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Maximal ϕ-inequalities for demimartingales

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Abstract

In this paper, we establish some maximal ϕ-inequalities for demimartingales that generalize the results of Wang (Stat. Probab. Lett. 66, 347-354, 2004) and Wang et al. (J. Inequal. Appl. 2010(838301), 11, 2010) and improve Doob's type inequality for demimartingales in some cases.

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1. Introduction

Definition 1.1 Let S_1, S_2, ... be an L^1 sequence of random variables. Assume that for $j = 1, 2, \ldots$

$$E\{(S_{j+1} - S_j)f(S_1, \ldots, S_j)\} \geq 0 \tag{1.1}$$

for all componentwise nondecreasing functions f such that the expectation is defined. Then $\{S_j, j \geq 1\}$ is called a demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_j, j \geq 1\}$ is called a demisubmartingale.

Remark. If the function f is not required to be nondecreasing, then the condition (1.1) is equivalent to the condition that $\{S_j, j \geq 1\}$ is a martingale with the natural choice of σ-algebras. If the function f is assumed to be nonnegative and not necessarily nondecreasing, then the condition (1.1) is equivalent to the condition that $\{S_j, j \geq 1\}$ is a submartingale with the natural choice of σ -algebras. A martingale with the natural choice of σ-algebras is a demimartingale. It can be checked that a submartingale is a demisubmartingale (cf. [[1], Proposition 1]). However, there are stochastic processes that are demimartingales but not martingales with the natural choice of σ-algebras (cf. [[1], example A], [[2], p. 10]). Definition 1.1 is due to Newman and Wright [3].

Relevant to the notion of demimartingales is the notion of positive dependence. To that end, we have the following definition.

Definition 1.2 A finite collection of random variables X_1, X_2, ..., X_m is said to be associated if

$$\mathrm{Cov}\{f(X_1, X_2, \ldots, X_m), g(X_1, X_2, \ldots, X_m)\} \geq 0$$

for any two componentwise nondecreasing functions f, g on R^m such that the covariance is defined. An infinite collection is associated if every finite subcollection is associated.

Remark. Associated random variables were introduced by Esary et al. [4] and have been found many applications especially in reliability theory. Proposition 2 of Newman and Wright [3] shows that the partial sum of a sequence of mean zero associated random variables is a demimartingale.

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The connection between demimartingales and martingales pointed out in the previous remark raises the question whether certain results and especially maximal inequalities valid for martingales are also valid for demimartingales. Newman and Wright [3] have extended various results including Doob's maximal inequality and Doob's upcrossing inequality to the case of demimartingales. Christofides [5] showed that Chow's maximal inequality for (sub)martingales can be extended to the case of demi(sub)martingales. Prakasa Rao [6] derived a Whittle-type inequality for demisubmartingales. Wang [7] obtained Doob's type inequality for more general demimartingales. Prakasa Rao [8] established some maximal inequalities for demisubmartingales. Wang et al. [9] established some maximal inequalities for demimartingales that generalize the results of Wang [7]. In this paper, we establish some maximal ϕ-inequalities for demimartingales that generalize the results of Wang [7] and Wang et al. [9], and improve Doob's type inequality for demimartingales in some cases.

2. Demimartingales inequalities

Let $\mathcal{C}$ denote the class of *Orlicz* functions, that is, unbounded, nondecreasing convex functions $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$. Let $\mathcal{C}'$ denote the set of $\phi \in \mathcal{C}$ such that $\frac{\phi'(x)}{x}$ is integrable at 0. Given $\phi \in \mathcal{C}$ and $a \geq 0$, define

$$\Phi_a(x) = \int_a^x \int_a^s \frac{\phi'(r)}{r} \mathrm{d}r\mathrm{d}s, \quad x > 0.$$

Denote $\Phi(x) = \Phi_0(x)$, $x > 0$.

We now prove a maximal ϕ-inequality for demimartingales.

Theorem 2.1. Let $S_1, S_2, \ldots$ be a demimartingale and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\phi \in \mathcal{C}'$ and $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive numbers, define $S_n^* = \max_{1 \leq k \leq n} c_k g(S_k)$. Then

$$E[\phi(S_n^*)] \leq \left(E\left[\sum_{j=1}^n c_j (g(S_j) - g(S_{j-1})) \right]^p \right)^{\frac{1}{p}} (E[\Phi'(S_n^*)]^q)^{\frac{1}{q}}, \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Proof. By Fubini theorem and Theorem 2.1 in [7] we have

$$\begin{aligned} E[\phi(S_n^*)] &= \int_0^{+\infty} \phi'(t) P(S_n^* \geq t) \mathrm{d}t \\ &\leq \int_0^{+\infty} \frac{\phi'(t)}{t} E\left[\sum_{j=1}^n c_j (g(S_j) - g(S_{j-1})) \chi_{\{S_n^* \geq t\}} \right] \mathrm{d}t \\ &= E\left[\int_0^{S_n^*} \frac{\phi'(t)}{t} \sum_{j=1}^n c_j (g(S_j) - g(S_{j-1})) \mathrm{d}t \right] \\ &= E\left[\sum_{j=1}^n c_j (g(S_j) - g(S_{j-1})) \Phi'(S_n^*) \right] \\ &\leq \left(E\left[\sum_{j=1}^n c_j (g(S_j) - g(S_{j-1})) \right]^p \right)^{\frac{1}{p}} (E[(\Phi'(S_n^*))^q])^{\frac{1}{q}}. \end{aligned}$$

The last inequality follows from the *Hölder*'s inequality.

Remark. Let $\phi(x) = x^p$, $p > 1$ in Theorem 2.1, then $\Phi(x) = \dfrac{x^p}{p-1}$. Hence

$$E[(S_n^*)^p] \le \frac{p}{p-1}\left(E\left[\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))\right]^p\right)^{\frac{1}{p}} (E[(S_n^*)^p])^{\frac{1}{q}}.$$

Let $E[(S_n^*)^p] < +\infty$. We get

$$E[(S_n^*)^p] \le \left(\frac{p}{p-1}\right)^p E\left[\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))\right]^p,$$

which is the inequality (2.1) of Theorem 2.1 in [9].

Let $\phi(x) = (x - 1)^+ = \max\{0, x - 1\}$ in Theorem 2.1. Then $\phi(x) = \int_0^x \chi_{\{s \ge 1\}} \mathrm{d}s$. Hence $\Phi'(x) = \int_0^x \frac{\phi'(r)}{r} \mathrm{d}r$. Therefore

$$\begin{aligned} E[S_n^* - 1] &\le E[S_n^* - 1]^+ \\ &\le E\left[\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1})) \int_0^{S_n^*} \frac{\chi_{\{r \ge 1\}}}{r} \mathrm{d}r\right] \\ &= E\left[\left(\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))\right) \ln^+ S_n^*\right], \end{aligned}$$

which is the inequality (2.6) in [9]. By the inequality

$$a \ln^+ b \le a \ln^+ a + be^{-1}, \quad a \ge 0,\ b > 0,$$

we have

$$E[S_n^*] \le \frac{e}{e-1}\left(1 + E\left[\left(\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))\right) \ln^+ \left(\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))\right)\right]\right), \qquad (2.2)$$

which is the inequality (2.2) of Theorem 2.1 in [9]. Let $c_j = 1$, $j \ge 1$ in inequality (2.2), the inequality (2.10) in [9] is obtained immediately. Let $g(x) = |x|$ in inequality (2.2) we have

$$E\left[\max_{1 \le k \le n} c_k |S_k|\right] \le \frac{e}{e-1}\left(1 + E\left[\left(\sum_{j=1}^{n} c_j(|S_j| - |S_{j-1}|)\right) \ln^+ \left(\sum_{j=1}^{n} c_j(|S_j| - |S_{j-1}|)\right)\right]\right), \qquad (2.3)$$

which is the inequality (2.10) in [7]. Let $c_j = 1$, $j \ge 1$ in inequality (2.3) we have

$$E\left[\max_{1 \le k \le n} |S_k|\right] \le \frac{e}{e-1}\left(1 + E\left[|S_n| \ln^+ |S_n|\right]\right), \qquad (2.4)$$

which is the inequality (2.11) in [9].

Corollary 2.1. Let S_1, S_2, ... be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\phi \in \mathcal{C}'$. Then

$$E\left[\phi\left(\max_{1\le k\le n} g(S_k)\right)\right] \le (E[g(S_n)]^p)^{\frac{1}{p}}\left(E\left[\Phi'\left(\max_{1\le k\le n} g(S_k)\right)\right]^q\right)^{\frac{1}{q}}, \tag{2.5}$$

Where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

Proof. Let $c_k = 1$, $k \ge 1$ in Theorem 2.1 we get (2.5) immediately.

Remark. Let $\phi(x) = x^p$, $p > 1$ in Corollary 2.1, then $\Phi(x) = \frac{x^p}{p-1}$. Hence

$$E\left[\max_{1\le k\le n} g(S_k)\right]^p \le \frac{p}{p-1}(E[g(S_n)]^p)^{\frac{1}{p}}\left(E\left[\max_{1\le k\le n} g(S_k)\right]^p\right)^{\frac{1}{q}}.$$

Let $E[\max_{1\le k\le n} g(S_k)]^p < +\infty$. We get

$$E\left[\max_{1\le k\le n} g(S_k)\right]^p \le \left(\frac{p}{p-1}\right)^p E[g(S_n)]^p,$$

which is the inequality (2.9) in [9]. Let $g(x) = |x|$ in the above inequality we get

$$E\left[\max_{1\le k\le n} |S_k|\right]^p \le \left(\frac{p}{p-1}\right)^p E[|S_n|]^p,$$

which is the inequality (2.11) in [9].

Corollary 2.2. Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $\{c_k, k \ge 1\}$ be a non-increasing sequence of positive numbers. Let $\phi \in \mathcal{C}'$. Then

$$E\left[\phi\left(\max_{1\le k\le n} c_k|S_k|\right)\right] \le \left(E\left[\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)\right]^p\right)^{\frac{1}{p}}\left(E\left[\Phi'\left(\max_{1\le k\le n} c_k|S_k|\right)\right]^q\right)^{\frac{1}{q}}, \tag{2.6}$$

Where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

Proof. Let $g(x) = |x|$ in Theorem 2.1, inequality (2.6) is obtained immediately.

Remark. Let $\phi(x) = x^p$, $p > 1$ in Corollary 2.2, then $\Phi(x) = \frac{x^p}{p-1}$. Hence

$$E\left[\max_{1\le k\le n} c_k|S_k|\right]^p \le \frac{p}{p-1}\left(E\left[\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)\right]^p\right)^{\frac{1}{p}}\left(E\left[\max_{1\le k\le n} c_k|S_k|\right]^p\right)^{\frac{1}{q}}.$$

Let $E[\max_{1\le k\le n} c_k|S_k|]^p < +\infty$. We get

$$E\left[\max_{1\le k\le n} c_k|S_k|\right]^p \le q^p E\left[\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)\right]^p,$$

which is the inequality (2.9) in [7].

We now prove some other maximal ϕ-inequalities for demimartingales following the techniques in [8].

Theorem 2.2 Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\{c_k, k \ge 1\}$ be a nonincreasing sequence of positive numbers and $\phi \in \mathcal{C}$. Then

$$\begin{aligned} P\left(\max_{1\le k\le n} c_k g(S_k) \ge t\right) &\le \frac{\lambda}{(1-\lambda)t}\int_t^{+\infty} P\left(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1})) > \lambda s\right)\mathrm{d}s \\ &= \frac{\lambda}{(1-\lambda)t}E\left(\frac{\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))}{\lambda} - t\right)^{+} \end{aligned} \tag{2.7}$$

for all $n \ge 1$, $t > 0$ and $0 < \lambda < 1$. Furthermore,

$$\begin{aligned} E\left[\phi\left(\max_{1\le k\le n} c_k g(S_k)\right)\right] \le \phi(b) + \frac{\lambda}{1-\lambda}\int_{\{\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))>\lambda b\}} &\left(\Phi_a\left(\frac{\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))}{\lambda}\right)\right. \\ &\left. -\Phi_a(b) - \Phi'_a(b)\left(\frac{\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))}{\lambda} - b\right)\right)\mathrm{d}P \end{aligned} \tag{2.8}$$

for $n \ge 1$, $a > 0$, $b > 0$ and $0 < \lambda < 1$.

Proof. Let $t > 0$ and $0 < \lambda < 1$. Theorem 2.1 in [7] implies

$$\begin{aligned} P\left(\max_{1\le k\le n} c_k g(S_k) \ge t\right) &\le \frac{1}{t}E\left[\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\chi_{\{\max_{1\le k\le n} c_k g(S_k)\ge t\}}\right] \\ &= \frac{1}{t}\int_{\{\max_{1\le k\le n} c_k g(S_k)\ge t\}} \sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\mathrm{d}P \\ &= \frac{1}{t}\int_0^{+\infty} P\left(\max_{1\le k\le n} c_k g(S_k)\ge t, \sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1})) > s\right)\mathrm{d}s \\ &\le \frac{1}{t}\int_0^{\lambda t} P\left(\max_{1\le k\le n} c_k g(S_k)\ge t\right)\mathrm{d}s + \frac{1}{t}\int_{\lambda t}^{+\infty} P(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1})) > s)\mathrm{d}s \\ &= \lambda P\left(\max_{1\le k\le n} c_k g(S_k)\ge t\right) + \frac{\lambda}{t}\int_t^{+\infty} P\left(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1})) > \lambda s\right)\mathrm{d}s. \end{aligned}$$

Rearranging the last inequality, we get that

$$\begin{aligned} P\left(\max_{1\le k\le n} c_k g(S_k) \ge t\right) &\le \frac{\lambda}{(1-\lambda)t}\int_t^{+\infty} P\left(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1})) > \lambda s\right)\mathrm{d}s \\ &= \frac{\lambda}{(1-\lambda)t}E\left(\frac{\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))}{\lambda} - t\right)^{+} \end{aligned}$$

for all $n \ge 1$, $t > 0$ and $0 < \lambda < 1$.

Let $b > 0$. By inequality (2.7), then

$$\begin{aligned}
E\left[\phi\left(\max_{1\le k\le n} c_k g(S_k)\right)\right] &= \int_0^{+\infty} \phi'(t) P\left(\max_{1\le k\le n} c_k g(S_k) \ge t\right) \mathrm{d}t \\
&= \int_0^{b} \phi'(t) P\left(\max_{1\le k\le n} c_k g(S_k) \ge t\right) \mathrm{d}t + \int_b^{+\infty} \phi'(t) P\left(\max_{1\le k\le n} c_k g(S_k) \ge t\right) \mathrm{d}t \\
&\le \phi(b) + \int_b^{+\infty} \phi'(t) P\left(\max_{1\le k\le n} c_k g(S_k) \ge t\right) \mathrm{d}t \\
&\le \phi(b) + \frac{\lambda}{1-\lambda} \int_b^{+\infty} \frac{\phi'(t)}{t} \left[\int_t^{+\infty} P\left(\sum_{j=1}^n c_j(g(S_j) - g(S_{j-1})) > \lambda s\right) \mathrm{d}s\right] \mathrm{d}t \\
&= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^{+\infty} \int_b^{s} \frac{\phi'(t)}{t} \mathrm{d}t P\left(\sum_{j=1}^n c_j(g(S_j) - g(S_{j-1})) > \lambda s\right) \mathrm{d}s \\
&= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^{+\infty} (\Phi'_a(s) - \Phi'_a(b)) P\left(\sum_{j=1}^n c_j(g(S_j) - g(S_{j-1})) > \lambda s\right) \mathrm{d}s \\
&= \phi(b) + \frac{\lambda}{1-\lambda} \int_{\{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))>\lambda b\}} \left(\Phi_a\left(\frac{\sum_{j=1}^n c_j(g(S_j) - g(S_{j-1}))}{\lambda}\right)\right. \\
&\quad \left. - \Phi_a(b) - \Phi'_a(b)\left(\frac{\sum_{j=1}^n c_j(g(S_j) - g(S_{j-1}))}{\lambda} - b\right)\right) \mathrm{d}P
\end{aligned}$$

for $n \ge 1$, $a > 0$, $b > 0$, $t > 0$ and $0 < \lambda < 1$.

Corollary 2.3. Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\phi \in \mathcal{C}$. Then

$$P\left(\max_{1\le k\le n} g(S_k) \ge t\right) \le \frac{\lambda}{(1-\lambda)t} \int_t^{+\infty} P(g(S_n) > \lambda s) \mathrm{d}s = \frac{\lambda}{(1-\lambda)t} E\left(\frac{g(S_n)}{\lambda} - t\right)^+$$

for all $n \ge 1$, $t > 0$ and $0 < \lambda < 1$. Furthermore,

$$E\left[\phi\left(\max_{1\le k\le n} g(S_k)\right)\right] \le \phi(b) + \frac{\lambda}{1-\lambda} \int_{\{g(S_n)>\lambda b\}} \left(\Phi_a\left(\frac{g(S_n)}{\lambda}\right) - \Phi_a(b) - \Phi'_a(b)\left(\frac{g(S_n)}{\lambda} - b\right)\right) \mathrm{d}P$$

for all $n \ge 1$, $a > 0$, $b > 0$ and $0 < \lambda < 1$.

Proof. Let $c_k = 1$, $k \ge 1$ in Theorem 2.2, Corollary 2.3 follows.

As a special case of Corollary 2.3 is the following corollary.

Corollary 2.4. Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $\phi \in \mathcal{C}$. Then

$$P\left(\max_{1\le k\le n} |S_k| \ge t\right) \le \frac{\lambda}{(1-\lambda)t} \int_t^{+\infty} P(|S_n| > \lambda s) \mathrm{d}s = \frac{\lambda}{(1-\lambda)t} E\left(\frac{|S_n|}{\lambda} - t\right)^+$$

for all $n \ge 1$, $t > 0$ and $0 < \lambda < 1$. Furthermore,

$$E\left[\phi\left(\max_{1\le k\le n} |S_k|\right)\right] \le \phi(b) + \frac{\lambda}{1-\lambda} \int_{\{|S_n|>\lambda b\}} \left(\Phi_a\left(\frac{|S_n|}{\lambda}\right) - \Phi_a(b) - \Phi'_a(b)\left(\frac{|S_n|}{\lambda} - b\right)\right) \mathrm{d}P$$

for all $n \ge 1$, $a > 0$, $b > 0$ and $0 < \lambda < 1$.

Remark. Theorem 3.1 in [8] is generalized in the case of demimartingales.

As a special case of Theorem 2.2 is the following theorem.

Theorem 2.3 Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive numbers and $\phi \in \mathcal{C}$. Then

$$E\left[\phi\left(\max_{1\leq k\leq n} c_k g(S_k)\right)\right] \leq \phi(a) + \frac{\lambda}{1-\lambda} E\left[\Phi_a\left(\frac{\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))}{\lambda}\right)\right] \tag{2.9}$$

for all $n \geq 1$, $a > 0$ and $0 < \lambda < 1$. Let $\lambda = \frac{1}{2}$ in (2.9). Then

$$E\left[\phi\left(\max_{1\leq k\leq n} c_k g(S_k)\right)\right] \leq \phi(a) + E\left[\Phi_a\left(2\sum_{j=1}^{n} c_j(g(S_j) - g(S_{j-1}))\right)\right]$$

for $a > 0$, $n \geq 1$.

Proof. Theorem 2.3 follows from Choosing $b = a$ in (2.8) and observing that $\Phi_a(a) = \Phi_a'(a) = 0$.

Let $c_k = 1$, $k \geq 1$ in Theorem 2.3 we have the following corollary.

Corollary 2.5. Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\phi \in \mathcal{C}$. Then

$$E\left[\phi\left(\max_{1\leq k\leq n} g(S_k)\right)\right] \leq \phi(a) + \frac{\lambda}{1-\lambda} E\left[\Phi_a\left(\frac{g(S_n)}{\lambda}\right)\right]$$

for all $n \geq 1$, $a > 0$, $0 < \lambda < 1$ and

$$E\left[\phi\left(\max_{1\leq k\leq n} g(S_k)\right)\right] \leq \phi(a) + E[\Phi_a(2g(S_n))]$$

for $a > 0$, $n \geq 1$.

As a special case of Corollary 2.5 is the following Corollary.

Corollary 2.6. Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $\phi \in \mathcal{C}$. Then

$$E\left[\phi\left(\max_{1\leq k\leq n} |S_k|\right)\right] \leq \phi(a) + \frac{\lambda}{1-\lambda} E\left[\Phi_a\left(\frac{|S_n|}{\lambda}\right)\right].$$

for all $n \geq 1$, $a > 0$, $0 < \lambda < 1$ and

$$E\left[\phi\left(\max_{1\leq k\leq n} |S_k|\right)\right] \leq \phi(a) + E[\Phi_a(2|S_n|)].$$

for $a > 0$, $n \geq 1$.

Remark. Theorem 3.2 in [8] is generalized in the case of demimartingales.

Theorem 2.4 Let $S_1, S_2, \ldots$ be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive numbers. Then

$$E\left[\max_{1\le k\le n} c_k g(S_k)\right] \le b + \frac{b}{b-1}\left(E\left[\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\ln^+\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\right] - E\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))-1\right]^+\right), \quad b>1,\ n\ge 1. \tag{2.10}$$

Proof. Let $\phi(x) = x$ in Theorem 2.2. Then $\Phi_1(x) = x \ln x - x + 1$, $\Phi_1'(x) = \ln x$. Hence

$$\begin{aligned} E\left[\max_{1\le k\le n} c_k g(S_k)\right] &\le b + \frac{\lambda}{1-\lambda}\int_{\{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))>\lambda b\}}\Bigg(\frac{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))}{\lambda} \\ &\quad\times \ln\frac{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))}{\lambda} - \frac{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))}{\lambda} + 1 \\ &\quad -b\ln b + b - 1 - \frac{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))}{\lambda}\ln b + b\ln b)\Bigg)\mathrm{d}P \\ &= b + \frac{1}{1-\lambda}\int_{\{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))>\lambda b\}}\left(\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\ln\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\right. \\ &\quad \left. - \left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)(\ln\lambda + \ln b + 1) + \lambda b\right)\mathrm{d}P \end{aligned}$$

for all $n \ge 1$, $b > 0$ and $0 < \lambda < 1$. Let $b > 1$, $\lambda = \frac{1}{b}$. Therefore

$$\begin{aligned} E\left[\max_{1\le k\le n} c_k g(S_k)\right] &\le b + \frac{b}{b-1}\int_{\{\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))>1\}}\left(\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\right. \\ &\quad \left.\times \ln\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right) - \sum_{j=1}^n c_j(g(S_j)-g(S_{j-1})) + 1\right)\mathrm{d}P \\ &= b + \frac{b}{b-1}E\left[\int_1^{\max(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1})),1)} \ln x\,\mathrm{d}x\right] \end{aligned} \tag{2.11}$$

for all $b > 1$ and $n \ge 1$. Since

$$\int_1^x \ln y\,\mathrm{d}y = x\ \ln^+\ x - (x-1), x \ge 1,$$

the inequality (2.11) can be rewritten in the form

$$E\left[\max_{1\le k\le n} c_k g(S_k)\right] \le b + \frac{b}{b-1}\left(E\left[\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\ln^+\left(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))\right)\right] - E\left[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))-1\right]^+\right), \quad b>1,\ n\ge 1.$$

Corollary 2.7. Let S_1, S_2, ... be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\{c_k, k \ge 1\}$ be a nonincreasing sequence of positive numbers. Then

$$E\left[\max_{1\le k\le n} c_k g(S_k)\right] \le \frac{1+E\left[\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))-1\right]^{+}}{E\left[\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))-1\right]^{+}} \times E\left[\left(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\right)\ln^{+}\left(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\right)\right]. \tag{2.12}$$

Proof. Let $b = E\left[\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))-1\right]^{+} + 1$ in (2.10). Then we get (2.12).

Corollary 2.8. Let S_1, S_2, ... be a demimartingale with $S_0 = 0$ and $g(.)$ be a nonnegative convex function such that $g(0) = 0$. Let $\{c_k, k \ge 1\}$ be a nonincreasing sequence of positive numbers. Then

$$E\left[\max_{1\le k\le n} c_k g(S_k)\right] \le e + \frac{e}{e-1}\left(E\left[\left(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\right)\ln^{+}\left(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\right)\right] - E\left[\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))-1\right]^{+}\right), \quad n \ge 1. \tag{2.13}$$

Proof. Let $b = e$ in (2.10). Then we get (2.13).

Remark. Inequality (2.13) is a sharper inequality than inequality (2.2) in [9] when

$$E\left[\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))-1\right]^{+} \ge e-2.$$

Corollary 2.9. Let S_1, S_2, ... be a demimartingale with $S_0 = 0$ and $\{c_k, k \ge 1\}$ be a nonincreasing sequence of positive numbers. Then

$$E\left[\max_{1\le k\le n} c_k |S_k|\right] \le e + \frac{e}{e-1}\left(E\left[\left(\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)\right)\ln^{+}\left(\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)\right)\right] - E\left[\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)-1\right]^{+}\right). \tag{2.14}$$

Proof. Let $g(x) = |x|$ in (2.13). Then we get (2.14).

Remark. Inequality (2.14) is a sharper inequality than inequality (2.10) in [7] when

$$E\left[\sum_{j=1}^{n} c_j(|S_j|-|S_{j-1}|)-1\right]^{+} \ge e-2.$$

Corollary 2.10. Let S_1, S_2, ... be a demimartingale with $S_0 = 0$. Then

$$E\left[\max_{1\le k\le n} |S_k|\right] \le b + \frac{b}{b-1}\left(E\left[|S_n|\ln^{+}|S_n|\right] - E[|S_n|-1]^{+}\right), \quad b > 1,\ n \ge 1. \tag{2.15}$$

Proof. Let $c_j = 1$, $j \ge 1$ and $g(x) = |x|$ in Theorem 2.4. We get inequality (2.15).

Remark. The inequality (3.22) in [8] is generalized in the case of demimartingales.

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References
1. Wood, TE: Sample paths of demimartingales. In: Seynal D, Weron A (eds.) Probability Theory on Vector Spaces III, LNM. pp. 365–373. Springer, New York (1984)
2. Bulinski, A, Shashkin, A: Limit Theorems for Associated Random Fields and Related Systems. World Scientific, Singapore (2007)
3. Newman, CM, Wright, AL: Associated random variables and martingales inequalities. Z Wahrsch Verw Geb. **59**(3), 361–371 (1982). doi:10.1007/BF00532227
4. Esary, J, Proschan, F, Walkup, D: Association of random variables with applications. Ann Math Stat. **38**(5), 1466–1474 (1967). doi:10.1214/aoms/1177698701
5. Christofides, TC: Maximal inequalities for demimartingales and a strong law of large numbers. Stat Probab Lett. **50**(4), 357–363 (2000). doi:10.1016/S0167-7152(00)00116-4
6. Prakasa Rao, BLS: Whittle type inequality for demisubmartingales. Proc Am Math Soc. **130**(12), 3719–3724 (2002). doi:10.1090/S0002-9939-02-06517-6
7. Wang, JF: Maximal inequalities for associated random variables and demimartingales. Stat Probab Lett. **66**(3), 347–354 (2004). doi:10.1016/j.spl.2003.10.021
8. Prakasa Rao, BLS: On some maximal inequality for demisubmartingales and *N*-demisuper martingales. J Inequal Pure Appl Math **8**(4), 1–17 (2007). Article 112
9. Wang, XJ, Hu, SH, Zhao, T., *et al*: Doob's type inequality and strong law of large numbers for demimartingales. J Inequal Appl. (2010). doi:10.1155/2010/838301

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