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On nonlinear stability in various random normed spaces

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Abstract

In this article, we prove the nonlinear stability of the quartic functional equation

$$16f(x+4y)+f(4x-y)=306\left[9f\left(x+\frac{y}{3}\right)+f(x+2y)\right]+136f(x-y)-1394f(x+y)+425f(y)-1530f(x)$$

in the setting of random normed spaces Furthermore, the interdisciplinary relation among the theory of random spaces, the theory of non-Archimedean space, the theory of fixed point theory, the theory of intuitionistic spaces and the theory of functional equations are also presented in the article.

Keywords: generalized Hyers-Ulam stability, quartic functional equation, random normed space, intuitionistic random normed space

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [1] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [2]. Subsequently, this result of Hyers was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The article of Rassias [4] has provided a lot of influence in the development of what we now call *generalized Ulam-Hyers stability* of functional equations. We refer the interested readers for more information on such problems to the article [5-17].

Recently, Alsina [18], Chang, et al. [19], Mirmostafaee et al. [20], [21], Miheț and Radu [22], Miheț et al. [23], [24], [25], [26], Baktash et al. [27], Eshaghi et al. [28], Saadati et al. [29], [30] investigated the stability in the settings of fuzzy, probabilistic, and random normed spaces.

In this article, we study the stability of the following functional equation

$$16f(x+4y)+f(4x-y)=306\left[9f\left(x+\frac{y}{3}\right)+f(x+2y)\right]+136f(x-y)-1394f(x+y)+425f(y)-1530f(x) \tag{1.1}$$

in the various random normed spaces via different methods. Since ax^4 is a solution of above functional equation, we say it *quartic functional equation*.

2. Preliminaries

In this section, we recall some definitions and results which will be used later on in the article.

A *triangular norm* (shorter *t-norm*) is a binary operation on the unit interval [0, 1], i.e., a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $a, b, c \in [0, 1]$ the following four axioms satisfied:

(i) $T(a, b) = T(b, a)$ (commutativity);
(ii) $T(a, (T(b, c))) = T(T(a, b), c)$ (associativity);
(iii) $T(a, 1) = a$ (boundary condition);
(iv) $T(a, b) \leq T(a, c)$ whenever $b \leq c$ (monotonicity).

Basic examples are the Lukasiewicz t-norm T_L, $T_L(a, b) = \max(a + b - 1, 0)$ $\forall a, b \in [0, 1]$ and the t-norms T_P, T_M, T_D, where $T_P(a, b) := ab$, $T_M(a, b) := \min\{a, b\}$,

$$T_D(a,b) := \begin{cases} \min(a,b), & \text{if } \max(a,b) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

If T is a t-norm then $x_T^{(n)}$ is defined for every $x \in [0, 1]$ and $n \in N \cup \{0\}$ by 1, if $n = 0$ and $T(x_T^{(n-1)}, x)$, if $n \geq 1$. A t-norm T is said to be *of Hadžić-type* (we denote by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n\in N}$ is equicontinuous at $x = 1$ (cf. [31]).

Other important triangular norms are (see [32]):

-the *Sugeno-Weber family* $\{T_\lambda^{SW}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{SW} = T_D$, $T_\infty^{SW} = T_P$ and

$$T_\lambda^{SW}(x,y) = \max\left(0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right)$$

if $\lambda \in (-1, \infty)$.

-the *Domby family* $\{T_\lambda^D\}_{\lambda\in[0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M , if $\lambda = \infty$ and

$$T_\lambda^D(x,y) = \frac{1}{1 + \left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{1/\lambda}}$$

if $\lambda \in (0, \infty)$.

-the *Aczel-Alsina family* $\{T_\lambda^{AA}\}_{\lambda\in[0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M , if $\lambda = \infty$ and

$$T_\lambda^{AA}(x,y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}$$

if $\lambda \in (0, \infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1, ..., x_n) \in [0, 1]^n$ the value $T(x_1, ..., x_n)$ defined by

$$\mathrm{T}_{i=1}^0 x_i = 1, \mathrm{T}_{i=1}^n x_i = T(\mathrm{T}_{i=1}^{n-1} x_i, x_n) = T(x_1, \ldots, x_n).$$

T can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in [0, 1] the value

$$\mathrm{T}_{i=1}^\infty x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^n x_i. \tag{2.1}$$

The limit on the right side of (2.1) exists since the sequence $\{\mathrm{T}_{i=1}^{n} x_i\}_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

Proposition 2.1. [32] *(i) For $T \geq T_L$ the following implication holds:*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

(ii) If T is of Hadžić-type then

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1$$

for every sequence $\{x_n\}_{n\in N}$ in [0, 1] *such that* $\lim_{n\to\infty} x_n = 1$.

(iii) If $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1-x_n)^{\alpha} < \infty.$$

(iv) If $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, then

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

Definition 2.2. [33] A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$, $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.3. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) An RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.4. [34]*If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$*almost everywhere.*

3. Non-Archimedean random normed space

By a *non-Archimedean field* we mean a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from K into $[0, \infty]$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. Let $\mathcal{X}$ be a vector space over a field $\mathcal{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$.

A function $||\cdot|| : \mathcal{X} \to [0, \infty]$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $||x|| = 0$ if and only if $x = 0$;
(ii) for any $r \in \mathcal{K}$, $x \in \mathcal{X}$, $||rx|| = ||r|| ||x||$;
(iii) the strong triangle inequality (ultrametric); namely,

$$||x + y|| \leq \max\{||x||, ||y||\} \quad (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, ||\cdot||)$ is called a non-Archimedean normed space. Due to the fact that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \qquad (n > m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [35] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any non-zero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on Q. The completion of Q with respect to the metric $d(x, y) = |x - y|_p$ is denoted by Q_p, which is called the p-adic number field.

Throughout the article, we assume that $\mathcal{X}$ is a vector space and $\mathcal{Y}$ is a complete non-Archimedean normed space.

Definition 3.1. A *non-Archimedean random normed space* (briefly, non-Archimedean RN-space) is a triple $(\mathcal{X}, \mu, T)$, where X is a linear space over a non-Archimedean field $\mathcal{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(NA-RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(NA-RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in \mathcal{X}$, $t > 0$, $\alpha \neq 0$;
(NA-RN3) $\mu_{x+y}(\max\{t, s\}) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in \mathcal{X}$ and $t, s \geq 0$.
It is easy to see that if (NA-RN3) holds then so is
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$.

As a classical example, if $(\mathcal{X}, ||.||)$ is a non-Archimedean normed linear space, then the triple $(\mathcal{X}, \mu, T_M)$, where

$$\mu_x(t) = \begin{cases} 0 & t \leq ||x|| \\ 1 & t > ||x|| \end{cases}$$

is a non-Archimedean RN-space.

Example 3.2. Let $(\mathcal{X}, ||.||)$ be is a non-Archimedean normed linear space. Define

$$\mu_x(t) = \frac{t}{t + ||x||}, \quad \forall x \in \mathcal{X} \quad t > 0.$$

Then $(\mathcal{X}, \mu, T_M)$ is a non-Archimedean RN-space.

Definition 3.3. Let $(\mathcal{X}, \mu, T)$ be a non-Archimedean RN-space. Let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \mu_{x_n-x}(t) = 1$$

for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ in $\mathcal{X}$ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $\mu_{x_{n+p}-x_n}(t) > 1 - \varepsilon$.

If each Cauchy sequence is convergent, then the random norm is said to be *complete* and the non-Archimedean RN-space is called a non-Archimedean *random Banach space*.

Remark 3.4. [36] Let $(\mathcal{X}, \mu, T_M)$ be a non-Archimedean RN-space, then

$$\mu_{x_{n+p}-x_n}(t) \geq \min\{\mu_{x_{n+j+1}-x_{n+j}}(t) : j = 0, 1, 2, \ldots, p-1\}$$

So, the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have

$$\mu_{x_{n+1}-x_n}(t) > 1 - \varepsilon.$$

4. Generalized Ulam-Hyers stability for a quartic functional equation in non-Archimedean RN-spaces

Let $\mathcal{K}$ be a non-Archimedean field, $\mathcal{X}$ a vector space over $\mathcal{K}$ and let $(\mathcal{Y}, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$.

We investigate the stability of the quartic functional equation

$$\begin{aligned} 16f(x+4y) + f(4x-y) = 306\left[9f\left(x+\frac{y}{3}\right) + f(x+2y)\right] \\ + 136f(x-y) - 1394f(x+y) + 425f(y) - 1530f(x), \end{aligned}$$

where f is a mapping from $\mathcal{X}$ to $\mathcal{Y}$ and $f(0) = 0$.

Next, we define a random approximately quartic mapping. Let Ψ be a distribution function on $\mathcal{X} \times \mathcal{X} \times [0, \infty]$ such that $\Psi(x, y, \cdot)$ is symmetric, nondecreasing and

$$\Psi(cx, cx, t) \geq \Psi\left(x, x, \frac{t}{|c|}\right) \quad (x \in \mathcal{X}, \quad c \neq 0).$$

Definition 4.1. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is said to be *Ψ-approximately quartic* if

$$\begin{aligned} &\mu_{16f(x+4y)+f(4x-y)-306\left[9f\left(x+\frac{y}{3}\right)+f(x+2y)\right]-136f(x-y)+1394f(x+y)-425f(y)+1530f(x)}(t) \\ &\quad \geq \Psi(x, y, t) \quad (x, y \in \mathcal{X}, \quad t > 0). \end{aligned} \tag{4.1}$$

In this section, we assume that $4 \neq 0$ in $\mathcal{K}$ (i.e., characteristic of $\mathcal{K}$ is not 4). Our main result, in this section, is the following:

Theorem 4.2. *Let $\mathcal{K}$ be a non-Archimedean field, $\mathcal{X}$ a vector space over $\mathcal{K}$ and let $(\mathcal{Y}, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a Ψ-approximately quartic mapping. If for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 3$ with $|4^k| < \alpha$,*

$$\Psi(4^{-k}x, 4^{-k}y, t) \geq \Psi(x, y, \alpha t) \quad (x \in \mathcal{X}, \quad t > 0) \tag{4.2}$$

and

$$\lim_{n\to\infty} \mathrm{T}_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1 \quad (x \in \mathcal{X}, \quad t > 0), \tag{4.3}$$

then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mathrm{T}_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|4|^{ki}}\right) \tag{4.4}$$

for all $x \in X$ *and* $t > 0$, *where*

$$M(x,t) := T(\Psi(x,0,t), \Psi(4x,0,t), \cdots, \Psi(4^{k-1}x,0,t)) \quad (x \in \mathcal{X}, \quad t > 0).$$

Proof. First, we show by induction on j that for each $x \in \mathcal{X}$, $t > 0$ and $j \geq 1$,

$$\mu_{f(4^j x)-256^j f(x)}(t) \geq M_j(x,t) := T(\Psi(x,0,t), \cdots, \Psi(4^{j-1}x,0,t)). \tag{4.5}$$

Putting $y = 0$ in (4.1), we obtain

$$\mu_{f(4x)-256f(x)}(t) \geq \Psi(x,0,t) \qquad (x \in \mathcal{X}, \quad t > 0).$$

This proves (4.5) for $j = 1$. Assume that (4.5) holds for some $j \geq 1$. Replacing y by 0 and x by $4^j x$ in (4.1), we get

$$\mu_{f(4^{j+1}x)-256f(4^j x)}(t) \geq \Psi(4^j x,0,t) \quad (x \in \mathcal{X}, \quad t > 0).$$

Since $|256| \leq 1$,

$$\begin{aligned}
\mu_{f(4^{j+1}x)-256^{j+1}f(x)}(t) &\geq T\left(\mu_{f(4^{j+1}x)-256f(4^j x)}(t), \mu_{256f(4^j x)-256^{j+1}f(x)}(t)\right) \\
&= T\left(\mu_{f(4^{j+1}x)-256f(4^j x)}(t), \mu_{f(4^j x)-256^j f(x)}\left(\tfrac{t}{|256|}\right)\right) \\
&\geq T\left(\mu_{f(4^{j+1}x)-256f(4^j x)}(t), \mu_{f(4^j x)-256^j f(x)}(t)\right) \\
&\geq T(\Psi(4^j x,0,t), M_j(x,t)) \\
&= M_{j+1}(x,t)
\end{aligned}$$

for all $x \in \mathcal{X}$. Thus (4.5) holds for all $j \geq 1$. In particular

$$\mu_{f(4^k x)-256^k f(x)}(t) \geq M(x,t) \quad (x \in \mathcal{X}, \quad t > 0). \tag{4.6}$$

Replacing x by $4^{-(kn+k)}x$ in (4.6) and using inequality (4.2), we obtain

$$\begin{aligned}
\mu_{f\left(\frac{x}{4^{kn}}\right)-256^k f\left(\frac{x}{4^{kn+k}}\right)}(t) &\geq M\left(\frac{x}{4^{kn+k}}, t\right) \\
&\geq M(x, \alpha^{n+1}t) \quad (x \in \mathcal{X}, \quad t > 0, \quad n = 0,1,2,\ldots).
\end{aligned} \tag{4.7}$$

Then

$$\mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-(4^{4k})^{n+1} f\left(\frac{x}{(4^k)^{n+1}}\right)}(t) \geq M\left(x, \frac{\alpha^{n+1}}{|(4^{4k})^n|}t\right) \quad (x \in \mathcal{X}, \quad t > 0, \quad n = 0,1,2,\ldots).$$

Hence,

$$\begin{aligned}
&\mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-(4^{4k})^{n+p} f\left(\frac{x}{(4^k)^{n+p}}\right)}(t) \\
&\quad\geq T_{j=n}^{n+p}\left(\mu_{(4^{4k})^j f\left(\frac{x}{(4^k)^j}\right)-(4^{4k})^{j+p} f\left(\frac{x}{(4^k)^{j+p}}\right)}(t)\right) \\
&\quad\geq T_{j=n}^{n+p} M\left(x, \tfrac{\alpha^{j+1}}{|(4^{4k})^j|}t\right) \\
&\quad\geq T_{j=n}^{n+p} M\left(x, \tfrac{\alpha^{j+1}}{|(4^k)^j|}t\right) \quad (x \in \mathcal{X}, \quad t > 0, \quad n = 0,1,2,\ldots).
\end{aligned}$$

Since $\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^{j+1}}{|(4^k)^j|}t\right) = 1 \quad (x \in \mathcal{X}, \quad t > 0)$, $\left\{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)\right\}_{n\in N}$, is a Cauchy sequence in the non-Archimedean random Banach space $(\mathcal{Y}, \mu, T)$. Hence, we can define a mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{n\to\infty} \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right) - Q(x)}(t) = 1 \quad (x \in X, \quad t > 0). \tag{4.8}$$

Next, for each $n \geq 1$, $x \in \mathcal{X}$ and $t > 0$,

$$\begin{aligned}
\mu_{f(x)-(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)}(t) &= \mu_{\sum_{i=0}^{n-1} (4^{4k})^i f\left(\frac{x}{(4^k)^i}\right) - (4^{4k})^{i+1} f\left(\frac{x}{(4^k)^{i+1}}\right)}(t) \\
&\geq T_{i=0}^{n-1}\left(\mu_{(4^{4k})^i f\left(\frac{x}{(4^k)^i}\right) - (4^{4k})^{i+1} f\left(\frac{x}{(4^k)^{i+1}}\right)}(t)\right) \\
&\geq T_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1} t}{|4^{4k}|^i}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_{f(x)-Q(x)}(t) &\geq T\left(\mu_{f(x)-(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)}(t), \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right) - Q(x)}(t)\right) \\
&\geq T\left(T_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1} t}{|4^{4k}|^i}\right), \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right) - Q(x)}(t)\right).
\end{aligned}$$

By letting $n \to \infty$, we obtain

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|4^k|^i}\right).$$

This proves (4.4).

As T is continuous, from a well-known result in probabilistic metric space (see e.g., [[34], Chapter 12]), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mu_{(4^k)^n \cdot 16 f(4^{-kn}(x+4y)) + (4^k)^n f(4^{-kn}(4x-y)) - 306\left[(4^k)^n \cdot 9 f(4^{-kn}(x+\frac{y}{3})) + (4^k)^n f(4^{-kn}(x+2y))\right] - 136(4^k)^n f(4^{-kn}(x-y)) + 1394(4^k)^n f(4^{-kn}(x+y)) - 425(4^k)^n f(4^{-kn} y) + 1530(4^k)^n f(4^{-kn} x)}(t) \\
&= \mu_{16Q(x+4y)+Q(4x-y)-306\left[9Q\left(x+\frac{y}{3}\right)+Q(x+2y)\right]-136Q(x-y)+1394Q(x+y)-425Q(y)+1530Q(x)}(t)
\end{aligned}$$

for almost all $t > 0$.

On the other hand, replacing x, y by $4^{-kn}x$, $4^{-kn}y$, respectively, in (4.1) and using (NA-RN2) and (4.2), we get

$$\begin{aligned}
&\mu_{(4^k)^n \cdot 16 f(4^{-kn}(x+4y)) + (4^k)^n f(4^{-kn}(4x-y)) - 306\left[(4^k)^n \cdot 9 f(4^{-kn}(x+\frac{y}{3})) + (4^k)^n f(4^{-kn}(x+2y))\right] - 136(4^k)^n f(4^{-kn}(x-y)) + 1394(4^k)^n f(4^{-kn}(x+y)) - 425(4^k)^n f(4^{-kn} y) + 1530(4^k)^n f(4^{-kn} x)}(t) \\
&\geq \Psi\left(4^{-kn}x, 4^{-kn}y, \frac{t}{|4^k|^n}\right) \geq \Psi\left(x, y, \frac{\alpha^n t}{|4^k|^n}\right)
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $t > 0$. Since $\lim_{n\to\infty} \Psi\left(x, y, \frac{\alpha^n t}{|4^k|^n}\right) = 1$, we infer that Q is a quartic mapping.

If $Q' : \mathcal{X} \to \mathcal{Y}$ is another quartic mapping such that $\mu_{Q'(x)-f(x)}(t) \geq M(x, t)$ for all $x \in \mathcal{X}$ and $t > 0$, then for each $n \in N$, $x \in \mathcal{X}$ and $t > 0$,

$$\mu_{Q(x)-Q'(x)}(t) \geq T\left(\mu_{Q(x)-(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)}(t), \mu_{(4^{4k})^n f\left(\frac{x}{(4^k)^n}\right)-Q'(x)}(t), t\right).$$

Thanks to (4.8), we conclude that $Q = Q'$. □

Corollary 4.3. *Let $\mathcal{K}$ be a non-Archimedean field, $\mathcal{X}$ a vector space over $\mathcal{K}$ and let $(\mathcal{Y}, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a Ψ-approximately quartic mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 3$, with $|4^k| < \alpha$,*

$$\Psi(4^{-k}x, 4^{-k}y, t) \geq \Psi(x, y, \alpha t) \quad (x \in \mathcal{X}, \quad t > 0),$$

then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|4|^{ki}}\right)$$

for all $x \in \mathcal{X}$ and all $t > 0$, where

$$M(x, t) := T(\Psi(x, 0, t), \Psi(4x, 0, t), \cdots, \Psi(4^{k-1}x, 0, t)) \quad (x \in \mathcal{X}, \quad t > 0).$$

Proof. Since

$$\lim_{n\to\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1 \quad (x \in \mathcal{X}, \quad t > 0)$$

and T is of Hadžić type, from Proposition 2.1, it follows that

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1 \quad (x \in \mathcal{X}, \quad t > 0).$$

Now we can apply Theorem 4.2 to obtain the result. □

Example 4.4. Let $(\mathcal{X}, \mu, T_M)$ non-Archimedean random normed space in which

$$\mu_x(t) = \frac{t}{t + ||x||}, \quad \forall x \in \mathcal{X}, \quad t > 0,$$

and $(\mathcal{Y}, \mu, T_M)$ a complete non-Archimedean random normed space (see Example 3.2). Define

$$\Psi(x, y, t) = \frac{t}{1 + t}.$$

It is easy to see that (4.2) holds for $\alpha = 1$. Also, since

$$M(x, t) = \frac{t}{1 + t},$$

we have

$$\begin{aligned}\lim_{n\to\infty} T_{M,j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} T_{M,j=n}^{m} M\left(x, \frac{t}{|4|^{kj}}\right)\right) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t + |4^k|^n}\right) \\ &= 1, \quad \forall x \in \mathcal{X}, \quad t > 0.\end{aligned}$$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a Ψ-approximately quartic mapping. Thus all the conditions of Theorem 4.2 hold and so there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{t}{t + |4^k|}.$$

5. Fixed point method for random stability of the quartic functional equation

In this section, we apply a fixed point method for achieving random stability of the quartic functional equation. The notion of generalized metric space has been introduced by Luxemburg [37], by allowing the value $+\infty$ for the distance mapping. The following lemma (Luxemburg-Jung theorem) will be used in the proof of Theorem 5.3.

Lemma 5.1. [38]. *Let (X, d) be a complete generalized metric space and let $A : X \to X$ be a strict contraction with the Lipschitz constant k such that $d(x_0, A(x_0)) < +\infty$ for some $x_0 \in X$. Then A has a unique fixed point in the set $Y := \{y \in X, d(x_0, y) < \infty\}$ and the sequence $(A^n(x))_{n\in N}$ converges to the fixed point x^* for every $x \in Y$. Moreover, $d(x_0, A(x_0)) \leq \delta$ implies $d(x^*, x_0) \leq \frac{\delta}{1-k}$.*

Let X be a linear space, (Y, ν, T_M) a complete RN-space and let G be a mapping from $X \times R$ into $[0, 1]$, such that $G(x, .) \in D^+$ for all x. Consider the set $E := \{g : X \to Y, g(0) = 0\}$ and the mapping d_G defined on $E \times E$ by

$$d_G(g, h) = \inf\{u \in R^+, \nu_{g(x)-h(x)}(ut) \geq G(x, t) \text{ for all } x \in X \text{ and } t > 0\}$$

where, as usual, $\inf \emptyset = +\infty$. The following lemma can be proved as in [22]:

Lemma 5.2. cf. [22,39]*d_G is a complete generalized metric on E.*

Theorem 5.3. *Let X be a real linear space, t f a mapping from X into a complete RN-space (Y, μ, T_M) with $f(0) = 0$ and let $\Phi : X^2 \to D^+$ be a mapping with the property*

$$\exists \alpha \in (0, 256) : \Phi_{4x,4y}(\alpha t) \geq \Phi_{x,y}(t), \ \forall x, y \in X, \quad \forall t > 0. \tag{5.1}$$

If

$$\mu_{16f(x+4y)+f(4x-y)-306\left[9f\left(x+\frac{y}{3}\right)+f(x+2y)\right]-136f(x-y)+1394f(x+y)-425f(y)+1530f(x)}(t) \geq \Phi_{x,y}(t), \ \forall x, y \in X, \tag{5.2}$$

then there exists a unique quartic mapping $g : X \to Y$ such that

$$\mu_{g(x)-f(x)}(t) \geq \Phi_{x,0}\,(Mt)\,, \ \forall x \in X, \quad \forall t > 0, \tag{5.3}$$

where

$$M = (256 - \alpha).$$

Moreover,

$$g(x) = \lim_{n\to\infty} \frac{f(4^n x)}{4^{4n}}.$$

Proof. By setting $y = 0$ in (5.2), we obtain

$$\mu_{f(4x)-256f(x)}(t) \geq \Phi_{x,0}(t)$$

for all $x \in X$, whence

$$\begin{aligned}\mu_{\frac{1}{256}f(4x)-f(x)}(t) &= \mu_{\frac{1}{256}(f(4x)-256f(x))}(t)\\ &= \mu_{f(4x)-256f(x)}(256t)\\ &\geq \Phi_{x,0}(256t), \quad \forall x \in X, \quad \forall t > 0.\end{aligned}$$

Let

$$G(x,t) := \Phi_{x,0}(256t).$$

Consider the set

$$E := \{g : X \to Y, g(0) = 0\}$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g,h) = \inf\{u \in R^+, \mu_{g(x)-h(x)}(ut) \geq G(x,t) \text{ for all } x \in X \text{ and } t > 0\}.$$

By Lemma 5.2, (E, d_G) is a complete generalized metric space. Now, let us consider the linear mapping $J : E \to E$,

$$Jg(x) := \frac{1}{256}g(4x).$$

We show that J is a strictly contractive self-mapping of E with the Lipschitz constant $k = \alpha/256$.

Indeed, let $g, h \in E$ be mappings such that $d_G(g, h) < \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq G(x,t), \forall x \in X, \quad \forall t > 0,$$

whence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}\left(\frac{\alpha}{256}\varepsilon t\right) &= \mu_{\frac{1}{256}(g(4x)-h(4x))}\left(\frac{\alpha}{256}\varepsilon t\right)\\ &= \mu_{g(4x)-h(4x)}(\alpha\varepsilon t)\\ &\geq G(4x,\alpha t) \quad (x \in X, \quad t > 0).\end{aligned}$$

Since $G(4x, \alpha t) \geq G(x, t)$, $\mu_{Jg(x)-Jh(x)}(\frac{\alpha}{256}\varepsilon t) \geq G(x,t)$, that is,

$$d_G(g,h) < \varepsilon \Rightarrow d_G(Jg,Jh) \leq \frac{\alpha}{256}\varepsilon.$$

This means that

$$d_G(Jg,Jh) \leq \frac{\alpha}{256}d_G(g,h)$$

for all g, h in E.

Next, from

$$\mu_{f(x)-\frac{1}{256}f(4x)}(t) \geq G(x,t)$$

it follows that $d_G(f, Jf) \leq 1$. Using the Luxemburg-Jung theorem, we deduce the existence of a fixed point of J, that is, the existence of a mapping $g : X \to Y$ such that $g(4x) = 256g(x)$ for all $x \in X$.

Since, for any $x \in X$ and $t > 0$,

$$d_G(u, v) < \varepsilon \Rightarrow \mu_{u(x)-v(x)}(t) \geq G\left(x, \frac{t}{\varepsilon}\right),$$

from $d_G(J^n f, g) \to 0$, it follows that $\lim_{n\to\infty} \frac{f(4^n x)}{4^{4n}} = g(x)$ for any $x \in X$.

Also, $d_G(f, g) \leq \frac{1}{1-L} d(f, Jf)$ implies the inequality $d_G(f, g) \leq \dfrac{1}{1 - \frac{\alpha}{256}}$ from which it immediately follows $\nu_{g(x)-f(x)}(\frac{256}{256-\alpha} t) \geq G(x, t)$ for all $t > 0$ and all $x \in X$. This means that

$$\mu_{g(x)-f(x)}(t) \geq G\left(x, \frac{256 - \alpha}{256} t\right), \quad \forall x \in X, \quad \forall t > 0.$$

It follows that

$$\mu_{g(x)-f(x)}(t) \geq \Phi_{x,0}((256 - \alpha)t) \quad \forall x \in X, \quad \forall t > 0.$$

The uniqueness of g follows from the fact that g is the unique fixed point of J with the property: there is $C \in (0, \infty)$ such that $\mu_{g(x)-f(x)}(Ct) \geq G(x, t)$ for all $x \in X$ and all $t > 0$, as desired. □

6. Intuitionistic random normed spaces

Recently, the notation of intuitionistic random normed space introduced by Chang et al. [19]. In this section, we shall adopt the usual terminology, notations, and conventions of the theory of intuitionistic random normed spaces as in [22], [31], [33], [34], [40], [41], [42].

Definition 6.1. A *measure distribution function* is a function $\mu : R \to [0, 1]$ which is left continuous, non-decreasing on R, $\inf_{t\in R} \mu(t) = 0$ and $\sup_{t\in R} \mu(t) = 1$.

We will denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 1, \text{ if } t > 0. \end{cases}$$

If X is a nonempty set, then $\mu : X \to D$ is called a *probabilistic measure* on X and $\mu(x)$ is

denoted by μ_x.

Definition 6.2. A *non-measure distribution function* is a function $\nu : R \to [0, 1]$ which is right continuous, non-increasing on R, $\inf_{t\in R} \nu(t) = 0$ and $\sup_{t\in R} \nu(t) = 1$.

We will denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t) = \begin{pmatrix} 1, \text{ if } t \leq 0, \\ 0, \text{ if } t > 0. \end{pmatrix}$$

If X is a nonempty set, then $\nu : X \to B$ is called a *probabilistic non-measure* on X and $\nu(x)$ is denoted by ν_x.

Lemma 6.3. [43], [44]*Consider the set L^* and operation $\leq_{L^*}$defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$*is a complete lattice.*

We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. In Section 2, we presented classical t-norm. Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 6.4. [44] A triangular norm (t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions:

(a) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition);
(b) $(\forall (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);
(c) $(\forall (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);
(d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y'))$ (monotonicity).

If $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}, then $\mathcal{T}$ is said to be a *continuous t-norm*.

Definition 6.5. [44] A continuous t-norm $\mathcal{T}$ on L^* is said to be *continuous t-representable* if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on [0, 1] such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

are continuous t-representable for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$.

Now, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \ldots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \ldots, x^{(n)}), x^{(n+1)}), \quad \forall n \geq 2, \quad x^{(i)} \in L^*.$$

Definition 6.6. A *negator* on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(1_{L^*}) = 0_{L^*}$and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then $\mathcal{N}$ is called an *involutive negator*. A *negator* on [0, 1] is a decreasing function $N : [0, 1] \to [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the *standard negator* on [0, 1] defined by

$$N_s(x) = 1 - x, \quad \forall x \in [0, 1].$$

Definition 6.7. Let μ and ν be measure and non-measure distribution functions from $X \times (0, +\infty)$ to [0, 1] such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. The triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic random normed space* (briefly IRN-space) if X is a vector space, $\mathcal{T}$ is continuous t-representable and $\mathcal{P}_{\mu,\nu}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

(a) $\mathcal{P}_{\mu,\nu}(x, 0) = 0_{L^*}$;
(b) $\mathcal{P}_{\mu,\nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;
(c) $\mathcal{P}_{\mu,\nu}(\alpha x, t) = \mathcal{P}_{\mu,\nu}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{P}_{\mu,\nu}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu,\nu}(x, t), \mathcal{P}_{\mu,\nu}(y, s))$.

In this case, $\mathcal{P}_{\mu,\nu}$ is called an *intuitionistic random norm*. Here,

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)).$$

Example 6.8. Let $(X, ||\cdot||)$ be a normed space. Let $\mathcal{T}(a,b) = (a_1b_1, \min(a_2+b_2, 1))$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ and let μ, ν be measure and non-measure distribution functions defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t+||x||}, \frac{||x||}{t+||x||}\right), \quad \forall t \in R^+.$$

Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IRN-space.

Definition 6.9. (1) A sequence $\{x_n\}$ in an IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a *Cauchy sequence* if, for any $\varepsilon > 0$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon), \quad \forall n, m \geq n_0,$$

where N_s is the standard negator.

(2) The sequence $\{x_n\}$ is said to be *convergent* to a point $x \in X$ (denoted by $x_n \xrightarrow{\mathcal{P}_{\mu,\nu}} x$) if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.

(3) An IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *complete* if every Cauchy sequence in X is convergent to a point $x \in X$.

7. Stability results in intuitionistic random normed spaces

In this section, we prove the generalized Ulam-Hyers stability of the quartic functional equation in intuitionistic random normed spaces.

Theorem 7.1. *Let X be a linear space and let $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete IRN-space. Let $f: X \to Y$ be a mapping with $f(0) = 0$ for which there are $\xi, \zeta : X^2 \to D^+$, where $\xi(x, y)$ is denoted by $\xi_{x,y}$ and $\zeta_{(x,y)}$ is denoted by $\zeta_{x,y}$, further, $(\xi_{x,y}(t), \zeta_{x,y}(t))$ is denoted by $Q_{\xi,\zeta}(x, y, t)$, with the property:*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(16f(x+4y) + f(4x-y) - 306[9f\left(x+\frac{y}{3}\right) + f(x+2y)] \\ &\quad -136f(x-y) + 1394f(x+y) - 425f(y) + 1530f(x), t) \\ &\quad \geq_{L^*} Q_{\xi,\zeta}(x,y,t).\end{aligned} \tag{7.1}$$

If

$$\mathcal{T}_{i=1}^{\infty}(Q_{\xi,\zeta}(4^{n+i-1}x, 0, 4^{4n+3i+3}t)) = 1_{L^*} \tag{7.2}$$

and

$$\lim_{n\to\infty} Q_{\xi,\zeta}(4^n x, 4^n y, 4^{4n} t) = 1_{L^*} \tag{7.3}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q: X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - Q(x), t) \geq_{L^*} \mathcal{T}_{i=1}^{\infty}(Q_{\xi,\zeta}(4^{i-1}x, 0, 4^{3i+3}t)). \tag{7.4}$$

Proof. Putting $y = 0$ in (7.1), we have

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(4x)}{256} - f(x), t\right) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 4^4 t). \tag{7.5}$$

Therefore, it follows that

$$\mathcal{P}_{\mu,\nu}(\frac{f(4^{k+1}x)}{4^{4(k+1)}} - \frac{f(4^k x)}{4^{4k}}, \frac{t}{4^{4k}}) \geq_{L^*} Q_{\xi,\zeta}(4^k x, 0, 4^4 t), \tag{7.6}$$

which implies that

$$\mathcal{P}_{\mu,\nu}(\frac{f(4^{k+1}x)}{4^{4(k+1)}} - \frac{f(4^k x)}{4^{4k}}, t) \geq_{L^*} Q_{\xi,\zeta}(4^k x, 0, 4^{4(k+1)} t), \tag{7.7}$$

that is,

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(4^{k+1}x)}{4^{4(k+1)}} - \frac{f(4^k x)}{4^{4k}}, \frac{t}{4^{k+1}}\right) \geq_{L*} Q_{\xi,\zeta}(4^k x, 0, 4^{4(k+1)} t) \tag{7.8}$$

for all $k \in N$ and all $t > 0$. As $1 > 1/4 + \ldots + 1/4^n$, from the triangle inequality, it follows

$$\begin{aligned} \mathcal{P}_{\mu,\nu}\left(\frac{f(4^n x)}{256^n} - f(x), t\right) &\geq_{L^*} \mathcal{T}_{k=0}^{n-1}\left(\mathcal{P}_{\mu,\nu}\left(\frac{f(4^{k+1}x)}{4^{4(k+1)}} - \frac{f(4^k x)}{4^{4k}}, \sum_{k=0}^{n-1} \frac{1}{4^{k+1}} t\right)\right) \\ &\geq_{L^*} \mathcal{T}_{i=1}^{n}(Q_{\xi,\zeta}(4^{i-1}x, 0, 4^{3i+3}t)). \end{aligned} \tag{7.9}$$

In order to prove convergence of the sequence $\{\frac{f(4^n x)}{256^n}\}$, replacing x with $4^m x$ in (7.9), we get that for $m, n > 0$

$$\mathcal{P}_{\mu,\nu}(\frac{f(4^{n+m}x)}{256^{(n+m)}} - \frac{f(4^m x)}{256^m}, t) \geq_{L^*} \mathcal{T}_{i=1}^{n}(Q_{\xi,\zeta}(4^{i+m-1}x, 0, 4^{3i+4m+3}t)). \tag{7.10}$$

Since the right-hand side of the inequality tends 1_{L^*} as m tends to infinity, the sequence $\{\frac{f(4^n x)}{4^{4n}}\}$ is a Cauchy sequence. So we may define $Q(x) = \lim_{n\to\infty} \frac{f(4^n x)}{4^{4n}}$ for all $x \in X$.

Now, we show that Q is a quartic mapping. Replacing x, y with $4^n x$ and $4^n y$, respectively, in (7.1), we obtain

$$\begin{aligned} &\mathcal{P}_{\mu,\nu}(\frac{f(4^n(x+4y))}{256^n} + \frac{f(4^n(4x-y))}{256^n} - \frac{306[9f(4^n(x+\frac{y}{3})) + f(4^n(x+2y))}{256^n} \\ &\quad - \frac{136f(4^n(x-y))}{256^n} + \frac{1394f(4^n(x+y))}{256^n} - \frac{425f(4^n(y))}{256^n} + \frac{1530f(4^n(x))}{256^n}, t) \\ &\geq_{L*} Q_{\xi,\zeta}(4^n x, 4^n y, 4^{4n} t). \end{aligned} \tag{7.11}$$

Taking the limit as $n \to \infty$, we find that Q satisfies (1.1) for all $x, y \in X$.

Taking the limit as $n \to \infty$ in (7.9), we obtain (7.4).

To prove the uniqueness of the quartic mapping Q subject to (7.4), let us assume that there exists another quartic mapping Q' which satisfies (7.4). Obviously, we have $x \in X$ and all $n \in \mathbb{N}$. Hence it follows from (7.4) that

$$\begin{aligned} &\mathcal{P}_{\mu,\nu}(Q(x) - Q'(x), t) \\ &\geq_{L^*} \mathcal{P}_{\mu,\nu}(Q(4^n x) - Q'(4^n x), 4^{4n} t) \\ &\geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu,\nu}(Q(4^n x) - f(4^n x), 4^{4n-1} t), \mathcal{P}_{\mu,\nu}(f(4^n x) - Q'(4^n x), 4^{4n-1} t)) \\ &\geq_{L^*} \mathcal{T}(\mathcal{T}_{i=1}^{\infty}(Q_{\xi,\zeta}(4^{n+i-1}x, 0, 4^{4n+3i+3}t)), \mathcal{T}_{i=1}^{\infty}(Q_{\xi,\zeta}(4^{n+i-1}x, 0, 4^{4n+3i+3}t)) \end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in (7.4), we prove the uniqueness of Q. This completes the proof of the uniqueness, as desired. □

Corollary 7.2. *Let* $(X, \mathcal{P}'_{\mu',\nu'}, \mathcal{T})$*be an IRN-space and let* $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$*be a complete IRN-space. Let* $f : X \to Y$ *be a mapping such that*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(16f(x+4y)+f(4x-y)-306\left[9f\left(x+\frac{y}{3}\right)+f(x+2y)\right]\\&\quad -136f(x-y)+1394f(x+y)-425f(y)+1530f(x),t)\\&\qquad \geq_{L*}\mathcal{P}'_{\mu',\nu'}(x+y,t)\end{aligned}$$

for all $t > 0$ *in which*

$$\lim_{n\to\infty}\mathcal{T}_{i=1}^{\infty}(\mathcal{P}'_{\mu',\nu'}(x,4^{4n+3i+3}t))=1_{L^*}$$

for all $x, y \in X$. *Then there exists a unique quartic mapping* $Q : X \to Y$ *such that*

$$\mathcal{P}_{\mu,\nu}(f(x)-Q(x),t)\geq_{L*}\mathcal{T}_{i=1}^{\infty}(\mathcal{P}'_{\mu',\nu'}(x,4^{3i+3}t)).$$

Now, we give an example to illustrate the main result of Theorem 7.1 as follows.

Example 7.3. Let $(X, ||.||)$ be a Banach algebra, $(X, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ an IRN-space in which

$$\mathcal{P}_{\mu,\nu}(x,t)=\left(\frac{t}{t+||x||},\frac{||x||}{t+||x||}\right)$$

and let $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IRN-space for all $x \in X$. Define $f : X \to X$ by $f(x) = x^4 + x_0$, where x_0 is a unit vector in X. A straightforward computation shows that

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(16f(x+4y)+f(4x-y)-306\left[9f\left(x+\frac{y}{3}\right)+f(x+2y)\right]\\&\quad -136f(x-y)+1394f(x+y)-425f(y)+1530f(x),t)\\&\qquad \geq_{L*}\mathcal{P}_{\mu,\nu}(x+y,t),\quad \forall t>0.\end{aligned}$$

Also

$$\begin{aligned}\lim_{n\to\infty}M_{i=1}^{\infty}(\mathcal{P}_{\mu,\nu}(4^{n+i-1}x,4^{4n+3i+3}t))&=\lim_{n\to\infty}\lim_{m\to\infty}M_{i=1}^{m}(\mathcal{P}_{\mu,\nu}(x,4^{3n+2i+4}t))\\&=\lim_{n\to\infty}\lim_{m\to\infty}\mathcal{P}_{\mu,\nu}(x,4^{3n+6}t)\\&=\lim_{n\to\infty}\mathcal{P}_{\mu,\nu}(x,4^{3n+6}t)\\&=1_{L^*}.\end{aligned}$$

Therefore, all the conditions of 7.1 hold and so there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x)-Q(x),t)\geq_{L^*}\mathcal{P}_{\mu,\nu}(x,4^{6}t).$$

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Authors' contributions

All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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