Journal of Inequalities and Applications
a SpringerOpen Journal

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A Caccioppoli-type estimate for very weak solutions to obstacle problems with weight

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Abstract

This paper gives a Caccioppoli-type estimate for very weak solutions to obstacle problems of the $\mathcal{A}$-harmonic equation $\mathrm{div}\mathcal{A}(x, \nabla u) = 0$ with $|\mathcal{A}(x, \xi)| \approx w(x)|\xi|^{p-1}$, where $1 < p < \infty$ and $w(x)$ be a Muckenhoupt A_1 weight.

Mathematics Subject Classification (2000) 35J50, 35J60

Keywords: Caccioppoli-type estimate, very weak solution, obstacle problem, Muckenhoupt weight, A-harmonic equation

1 Introduction

Let w be a locally integrable non-negative function in R^n and assume that $0 < w < \infty$ almost everywhere. We say that w belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that w is an A_p weight, if there is a constant $A_p(w)$ such that

$$\sup_B \left(\frac{1}{|B|} \int_B w \mathrm{d}x \right) \left(\frac{1}{|B|} \int_B w^{1/(1-p)} \mathrm{d}x \right)^{p-1} = A_p(w) < \infty \tag{1.1}$$

for all balls B in R^n. We say that w belongs to A_1, or that w is an A_1 weight, if there is a constant $A_1(w)$ such that

$$\frac{1}{|B|} \int_B w \mathrm{d}x \leq A_1(w) \mathrm{essinf}_B w$$

for all balls B in R^n.

As customary, μ stands for the measure whose Radon-Nikodym derivative w is

$$\mu(E) = \int_E w \mathrm{d}x.$$

It is well known that $A_1 \subset A_p$ whenever $p > 1$, see [1]. We say that a weight w is doubling if there is a constant $C > 0$ such that

$$\mu(2B) \leq C\mu(B)$$

whenever $B \subset 2B$ are concentric balls in R^n, where $2B$ is the ball with the same center as B and with radius twice that of B. Given a measurable subset E of R^n, we will denote by $L^p(E, w)$, $1 < p < \infty$, the Banach space of all measurable functions f defined on E for which

$$||f||_{L^p(E,w)} = \left(\int_E |f(x)|^p w(x) \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

The weighted Sobolev class $W^{1,p}(E, w)$ consists of all functions f, and its first generalized derivatives belong to $L^p\ (E, w)$. The symbols $L^p_{\mathrm{loc}}(E, w)$ and $W^{1,p}_{\mathrm{loc}}(E, w)$ are self-explanatory.

If $x_0 \in \Omega$ and $t > 0$, then B_t denotes the ball of radius t centered at x_0. For the function $u(x)$ and $k > 0$, let $A_k = \{x \in \Omega : |u(x)| > k\}$, $A_{k,t} = A_k \cap B_t$. Let $T_k(u)$ be the usual truncation of u at level $k > 0$, that is

$$T_k(u) = \max\{-k, \min\{k, u\}\}.$$

Let Ω be a bounded regular domain in R^n, $n \geq 2$. By a regular domain, we understand any domain of finite measure for which the estimates for the Hodge decomposition in (2.1) and (2.2) are satisfied. A Lipschitz domain, for example, is regular. We consider the second-order degenerate elliptic equation (also called $\mathcal{A}$-harmonic equation or Leray-Lions equation)

$$\mathrm{div}\mathcal{A}(x, \nabla u) = 0 \tag{1.2}$$

where $\mathcal{A}(x, \xi) : \Omega \times \mathrm{R}^n \to \mathrm{R}^n$ is a carathéodory function satisfying the following assumptions

1. $\langle \mathcal{A}(x, \xi), \xi \rangle \geq \alpha w(x) |\xi|^p$,
2. $|\mathcal{A}(x, \xi)| \leq \beta w(x) |\xi|^{p-1}$,

where $0 < \alpha \leq \beta < \infty$, $w \in A_1$ and $w \geq k_0 > 0$. Suppose ψ is any function in Ω with values in the extended reals $[-\infty, +\infty]$ and that $\theta \in W^{1,r}(\Omega, w)$, $\max\{1, p-1\} < r \leq p$. Let

$$\mathcal{K}^r_{\psi,\theta} = \mathcal{K}^r_{\psi,\theta}(\Omega, w) = \{v \in W^{1,r}(\Omega, w) : v \geq \psi,\ \text{a.e. } x \in \Omega \text{ and } v - \theta \in W^{1,r}_0(\Omega, w)\}.$$

The function ψ is an obstacle, and θ determines the boundary values.

We introduce the Hodge decomposition for $|\nabla(v-u)|^{r-p}\nabla(v-u) \in L^{\frac{r}{r-p+1}}(\Omega, w)$, from Lemma 1 in Section 2,

$$|\nabla(v-u)|^{r-p}\nabla(v-u) = \nabla\varphi + H \tag{1.3}$$

and the following estimate holds

$$\|H\|_{L^{\frac{r}{r-p+1}}(\Omega,w)} \leq cA_p(w)^\gamma |r-p| \|\nabla(v-u)\|^{r-p+1}_{L^r(\Omega,w)}. \tag{1.4}$$

Definition 1 *A very weak solution to the* $\mathcal{K}^r_{\psi,\theta}$*-obstacle problem is a function* $u \in \mathcal{K}^r_{\psi,\theta}(\Omega, w)$ *such that*

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), |\nabla(v-u)|^{r-p}\nabla(v-u) \rangle \mathrm{d}x \geq \int_\Omega \langle \mathcal{A}(x, \nabla u), H \rangle \mathrm{d}x \tag{1.5}$$

whenever $v \in \mathcal{K}^r_{\psi,\theta}(\Omega, w)$ *and H comes from the Hodge decomposition (1.3).*

The local and global higher integrability of the derivatives in obstacle problems with $w(x) \equiv 1$ was first considered by Li and Martio [2] in 1994, using the so-called reverse Hölder inequality. Gao and Tian [3] gave a local regularity result for weak solutions to obstacle problem in 2004. Recently, regularity theory for very weak solutions of the $\mathcal{A}$-harmonic equations with $w(x) \equiv 1$ have been considered [4], and the regularity theory for very solutions of obstacle problems with $w(x) \equiv 1$ have been explored in [5]. This paper gives a Caccioppoli-type estimate for solutions to obstacle problems with weight, which is closely related to the local regularity theory for very weak solutions of the $\mathcal{A}$-harmonic equation (1.2).

Theorem *There exists $r_1 \in (p - 1, p)$ such that for arbitrary $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega, w)$ and $r_1 < r < p$, a solution u to the $\mathcal{K}^r_{\psi,\theta}$-obstacle problem with weight $w(x) \in A_1$ satisfies the following Caccioppoli-type estimate*

$$\int_{A_{k,\rho}} |\nabla u|^r d\mu \le C\left[\int_{A_{k,R}} |\nabla \psi|^r d\mu + \frac{1}{(R-\rho)^r}\int_{A_{k,R}} |u|^r \mathrm{d}\mu\right]$$

where $0 < \rho < R < +\infty$ and C is a constant depends only on n, p and β/α.

2 Preliminary Lemmas

The following lemma comes from [6] which is a Hodge decomposition in weighted spaces.

Lemma 1 *Let Ω be a regular domain of R^n and $w(x)$ be an A_1 weight. If $u \in W_0^{1,p-\varepsilon}(\Omega, w)$, $1 < p < \infty$, $-1 < \varepsilon < p - 1$, then there exist $\varphi \in W_0^{1,\frac{p-\varepsilon}{1-\varepsilon}}(\Omega, w)$ and a divergence-free vector field $H \in L^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega, w)$ such that*

$$|\nabla u|^{-\varepsilon}\nabla u = \nabla\varphi + H$$

and

$$\|\nabla\varphi\|_{L^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega,w)} \le cA_p(w)^\gamma \|\nabla u\|^{1-\varepsilon}_{L^{p-\varepsilon}(\Omega,w)} \tag{2.1}$$

$$\|H\|_{L^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega,w)} \le cA_p(w)^\gamma |\varepsilon| \|\nabla u\|^{1-\varepsilon}_{L^{p-\varepsilon}(\Omega,w)} \tag{2.2}$$

where γ depends only on p.

We also need the following lemma in the proof of the main theorem.

Lemma 2 [7]*Let $f(t)$ be a non-negative bounded function defined for $0 \le T_0 \le t \le T_1$. Suppose that for $T_0 \le t < s \le T_1$, we have*

$$f(t) \le A(s-t)^{-\alpha} + B + \theta f(s),$$

where A, B, α, θ are non-negative constants and $\theta < 1$. Then, there exist a constant c, depending only on α and θ, such that for every ρ, R, $T_0 \le \rho < R \le T_1$ we have

$$f(\rho) \le c[A(R-\rho)^{-\alpha} + B].$$

3 Proof of the main theorem

Let u be a very weak solution to the $\mathcal{K}^r_{\psi,\theta}$-obstacle problem. Let $B_{R_1} \subset\subset \Omega$ and $0 < R_0 \leq \tau < t \leq R_1$ be arbitrarily fixed. Fix a cut-off function $\phi \in C_0^\infty(B_t)$ such that

$$\operatorname{supp}\phi \subset B_t, \ \ 0 \leq \phi \leq 1, \ \ \phi = 1 \text{ in } B_\tau \text{ and } |\nabla\phi| \leq 2(t-\tau)^{-1}.$$

Consider the function

$$v = u - T_k(u) - \phi^r(u - \psi_k^+),$$

where $T_k(u)$ is the usual truncation of u at the level k defined in Section 1 and $\psi_k^+ = \max\{\psi, T_k(u)\}$. Now $v \in \mathcal{K}^r_{\psi - T_k(u), \theta - T_k(u)}(\Omega, w)$. Indeed,

$$v - (\theta - T_k(u)) = u - \theta - \phi^r(u - \psi_k^+) \in W_0^{1,r}(\Omega, w)$$

since $\phi \in C_0^\infty(\Omega)$ and

$$v - (\psi - T_k(u)) = (u - \psi) - \phi^r(u - \psi_k^+) \geq (1 - \phi^r)(u - \psi) \geq 0$$

a.e. in Ω. Let

$$E(v,u) = |\phi^r \nabla u|^{r-p}\phi^r\nabla u + |\nabla(v - u + T_k(u))|^{r-p}\nabla(v - u + T_k(u)). \tag{3.1}$$

From an elementary formula [[8], (4.1)]

$$||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y| \leq 2^\varepsilon \frac{1+\varepsilon}{1-\varepsilon}|X - Y|^{1-\varepsilon}, \ \ X, Y \in \mathrm{R}^n, \ \ 0 \leq \varepsilon < 1$$

and $\nabla v = \nabla(u - T_k(u)) - \phi^r\nabla(u - \psi_k^+) - r\phi^{r-1}\nabla\phi(u - \psi_k^+)$, we can derive that

$$|E(v,u)| \leq 2^{p-r}\frac{p-r+1}{r-p+1}|\phi^r\nabla u - \phi^r\nabla(u - \psi_k^+) - r\phi^{r-1}\nabla\phi(u - \psi_k^+)|^{r-p+1}. \tag{3.2}$$

From (3.1), we get that

$$\begin{aligned}\int_{A_{k,t}} \langle \mathcal{A}(x,\nabla u), |\phi^r\nabla u|^{r-p}\phi^r\nabla u\rangle \mathrm{d}x = &\int_{A_{k,t}} \langle \mathcal{A}(x,\nabla u), E(v,u)\rangle \mathrm{d}x \\ &- \int_{A_{k,t}} \langle \mathcal{A}(x,\nabla u), |\nabla(v-u)|^{r-p}\nabla(v-u)\rangle \mathrm{d}x.\end{aligned} \tag{3.3}$$

Now we estimate the left-hand side of (3.3),

$$\int_{A_{k,t}} \langle A(x,\nabla u), |\phi^r\nabla u|^{r-p}\phi^r\nabla u\rangle \mathrm{d}x \geq \int_{A_{k,\tau}} \langle A(x,\nabla u), |\nabla u|^{r-p}\nabla u\rangle \mathrm{d}x \geq \alpha \int_{A_{k,\tau}} |\nabla u|^r \mathrm{d}\mu. \tag{3.4}$$

Using (1.3), we get

$$|\nabla(v - u + T_k(u))|^{r-p}\nabla(v - u + T_k(u)) = \nabla\varphi + H \tag{3.5}$$

and (1.4) yields

$$\|H\|_{L^{\frac{r}{r-p+1}}(\Omega,w)} \leq cA_p(w)^\gamma|r-p|\|\nabla(v - u + T_k(u))\|^{r-p+1}_{L^r(\Omega,w)}. \tag{3.6}$$

Since u - $T_k(u)$ is a very weak solution to the $\mathcal{K}^r_{\psi - T_k(u), \theta - T_k(u)}$-obstacle problem, we derive, by

Definition 1, that

$$\int_{\Omega} \langle \mathcal{A}(x, \nabla(u-T_k(u))), |\nabla(v-u+T_k(u))|^{r-p}\nabla(v-u+T_k(u))\rangle \mathrm{d}x \geq \int_{\Omega} \langle \mathcal{A}(x, \nabla(u-T_k(u))), H\rangle \mathrm{d}x$$

that is

$$\int_{A_{k,t}} \langle \mathcal{A}(x, \nabla u), |\nabla(v-u)|^{r-p}\nabla(v-u)\rangle \mathrm{d}x \geq \int_{A_{k,t}} \langle \mathcal{A}(x, \nabla u), H\rangle \mathrm{d}x. \tag{3.7}$$

Combining the inequalities (3.3), (3.4) and (3.7), we obtain

$$\begin{aligned}
\alpha \int_{A_{k,\tau}} |\nabla u|^r \mathrm{d}\mu \leq & \int_{A_{k,t}} \langle A(x, \nabla u), E(v, u)\rangle \mathrm{d}x - \int_{A_{k,t}} \langle A(x, \nabla u), H\rangle \mathrm{d}x \\
\leq & \beta \frac{2^{p-r}(p-r+1)}{r-p+1} \int_{A_{k,t}} |\nabla u|^{p-1} |\phi^r \nabla \psi_k^+ - r\phi^{r-1}\nabla\phi(u-\psi_k^+)|^{r-p+1} \mathrm{d}\mu \\
& + \beta \int_{A_{k,t}} |\nabla u|^{p-1}|H| \mathrm{d}\mu \\
\leq & \beta \frac{2^{p-r}(p-r+1)}{r-p+1} \int_{A_{k,t}} |\nabla u|^{p-1} |\phi^r \nabla \psi|^{r-p+1} \mathrm{d}\mu \\
& + \beta \frac{2^{p-r}(p-r+1)}{r-p+1} \int_{A_{k,t}} |\nabla u|^{p-1} |r\phi^{r-1}\nabla\phi(u-\psi_k^+)|^{r-p+1} \mathrm{d}\mu \\
& + \beta \int_{A_{k,t}} |\nabla u|^{p-1}|H| \mathrm{d}\mu \\
\leq & \beta \frac{2^{p-r}(p-r+1)}{r-p+1} \left(\int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu \right)^{\frac{p-1}{r}} \left(\int_{A_{k,t}} |\nabla \psi|^r \mathrm{d}\mu \right)^{\frac{r-p+1}{r}} \\
& + \beta \frac{2^{p-r}(p-r+1)}{r-p+1} \left(\int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu \right)^{\frac{p-1}{r}} \left(\int_{A_{k,t}} |r\phi^{p-1}\nabla\phi(u-\psi_k^+)|^r \mathrm{d}\mu \right)^{\frac{r-p+1}{r}} \\
& + \beta \left(\int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu \right)^{\frac{p-1}{r}} \left(\int_{A_{k,t}} |H|^{\overline{\frac{r}{r-p+1}}} \mathrm{d}\mu \right)^{\frac{r-p+1}{r}}.
\end{aligned}$$

Let $c_1 = \frac{2^{p-r}(p-r+1)}{r-p+1}$, by (3.6) and Young's inequality

$$ab \leq \varepsilon a^{p'} + c_2(\varepsilon, p) b^p, \frac{1}{p'} + \frac{1}{p} = 1, \ a, \ b \geq 0, \ \varepsilon \geq 0, \ p \geq 1,$$

we can derive that

$$\begin{aligned}
\alpha \int_{A_{k,\tau}} |\nabla u|^r \mathrm{d}\mu \leq & \beta c_1 \varepsilon \int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu + \beta c_1 c_2(\varepsilon, p) \int_{A_{k,t}} |\nabla \psi|^r \mathrm{d}\mu \\
& + \beta c_1 \varepsilon \int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu + \beta c_1 c_2(\varepsilon, p) \int_{A_{k,t}} |r\phi^{r-1}\nabla\phi(u-\psi_k^+)|^r \mathrm{d}\mu \\
& + \beta c A_p(w)^{\gamma}(p-r)\varepsilon \int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu \\
& + \beta c A_p(w)^{\gamma}(p-r) c_2(\varepsilon, p) \int_{\Omega} |\nabla(v-u+T_k(u))|^r \mathrm{d}\mu,
\end{aligned}$$

where c is the constant given by Lemma 1. Since $v - u + T_k(u) = 0$ on $\Omega \backslash A_{k,t}$, by the equality

$$\nabla v = \nabla(u - T_k(u)) - \phi^r \nabla(u - \psi_k^+) - r\phi^{r-1}\nabla\phi(u - \psi_k^+),$$

we obtain that

$$\int_\Omega |\nabla(v - u + T_k(u))|^r \mathrm{d}\mu = \int_{A_{k,t}} |\nabla(v - u)|^r \mathrm{d}\mu$$

$$= \int_{A_{k,t}} |\phi^r \nabla(u - \psi_k^+) + r\phi^{r-1}\nabla\phi(u - \psi_k^+)|^r \mathrm{d}\mu$$

$$\leq 2^{r-1} \int_{A_{k,t}} |\nabla(u - \psi_k^+)|^r \mathrm{d}\mu + 2^{r-1} r \int_{Ak,t} |\nabla\phi(u - \psi_k^+)|^r \mathrm{d}\mu$$

$$\leq 2^{2r-2} \int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu + 2^{2r-2} \int_{A_{k,t}} |\nabla\psi|^r \mathrm{d}\mu + r2^{2r-2} \int_{A_{k,t}} \frac{|u^r|}{(t-\tau)^r} \mathrm{d}\mu.$$

Finally, we obtain

$$\begin{aligned}\int_{A_{k,\tau}} |\nabla u|^r \mathrm{d}\mu \leq & \frac{\beta(2c_1 + cA_p(w)^\gamma(p-r))\varepsilon + \beta cA_p(w)^\gamma c_2(\varepsilon,p)2^{2r-2}(p-r)}{\alpha} \int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu \\ & + \frac{\beta c_1 c_2(\varepsilon,p) + 2^{2r-2}\beta cA_p(w)^\gamma c_2(\varepsilon,p)(p-r)}{\alpha} \int_{A_{k,t}} |\nabla\psi|^r \mathrm{d}\mu \\ & + r\frac{\beta c_1 c_2(\varepsilon,p) + 2^{2r-1}\beta cA_p(w)^\gamma c_2(\varepsilon,p)(p-r)}{\alpha} \int_{A_{k,t}} \frac{|u|^r}{(t-\tau)^r} \mathrm{d}\mu.\end{aligned} \tag{3.8}$$

Now we want to eliminate the first term in the right-hand side containing ∇u. Choosing ε and r_1 such that

$$\eta = \frac{\beta(2c_1 + cA_p(w)^\gamma(p-r))\varepsilon + \beta cA_p(w)^\gamma c_2(\varepsilon,p)2^{2r-2}(p-r)}{\alpha} < 1$$

and let ρ, R be arbitrarily fixed with $R_0 \leq \rho < R \leq R_1$. Thus, from (3.8), we deduce that for every t and τ such that $\rho \leq \tau < t \leq R$, we have

$$\int_{A_{k,\tau}} |\nabla u|^r \mathrm{d}\mu \leq \eta \int_{A_{k,t}} |\nabla u|^r \mathrm{d}\mu + \frac{c_3}{\alpha} \int_{A_{k,t}} |\nabla\psi| \mathrm{d}\mu + \frac{c_4}{\alpha(t-\tau)^r} \int_{A_{k,t}} |u|^r \mathrm{d}\mu, \tag{3.9}$$

where

$$c_3 = \beta c_1 c_2(\varepsilon,p) + 2^{2r-2}\beta cA_p(w)^\gamma c_2(\varepsilon,p)(p-r)$$

and

$$c_4 = r\beta c_1 c_2(\varepsilon,p) + r2^{2r-1}\beta cA_p(w)^\gamma c_2(\varepsilon,p)(p-r).$$

Applying Lemma 2 in (3.9), we conclude that

$$\int_{A_{k,\rho}} |\nabla u|^r \mathrm{d}\mu \leq \frac{cc_3}{\alpha} \int_{A_{k,R}} |\nabla\psi|^r d\mu + \frac{cc_4}{\alpha(R-\rho)^r} \int_{A_{k,R}} |u|^r \mathrm{d}\mu,$$

where c is the constant given by Lemma 2. This ends the proof of the main theorem.

Acknowledgements
The authors would like to thank the referee of this paper for helpful suggestions.
Research supported by NSFC (10971224) and NSF of Hebei Province (A2011201011).

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Authors' contributions
GH gave Definition 1. QJ found Lemmas 1 and 2. Theorem 1 was proved by both authors. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 3 March 2011 Accepted: 17 September 2011 Published: 17 September 2011

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doi:10.1186/1029-242X-2011-58
Cite this article as: Hongya and Jinjing: **A Caccioppoli-type estimate for very weak solutions to obstacle problems with weight.** *Journal of Inequalities and Applications* 2011 **2011**:58.