\mu(z)$(weight) be a positive continuous function on $\mathbb{D}$ and $n \in \mathbb{N}_0$. The nth weighted space on the unit disk, denoted by $\omega^{(n)}_\mu(\mathbb{D})$, consists of all $f \in H(\mathbb{D})$ such that

$$b_{\omega^{(n)}_\mu(\mathbb{D})}(f) = \sup_{z \in \mathbb{D}} \mu(z) | f^{(n)}(z) | < \infty.$$

For $n = 0$, the space becomes the weighted-type space $H^\infty_\mu(\mathbb{D})$; for $n = 1$, the Bloch-type space $\mathcal{B}_\mu(\mathbb{D})$; and for $n = 2$, the Zygmud-type $Z_\mu(\mathbb{D})$. For more details about these spaces, we recommend the readers to ([1,2]).

The expression $b_{\omega^{(n)}_\mu(\mathbb{D})}(f)$ defines a semi-norm on the nth weighted space $\omega^{(n)}_\mu(\mathbb{D})$, while the natural norm is given by

$$\| f \|_{\omega^{(n)}_\mu(\mathbb{D})} = \sum_{j=0}^{n-1} \left| f^{(j)}(0) \right| + b_{\omega^{(n)}_\mu(\mathbb{D})}(f).$$

With this norm, $\omega_{\mu}^{(n)}(\mathbb{D})$ becomes a Banach space. The little *n*th weighted space, denoted by $\omega_{\mu,0}^{(n)}(\mathbb{D})$, is a closed subspace of $\omega_{\mu}^{(n)}(\mathbb{D})$, consisting of those *f* for which

$$\lim_{|z|\to 1} \mu(z) \mid f^{(n)}(z) \mid = 0.$$

Let ϕ be a non-constant analytic self-map of $\mathbb{D}$, $u(z) \in H(\mathbb{D})$, and $m \in \mathbb{N}$. The weighted differentiation composition operator $D_{\varphi,u}^{m}$ is defined by

$$D_{\varphi,u}^{m} f(z) = u(z) f^{(m)}(\varphi(z)),$$

for $z \in \mathbb{D}$, $f \in H(\mathbb{D})$. If $m = 1$, $u(z) = \phi'(z)$, then $D_{\varphi,u}^{m} = DC_{\varphi}$; if let $m = 1$, $u(z) = 1$, then $D_{\varphi,u}^{m} = C_{\varphi}D$.

Recently, there have been some interests in studying some particular cases of operators, such as DC_{ϕ}, $C_{\phi}D$ and $D_{\varphi,u}^{m}$, between different function spaces. From those studies, they gave some sufficient and necessary conditions for these operators to be bounded and compact. Concerning these results, we also recommend the interested readers to ([3-9]).

In this paper, we characterize the boundedness and compactness of the operator $D_{\varphi,u}^{m}$ from A_{α}^{p} to *n*th weighted space. For the case of $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu}^{(n)}$, we have the following results:

Theorem 1. *Assume that* $p > 0$, $\alpha > -1$, $n, m \in \mathbb{N}$, μ *is a weight on* $\mathbb{D}$, ϕ *is a non-constant analytic self-map of* $\mathbb{D}$, *and* $u \in H(\mathbb{D})$. *Then,*

(1*a*) $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu}^{(n)}$ *is bounded if and only if for each* $k \in \{0, 1, \ldots, n\}$

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}} < \infty. \tag{1}$$

(1*b*) $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu}^{(n)}$ *is compact if and only if* $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu}^{(n)}$ *is bounded and for each* $k \in \{0, 1, \ldots, n\}$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}} = 0. \tag{2}$$

For the case of $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu,0}^{(n)}$, our main results are the following:

Theorem 2. *Assume that* $p > 0$, $\alpha > -1$, $n, m \in \mathbb{N}$, μ *is a weight on* $\mathbb{D}$, ϕ *is a non-constant analytic self-map of* $\mathbb{D}$, *and* $u \in H(\mathbb{D})$. *Then,*

(2a) $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu,0}^{(n)}$ *is bounded if and only if* $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu}^{(n)}$ *is bounded and for each* $k \in \{0, 1, \ldots, n\}$,

$$\lim_{|z|\to 1} \mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right| = 0. \tag{3}$$

(2b) $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu,0}^{(n)}$ *is compact if and only if* $D_{\varphi,u}^{m} : A_{\alpha}^{p} \to \omega_{\mu,0}^{(n)}$ *is bounded and for each* $k \in \{0, 1, \ldots, n\}$,

$$\lim_{|z|\to 1} \frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}} = 0. \tag{4}$$

The organization of this paper is as follows: we give some lemmas in Section 2, and then prove Theorem 1 in Section 3 and Theorem 2 in Section 4, respectively.

Throughout this paper, we will use the symbol C to denote a finite positive number, and it may differ from one occurrence to the other.

2. Some Lemmas

Lemma 1 is a direct consequence of the well-known estimate in ([10], Proposition 1.4.10). Hence, we omit its proof.

Lemma 1. *Assume that* $p > 0$, $\alpha > -1$, $n \in \mathbb{N}$, $n > 0$, *and* $w \in \mathbb{D}$. *Then the function*

$$g_{w,n}(z) = \frac{\left(1-|w|^2\right)^n}{(1-\overline{w}z)^{n+\frac{2+\alpha}{p}}}$$

belongs to A_α^p. *Moreover,* $\sup_{w\in\mathbb{D}} \|g_{w,n}\|_{A_\alpha^p} < \infty$.

The next lemma comes from ([11]).

Lemma 2. *Assume that* $p > 0$, $\alpha > -1$, $n \in \mathbb{N}$, *and* $z \in \mathbb{D}$. *Then, there is a positive constant C independent of f such that*

$$\left|f^{(n)}(z)\right| \le C \frac{\|f\|_{A_\alpha^p}}{\left(1-|z|^2\right)^{n+\frac{2+\alpha}{p}}}.$$

Lemma 3. *Let* $p > 0$, $\alpha > -1$, $m \in \mathbb{N}$, $a = m + 1 + \frac{\alpha+2}{p}$ *and*

$$D_{n+1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n \\ & & \cdots & \\ \prod_{j=0}^{n-1}(a+j) & \prod_{j=0}^{n-1}(a+j+1) & \cdots & \prod_{j=0}^{n-1}(a+j+n) \end{vmatrix}.$$

Then, $D_{n+1} = \prod_{j=1}^{n} j!$.

Proof. With $a = m + 1 + \frac{\alpha+2}{p}$ and replacing n by $n + 1$ in ([12], Lemma 2.3), the lemma easily follows. □

The next lemma can be found in ([7], Lemma 4).

Lemma 4. *Assume* $n \in \mathbb{N}$, $g, u \in H(\mathbb{D})$ *and* ϕ *is an analytic self-map of* $\mathbb{D}$. *Then,*

$$\left(u(z)g(\varphi(z))\right)^{(n)} = \sum_{k=0}^{n} g^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_n^l u^{(n-l)}(z)\, B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right),$$

where

$$B_{l,k}(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}) = \sum_{k_1,\ldots,k_l} \frac{l!}{k_1!\ldots k_l!} \prod_{j=1}^{l} \left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j}, \tag{5}$$

and the sum in (5) is overall non-negative integers $k_1,\ldots, k_l$ *satisfying* $k_1 + k_2 + \ldots + k_1 = k$ *and* $k_1 + 2k_2 + \ldots + lk_l = l$.

By a proof in a standard way ([1], Proposition 3.11), we can get the next lemma.

Lemma 5. *Suppose $u \in H(\mathbb{D})$, $p > 0$, $\alpha > -1$, $n, m \in \mathbb{N}$ and ϕ is a non-constant analytic self-map of $\mathbb{D}$. Then the operator $D^m_{\varphi,u} : A^p_\alpha \to \omega^{(n)}_\mu$ is compact if and only if $D^m_{\varphi,u} : A^p_\alpha \to \omega^{(n)}_\mu$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in A^p_α which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $D^m_{\varphi,u} f_k \to 0$ in $\omega^{(n)}_\mu$ as $k \to \infty$.*

Lemma 6. *Suppose $n \in \mathbb{N}$ and μ is a radial weight such that $\lim_{|z|\to 1} \mu(z) = 0$. A closed set K in $\omega^{(n)}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z) \left| f^{(n)}(z) \right| = 0.$$

Proof. The proof of this Lemma is followed by standard arguments similar to those outlined in ([13]). We omit the details. □

3. The Proof of Theorem 1

(1*a*) Boundedness of $D^m_{\varphi,u}$.

We will prove the sufficiency first. Suppose that the conditions in (1) hold. Then, for any $f \in A^p_\alpha$, from Lemma 2 and Lemma 4, we obtain

$$\begin{aligned} &\mu(z) \left| \left(D^m_{\varphi,u} f\right)^{(n)}(z) \right| \\ &= \mu(z) \left| \sum_{k=0}^{n} f^{(k+m)}(\varphi(z)) \sum_{l=k}^{n} C^l_n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right) \right| \\ &\le \mu(z) \sum_{k=0}^{n} \left| f^{(k+m)}(\varphi(z)) \right| \left| \sum_{l=k}^{n} C^l_n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right) \right| \\ &\le C\|f\|_{A^p_\alpha} \sum_{k=0}^{n} \frac{\mu(z) \left| \sum_{l=k}^{n} C^l_n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right) \right|}{\left(1 - |\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}}. \end{aligned} \tag{6}$$

For $z = 0$ and every $d \in \{0, 1, \ldots, n-1\}$,

$$\begin{aligned} &\left| \left(D^m_{\varphi,u} f\right)^{(d)}(0) \right| \\ &= \left| \sum_{k=0}^{d} f^{(k+m)}(\varphi(0)) \sum_{l=k}^{d} C^l_d u^{(d-l)}(0) B_{l,k}\left(\varphi'(0), \varphi''(0), \ldots \varphi^{(l-k+1)}(0)\right) \right| \\ &\le C\|f\|_{A^p_\alpha} \sum_{k=0}^{d} \frac{\left| \sum_{l=k}^{d} C^l_d u^{(d-l)}(0) B_{l,k}\left(\varphi'(0), \varphi''(0), \ldots, \varphi^{(l-k+1)}(0)\right) \right|}{\left(1 - |\varphi(0)|^2\right)^{k+m+\frac{\alpha+2}{p}}} \end{aligned} \tag{7}$$

From (1), (6), and (7), we know that $D^m_{\varphi,u} : A^p_\alpha \to \omega^{(n)}_\mu$ is bounded.

Conversely, suppose that $D^m_{\varphi,u} : A^p_\alpha \to \omega^{(n)}_\mu$ is bounded. Then, there exists a constant C such that $\|D^m_{\varphi,u} f\|_{\omega^{(n)}_\mu} \le C\|f\|_{A^p_\alpha}$, for all $f \in A^p_\alpha$.

For a fixed $w \in \mathbb{D}$, and constants $c_1, c_2, \ldots, c_{n+1}$, set

$$g_w(z) = \sum_{j=1}^{n+1} \frac{c_j}{\left(j + \frac{2+\alpha}{p}\right)\left(j + \frac{2+\alpha}{p} + 1\right) \ldots \left(j + \frac{2+\alpha}{p} + m - 1\right)} \frac{\left(1 - |w|^2\right)^j}{\left(1 - \overline{w}z\right)^{j+\frac{2+\alpha}{p}}}. \tag{8}$$

Applying Lemma 1 and triangle inequality, it is easy to get that $g_w \in A_\alpha^p$ for every $w \in \mathbb{D}$. Moreover, we have that

$$\sup_{w\in\mathbb{D}} \|g_w\|_{A_\alpha^p} < \infty. \tag{9}$$

Now we show that for each $s \in \{m, m+1, \ldots, m+n\}$, there are constants $c_1, c_2, \ldots, c_{n+1}$, such that,

$$g_w^{(s)}(w) = \frac{\overline{w}^s}{\left(1-|w|^2\right)^{s+\frac{\alpha+2}{p}}}, \quad g_w^{(t)}(w) = 0, \quad t \in \{m, \ldots, m+n\}\backslash\{s\} \tag{10}$$

Indeed, by differentiating function g_w for each $s \in \{m, m+1, \ldots, m+n\}$, the system in (10) becomes

$$\begin{aligned}
&c_1 + c_2 + \cdots + c_{n+1} = 0\\
&(m+1+\tfrac{\alpha+2}{p})c_1 + (m+2+\tfrac{\alpha+2}{p})c_2 + \cdots + (m+n+1+\tfrac{\alpha+2}{p})c_{n+1} = 0\\
&\ldots\ldots\\
&(m+1+\tfrac{\alpha+2}{p})\ldots(s+\tfrac{\alpha+2}{p})c_1 + \cdots + (m+n+1+\tfrac{\alpha+2}{p})\ldots\left(n+s+\tfrac{\alpha+2}{p}\right)c_{n+1} = 1\\
&\ldots\ldots\\
&(m+1+\tfrac{\alpha+2}{p})\ldots(m+n+\tfrac{\alpha+2}{p})c_1 + \cdots + (m+n+1+\tfrac{\alpha+2}{p})\ldots\left(m+2n+\tfrac{\alpha+2}{p}\right)c_{n+1} = 0
\end{aligned} \tag{11}$$

By Lemma 3, the determinant of system (11) is different from zero, which implies the statement. For each $k \in \{0, 1, 2, \ldots, n\}$, we choose the corresponding family of functions that satisfy (10) with $s = m + k$ and denote it by $g_{w,k}$. For each fixed $k \in \{0, 1, \ldots, n\}$, the boundedness of the operator $D_{\varphi,u}^m : A_\alpha^p \to \omega_\mu^{(n)}$, along with Lemma 4 and (8) implies that for each $\phi(w) \neq 0$,

$$\begin{aligned}
&\frac{\mu(w)|\varphi(w)|^{m+k}\left|\sum_{l=k}^n C_n^l u^{(n-l)}(w) B_{l,k}\left(\varphi'(w), \varphi''(w), \ldots, \varphi^{(l-k+1)}(w)\right)\right|}{\left(1-|\varphi(w)|^2\right)^{k+m+\frac{\alpha+2}{p}}}\\
&\leq C \sup_{w\in D}\left\|D_{\varphi,u}^m(g_{\varphi(w),k})\right\|_{\omega_\mu^{(n)}} \leq C\left\|D_{\varphi,u}^m\right\|_{A_\alpha^p\to\omega_\mu^{(n)}}.
\end{aligned} \tag{12}$$

From (12), it follows that for each $k \in \{0, 1, \ldots, n\}$,

$$\begin{aligned}
&\sup_{|\varphi(z)|>\frac{1}{2}} \frac{\mu(z)\left|\sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}}\\
&\leq C\left\|D_{\varphi,u}^m\right\|_{A_\alpha^p\to\omega_\mu^{(n)}}.
\end{aligned} \tag{13}$$

Now we use the test functions

$$h_k(z) = z^{k+m}, \quad k = 0, 1, \ldots, n.$$

For each $k \in \mathbb{N}$, it is easy to get that

$$h_k \in A_\alpha^p, \quad \|h_k\|_{A_\alpha^p} \leq 1. \tag{14}$$

By applying Lemma 4 to the $h_0(z) = z^m$, we get

$$\begin{aligned}
&(D_{\varphi,u}^m h_0)^{(n)}(z)\\
&= h_0^{(m)}(\varphi(z)) \sum_{l=0}^n C_n^l u^{(n-l)}(z) B_{l,0}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l+1)}(z)\right)\\
&= m! \sum_{l=0}^n C_n^l u^{(n-l)}(z) B_{l,0}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l+1)}(z)\right),
\end{aligned}$$

which along with boundedness of the operator $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_\mu$ implies that

$$m!\sup_{z\in\mathbb{D}}\mu(z)\left|\sum_{l=0}^{n}C_n^l u^{(n-l)}(z)B_{l,0}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l+1)}(z)\right)\right|\le C\left\|D^m_{\varphi,u}\right\|_{A^p_\alpha\to\omega^{(n)}_\mu}. \quad (15)$$

Assume now that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}}\mu(z)\left|\sum_{l=j}^{n}C_n^l u^{(n-l)}(z)B_{l,j}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-j+1)}(z)\right)\right|\le C\left\|D^m_{\varphi,u}\right\|_{A^p_\alpha\to\omega^{(n)}_\mu}. \quad (16)$$

for $j \in \{0, 1,..., k - 1\}$, $k \le n$.

Apply Lemma 4 to the $h_k(z) = z^{m+k}$, and knowing that $z^{(s)} \equiv 0$ for $s > m + k$ and the boundedness of the operator $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_\mu$, we get

$$\begin{aligned}&\left|(D^m_{\varphi,u}h_k)^{(n)}(z)\right|\\ &\ge\left|\sum_{j=0}^{k-1}(m+k)\ldots(k-j+1)(\varphi(z))^{(k-j)}\sum_{l=j}^{n}C_n^l u^{(n-l)}(z)B_{l,j}\left(\varphi'(z),\ldots,\varphi^{(l-j+1)}(z)\right)\right|\\ &\quad-\left|(m+k)!\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|.\end{aligned} \quad (17)$$

Using hypothesis (16), we can know that

$$\sup_{z\in\mathbb{D}}\mu(z)\left|\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)\,B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|\le C\left\|D^m_{\varphi,u}\right\|_{A^p_\alpha\to\omega^{(n)}_\mu}. \quad (18)$$

for each $k \in \{0, 1,..., n\}$. Then, for each $k \in \{0, 1,..., n\}$,

$$\begin{aligned}&\sup_{|\varphi(z)|\le\frac{1}{2}}\frac{\mu(z)\left|\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}}\\ &\le C\sup_{z\in D}\mu(z)\left|\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots\varphi^{(l-k+1)}(z)\right)\right|.\\ &\le C\left\|D^m_{\varphi,u}\right\|_{A^p_\alpha\to\omega^{(n)}_\mu}\end{aligned} \quad (19)$$

From (13) and (19), we know that (1) holds.

($1b$) Compactness of $D^m_{\varphi,u}$.

Suppose $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_\mu$ is bounded and (2) holds. Then, by (1a), (1) holds. Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in A^p_α, such that, $\sup_{i\in\mathbb{N}}\|f_i\|_{A^p_\alpha}\le M$ and f_i converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that, for each $k \in \{0, 1,..., n\}$ and $\delta < |\phi(z)| < 1$,

$$\frac{\mu(z)\left|\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}}<\varepsilon. \quad (20)$$

From Lemma 2 and (20), we have

$$\begin{aligned}&\sup_{i\in\mathbb{N}}\ \mu(z)\left|\sum_{k=0}^{n} f_i^{(k+m)}\left(\varphi(z)\right)\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\dots\varphi^{(l-k+1)}(z)\right)\right|\\ &\le C\sup_{i\in\mathbb{N}}\|f_i\|_{A_\alpha^p}\sum_{k=0}^{n}\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\dots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}}\\ &\le CM\,(n+1)\,\varepsilon.\end{aligned} \tag{21}$$

If $|\phi(z)| \le r$, then by Cauchy's estimate and (19), we have

$$\begin{aligned}&\sup\mu(z)\left|\sum_{k=0}^{n} f_i^{(k+m)}(\varphi(z))\sum_{l=k}^{n} C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\dots\varphi^{(l-k+1)}(z)\right)\right|\\ &\le C\sum_{k=0}^{n}\sup_{|\varphi(z)|\le r}\left|f_i^{(m+k)}(\varphi(z))\right|\to 0,\ (i\to\infty)\end{aligned} \tag{22}$$

For $j = 0, 1, \dots, n - 1$, we have

$$\left|(D_{\varphi,u}^m f_i)^{(j)}(0)\right|\to 0,\ (i\to\infty) \tag{23}$$

Applying (21), (22), and (23), we know that $\|D_{\varphi,u}^m f_i\|_{\omega_\mu^{(n)}}\to 0,\ (i\to\infty)$. From Lemma 5, $D_{\varphi,u}^m : A_\alpha^p\to\omega_\mu^{(n)}$ is compact.

Conversely, suppose that $D_{\varphi,u}^m : A_\alpha^p\to\omega_\mu^{(n)}$ is compact, then $D_{\varphi,u}^m : A_\alpha^p\to\omega_\mu^{(n)}$ is bounded. Let $(z_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\phi(z_i)|\to 1$, $i\to\infty$. If such a sequence does not exist, then the condition in (2) is easily satisfied. Now, assume that when $\|\phi\|_\infty = 1$ and (2) does not hold, then there is $k\in \{0, 1, \dots, n\}$ and $\delta > 0$ such that

$$\frac{\mu(z_i)\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i)B_{l,k}\left(\varphi'(z_i),\varphi''(z_i),\dots\varphi^{(l-k+1)}(z_i)\right)\right|}{\left(1-|\varphi(z_i)|^2\right)^{k+m+\frac{\alpha+2}{p}}}\ge\delta.$$

Let $g_i(z) = g_{\varphi(z_i),k}(z)$, $i\in\mathbb{N}$, $k\in\ 0, 1,\dots, n$ be as in Theorem 1. Then, $\sup_{i\in\mathbb{N}}\|g_i\|_{A_\alpha^p}\le M$ and $g_i\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$. By the assumption and Lemma 5, we have that for $k\in \{0, 1,\dots, n\}$

$$\lim_{i\to\infty}\|D_{\varphi,u}^m g_{\varphi(z_i),k}\|_{\omega_\mu^{(n)}} = 0. \tag{24}$$

On the other hand, from (12), we obtain

$$\begin{aligned}&\|D_{\varphi,u}^m g_{\varphi(z_i),k}\|_{\omega_\mu^{(n)}}\\ &\ge\frac{\mu(z_i)\,|\varphi(z_i)|^{k+m}\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i)B_{l,k}\left(\varphi'(z_i),\varphi''(z_i),\dots\varphi^{(l-k+1)}(z_i)\right)\right|}{\left(1-|\varphi(z_i)|^2\right)^{k+m+\frac{\alpha+2}{p}}}\\ &>\frac{\delta}{3}\end{aligned} \tag{25}$$

for large enough i. From (24) and (25), this is a contradiction. So, (2) holds.

Now the proof of Theorem 1 is completed.

4. The Proof of Theorem 2

(2*a*) Boundedness of $D^m_{\varphi,u}$.

First, suppose that $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_\mu$ is bounded and (3) holds. For each polynomial $p(z)$, we obtain $|p^{(m+k)}(z)| \le C_p$, $z \in$ D, C_p is s constant depending on p.

And

$$\begin{aligned}
&\mu(z)\left|(D^m_{\varphi,u}p)^{(n)}(z)\right| \\
&= \mu(z)\left|\sum_{k=0}^{n} p^{(k+m)}(\varphi(z))\sum_{l=k}^{n} C^l_n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right| \\
&\le C_p\sum_{k=0}^{n} \mid \mu(z) \mid \left|\sum_{l=k}^{n} C^l_n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right| \\
&\to 0, |z| \to 1.
\end{aligned} \tag{26}$$

From (26), we have that for each polynomial $p(z)$, $D^m_{\varphi,u}p \in \omega^{(n)}_{\mu,0}$. Since the set of polynomials is dense in A^p_α, we have that for each $f \in A^p_\alpha$, there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$, such that $\| f - p_k \|_{A^p_\alpha} \to 0$ as $k \to \infty$. From the boundness of $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_\mu$, we have that

$$\left\|D^m_{\varphi,u}f - D^m_{\varphi,u}p_k\right\|_{\omega^{(n)}_\mu} \le \left\|D^m_{\varphi,u}\right\|_{A^p_\alpha\to\omega^{(n)}_\mu}\left\|f-p_k\right\|_{A^p_\alpha} \to 0, k \to \infty.$$

Then, $D^m_{\varphi,u}f \in \omega^{(n)}_{\mu,0}$, from which the boundedness of $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_{\mu,0}$ follows.

Conversely, suppose that $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_{\mu,0}$ is bounded. It is clear that $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_\mu$ is bounded. Then, taking the test functions $h_k(z) = z^{m+k}$ for each $k \in \{0, 1,..., n\}$, we obtain $D^m_{\varphi,u}z^{m+k} \in \omega^{(n)}_{\mu,0}$. By the proof of Theorem 1, for each $k \in \{0, 1,..., n\}$,

$$\lim_{|z|\to 1} \mu(z)\left|\sum_{l=k}^{n} C^l_n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots\varphi^{(l-k+1)}(z)\right)\right| = 0.$$

(2a) is completed.

(2*b*) Compactness of $D^m_{\varphi,u}$.

First, assume that $D^m_{\varphi,u}: A^p_\alpha \to \omega^{(n)}_{\mu,0}$ is compact, so it is bounded and (3) holds. Hence, if $||\phi||_\infty < 1$,

$$\begin{aligned}
&\frac{\mu(z)\left|\sum_{l=k}^{n} C^l_n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\dfrac{\alpha+2}{p}}} \\
&\le \frac{\mu(z)\left|\sum_{l=k}^{n} C^l_n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-\|\varphi\|^2_\infty\right)^{k+m+\frac{\alpha+2}{p}}} \\
&\to 0, (|z| \to 1)
\end{aligned} \tag{27}$$

If $||\phi||_\infty = 1$, since $D^m_{\varphi,u} : A^p_\alpha \to \omega^{(n)}_\mu$ is compact too and (2) holds, then for all $\varepsilon > 0$, there is an $r \in (0, 1)$, such that when $r < |\phi(z)| < 1$, for $k \in \{0, 1,..., n\}$,

$$\frac{\mu(z)\left|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}} < \varepsilon. \tag{28}$$

By (3), we know there is a $\delta \in (0, 1)$, such that $\delta < |z| < 1$, for $k \in \{0, 1,..., n\}$,

$$\begin{aligned}&\mu(z)\left|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|\\&<\varepsilon\left(1-r^2\right)^{k+m+\frac{\alpha+2}{p}}.\end{aligned} \tag{29}$$

Then, when $\delta < |z| < 1$ and $r < |\phi(z)| < 1$ for $k \in \{0, 1,..., n\}$, we get

$$\frac{\mu(z)\left|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{k+m+\frac{\alpha+2}{p}}} < \varepsilon. \tag{30}$$

In addition, when $|\phi(z)| \le r$ and $\delta < |z| < 1$, we have

$$\begin{aligned}&\frac{\mu(z)\left|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+m+\frac{\alpha+2}{p}}}\\&<\frac{\mu(z)\left|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-r^2\right)^{k+m+\frac{\alpha+2}{p}}} < \varepsilon.\end{aligned} \tag{31}$$

Combining (30) and (31), we know (4) holds.

Conversely, assume $D^m_{\varphi,u} : A^p_\alpha \to \omega^{(n)}_{\mu,0}$ is bounded and (4) holds. Taking the supremum in (6) for all f in the unit ball of A^p_α, and using the condition (4), we have $\lim_{|z|\to 1}\sup_{\|f\|_{A^p_\alpha}\le 1} \mu(z)\left|(D^m_{\varphi,u}f)^{(n)}(z)\right| = 0$, from which by Lemma 6, the compactness of $D^m_{\varphi,u} : A^p_\alpha \to \omega^{(n)}_{\mu,0}$ follows.

Now the proof of Theorem 2 is finished.

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Authors' contributions
LZ found the question and drafted the manuscript. HGZ joined in the discussion about the question and revised the paper. All authors read and approved the final manuscript

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