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Some results on the partial orderings of block matrices

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Abstract

Some results relating to the block matrix partial orderings and the submatrix partial orderings are given. Special attention is paid to the star ordering of a sum of two matrices and the minus ordering of matrix product. Several equivalent conditions for the minus ordering are established.

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1 Introduction

Let $C^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field C. The symbols A^*, $R(A)$, $R^{\perp}(A)$, $N(A)$ and $r(A)$ denote the conjugate transpose, the range, orthogonal complement space, the null space and the rank of a given matrix $A \in C^{m\times n}$.

Furthermore, $A^{\dagger}$ will stand for the Moore-Penrose inverse of A, i.e., the unique matrix satisfying the equations [1]:

$$AXA = A \quad XAX = X \quad (AX)^* = AX \quad (XA)^* = XA. \tag{1.1}$$

Matrix partial orderings defined in $C^{m\times n}$ are considered in this paper. First of them is the star ordering introduced by Drazin [2], which is determined by

$$A \overset{*}{\leq} B \Leftrightarrow A^*A = A^*B \text{ and } AA^* = BA^*, \tag{1.2}$$

and can alternatively be specified as

$$A \overset{*}{\leq} B \Leftrightarrow A^{\dagger}A = A^{\dagger}B \text{ and } AA^{\dagger} = BA^{\dagger}. \tag{1.3}$$

Modifying (1.2), Baksalary and Mitra [3] proposed the left-star and right-star orderings characterized as

$$A* \leq B \Leftrightarrow A^*A = A^*B \text{ (or } A^{\dagger}A = A^{\dagger}B) \text{ and } R(A) \subseteq R(B), \tag{1.4}$$

$$A \leq *B \Leftrightarrow AA^* = BA^* (\text{or } AA^{\dagger} = BA^{\dagger}) \text{ and } R(A^*) \subseteq R(B^*). \tag{1.5}$$

The second partial ordering of interest is minus (rank subtractivity) ordering devised by Hartwig [4] and independently by Nambooripad [5]. It can be characterized as

$$A \overset{-}{\leq} B \Leftrightarrow r(B - A) = r(B) - r(A), \tag{1.6}$$

or

$$A \overset{-}{\leq} B \Leftrightarrow AB^{\dagger}B = A,\ BB^{\dagger}A = A,\ \text{and } AB^{\dagger}A = A. \tag{1.7}$$

From (1.2), (1.4) and (1.5), it is seen that

$$A \overset{*}{\leq} B \Leftrightarrow A^* \overset{*}{\leq} B^*, \tag{1.8}$$

$$A* \leq B \Leftrightarrow A^* \leq *B^*. \tag{1.9}$$

Hartwig and Styan [6] considered the rank subtractivity and Schur complement, and shown that

$$A = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \overset{-}{\leq} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = B \Leftrightarrow C \overset{-}{\leq} E - FH^{-}G,$$

when the conditions $r\begin{pmatrix} F \\ H \end{pmatrix} = r(H) = r\begin{pmatrix} G & H \end{pmatrix}$ are required, and H^- is a inner generalized inverse of H (satisfying $HH^-H = H$).

Recently, the relationships between orderings defined in (1.2)-(1.7) and their powers with the emphasis laid on indicating classes of matrices were considered by several authors [7-9]. The results on matrix partial orderings and reverse order law were considered by Benitez et al. [10]. In this paper, we focus our attention on the partial orderings of block matrices. Special attention is paid to the star ordering of a sum of two matrices and the minus ordering of matrix product. To our knowledge, there is no article yet discussing these partial orderings in the literature.

If $A \prec C$, $B \prec D$, an interesting question is that whether the partitioned matrices $\begin{pmatrix} A & B \end{pmatrix}$ $\left(\text{or} \begin{pmatrix} A \\ B \end{pmatrix}\right)$ and $\begin{pmatrix} C & D \end{pmatrix}$ $\left(\text{or} \begin{pmatrix} C \\ D \end{pmatrix}\right)$ have the same orderings, and the solutions will be given in the following sections. Also, the relations between $A \overset{*}{\leq} C,\ B \overset{*}{\leq} D$ and $A + B \overset{*}{\leq} C + D,\ A \overset{-}{\leq} B$ and $CA \overset{-}{\leq} CB$ are considered.

2 Star partial ordering

In this section, we give some results on the star partial orderings of block matrices.

Theorem 1 *Let* $A, C \in \mathbb{C}^{m\times n}$ *and* $B, D \in \mathbb{C}^{m\times k}$ *be star-ordered as* $A \overset{*}{\leq} C,\ B \overset{*}{\leq} D$. *If* $R(A) = R(B)$, *then* $\begin{pmatrix} A & B \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} C & D \end{pmatrix}$.

Proof. On account of (1.2) and (1.3), since $A \overset{*}{\leq} C,\ B \overset{*}{\leq} D$ and $R(A) = R(B)$, so

$$\begin{aligned}
\begin{pmatrix} A & B \end{pmatrix}^* \begin{pmatrix} A & B \end{pmatrix} &= \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix} \\
&= \begin{pmatrix} A^*C & A^*BB^{\dagger}D \\ B^*AA^{\dagger}C & B^*D \end{pmatrix} \\
&= \begin{pmatrix} A^*C & (BB^{\dagger}A)^*D \\ (AA^{\dagger}B)^*C & B^*D \end{pmatrix} \\
&= \begin{pmatrix} A^*C & A^*D \\ B^*C & B^*D \end{pmatrix} \\
&= \begin{pmatrix} A & B \end{pmatrix}^* \begin{pmatrix} C & D \end{pmatrix},
\end{aligned}$$

and

$$\begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} A & B \end{pmatrix}^* = AA^* + BB^* = CA^* + DB^* = \begin{pmatrix} C & D \end{pmatrix}\begin{pmatrix} A & B \end{pmatrix}^*,$$

which according to (1.2) show that $\begin{pmatrix} A & B \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} C & D \end{pmatrix}$. □

For the left-star orderings, we have a similar result.

Theorem 2 *Let $A, C \in C^{m\times n}$ and $B, D \in C^{m\times k}$ be star-ordered as $A_* \leq C$, $B_* \leq D$. If $R(A) = R(B)$, then $\begin{pmatrix} A & B \end{pmatrix} * \leq \begin{pmatrix} C & D \end{pmatrix}$.*

Proof. In view of (1.4), according to the assumptions, we have

$$\begin{pmatrix} A & B \end{pmatrix}^* \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}^* \begin{pmatrix} C & D \end{pmatrix}.$$

On the other hand, on account of (1.4), from the conditions $A_* \leq C$ and $B_* \leq D$, we have $R(A) \subseteq R(C)$ and $R(B) \subseteq R(D)$, which imply that $R\begin{pmatrix} A & B \end{pmatrix} \subseteq R\begin{pmatrix} C & D \end{pmatrix}$. According to (1.4), we have $\begin{pmatrix} A & B \end{pmatrix} * \leq \begin{pmatrix} C & D \end{pmatrix}$. □

Theorem 3 *Let $A, C \in C^{m\times n}$ and $B, D \in C^{m\times k}$ be star-ordered as $\begin{pmatrix} A & B \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} C & D \end{pmatrix}$. If $A \overset{*}{\leq} C$ (or $B \overset{*}{\leq} D$), then $B \overset{*}{\leq} D$ (or $A \overset{*}{\leq} C$). Moreover, the condition $A \overset{*}{\leq} C$ (or $B \overset{*}{\leq} D$) can be replaced by $A \leq {}_*C$ (or $B \leq {}_*D$).*

Proof. The proof is trivial and therefore omitted.

Since $A \overset{*}{\leq} B$ and $A \leq {}_*B$ are equivalent to $A^* \overset{*}{\leq} B^*$ and $A^* * \leq B^*$, respectively, therefore, for the rowwise partitioned matrix we have the similar results.

Corollary 1 *Let $A, C \in C^{m\times n}$ and $B, D \in C^{k\times n}$ be star-ordered as $A \overset{*}{\leq} C$, $B \overset{*}{\leq} D$. If $R(A^*) = R(B^*)$, then $\begin{pmatrix} A \\ B \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} C \\ D \end{pmatrix}$.*

Corollary 2 *Let $A, C \in C^{m\times n}$ and $B, D \in C^{k\times n}$ be star-ordered as $A \leq {}_*C$, $B \leq {}_*D$. If $R(A^*) = R(B^*)$, then $\begin{pmatrix} A \\ B \end{pmatrix} \leq * \begin{pmatrix} C \\ D \end{pmatrix}$.*

Corollary 3 *Let $A, C \in C^{m\times n}$ and $B, D \in C^{k\times n}$ be star-ordered as $\begin{pmatrix} A \\ B \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} C \\ D \end{pmatrix}$. If $A_* \leq C$ (or $B_* \leq D$), then $B \overset{*}{\leq} D$ (or $A \overset{*}{\leq} C$).*

Specially, we present the following results without proofs.

Theorem 4 *Let $A, B \in C^{m\times n}$, $C \in C^{m\times k}$ and $D \in C^{k\times n}$. Then*

(1) *If $A \overset{*}{\leq} B$ and $R(C) \subseteq R(A)$, then $\begin{pmatrix} A & C \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} B & C \end{pmatrix}$ and $\begin{pmatrix} C & A \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} C & B \end{pmatrix}$. Moreover, both $\begin{pmatrix} A & C \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} B & C \end{pmatrix}$ and $\begin{pmatrix} C & A \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} C & B \end{pmatrix}$ imply $A \overset{*}{\leq} B$, even though $R(C) \not\subseteq R(A)$.*

(2) *If $A_* \leq B$ and $R(C) \subseteq R(A)$, then $\begin{pmatrix} A & C \end{pmatrix} * \leq \begin{pmatrix} B & C \end{pmatrix}$ and $\begin{pmatrix} C & A \end{pmatrix} * \leq \begin{pmatrix} C & B \end{pmatrix}$.*

(3) *If $A \overset{*}{\leq} B$ and $R(D^*) \subseteq R(A^*)$, then $\begin{pmatrix} A \\ D \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} B \\ D \end{pmatrix}$ and $\begin{pmatrix} D \\ A \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} D \\ B \end{pmatrix}$. Moreover, both $\begin{pmatrix} A \\ D \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} B \\ D \end{pmatrix}$ and $\begin{pmatrix} D \\ A \end{pmatrix} \overset{*}{\leq} \begin{pmatrix} D \\ B \end{pmatrix}$ imply $A \overset{*}{\leq} B$, even though $R(D^*) \not\subseteq R(A^*)$.*

(4) *If $A \leq {}_*B$ and $R(D^*) \subseteq R(A^*)$, then $\begin{pmatrix} A \\ D \end{pmatrix} \leq * \begin{pmatrix} B \\ D \end{pmatrix}$ and $\begin{pmatrix} D \\ A \end{pmatrix} \leq * \begin{pmatrix} D \\ B \end{pmatrix}$.*

Next, we use some examples to illustrate the above results. The case (1) shows that the condition $R(C) \subseteq R(A)$ is sufficient but not necessary. For example, we take the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to verify that $A \overset{*}{\leq} B$. For $C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $R(C) \not\subset R(A)$, and a simple computation shows that $(A\ C)^* \ (A\ C) \neq (A\ C)^* \ (B\ C)$. For $C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $R(C) \subset R(A)$, and we have $(A\ C) \overset{*}{\leq} (B\ C)$ as well as $(C\ A) \overset{*}{\leq} (C\ B)$. On the other hand, we take the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We can verify that $(A\ C) \overset{*}{\leq} (B\ C)$. Although $R(C) \not\subset R(A)$, we have $A \overset{*}{\leq} B$.

Mitra [11] pointed out that the star ordering has the property that if $C \overset{*}{\leq} A$ and $C \overset{*}{\leq} B$, then $2C \overset{*}{\leq} A + B$. Moreover, it is well known that the Löwner ordering has the property that for Hermitian nonnegative definite matrices A, B, C and D, if $A \leq_L C$ and $B \leq_L D$, then $A + B \leq_L C + D$. A direct consideration is to see whether the star ordering has the same property. And the solution is given in the following.

Theorem 5 *Let A, B, C, D* $\in \mathbb{C}^{m\times n}$, *and* $A \overset{*}{\leq} C,\ B \overset{*}{\leq} D$. *If* $R(A) = R(B)$ *and* $R(A^*) = R(B^*)$, *then* $A + B \overset{*}{\leq} C + D$.

Proof. The proof is trivial and therefore omitted. □

3 Minus partial ordering

In this section, we present some results on the minus orderings of the matrix product and block matrices. In our development, we will use the following preliminary results for our further discussion.

Lemma 1 [12]*Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$. *Then*

$$r(AB) = r(B) - \dim\left(R(B) \cap N(A)\right).$$

Baksalary et al. [13] established a formula for the Moore-Penrose inverse of a columnwise partitioned matrix. Here, we state it as given below.

Lemma 2 *Let* $A \in \mathbb{C}^{m\times n}$ *and be partioned as* $A = (A_1\ A_2)$. *Then the following statements are equivalent:*

(1) $A^\dagger = \begin{pmatrix} A_1^\dagger - A_1^\dagger A_2 (Q_1 A_2)^\dagger \\ A_2^\dagger - A_2^\dagger A_1 (Q_2 A_1)^\dagger \end{pmatrix}$,

(2) $R(A_1) \cap R(A_2) = \{0\}$,

where $Q_i = I_m - A_i A_i^\dagger,\ i = 1, 2$.

Lemma 3 [14]*Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, *such that* $R(B) \subseteq R(A)$. *Then*

$$(A\ B)^\dagger = \begin{pmatrix} A^\dagger - A^\dagger B M^{-1} B^* (A^\dagger)^* A^\dagger \\ M^{-1} B^* (A^\dagger)^* A^\dagger \end{pmatrix},$$

where $M = I + B^*(A^\dagger)^* A^\dagger B$.

It is easy to verify that, for a full column rank matrix C with proper size, the minus orders $A \overset{-}{\leq} B$ and $CA \overset{-}{\leq} CB$ are equivalent, but if C is not a full column rank matrix, this implication may be not true. The following theorem shows that when the implication is true.

Theorem 6 *Let A, $B \in C^{m\times n}$, $C \in C^{k\times m}$. Then any two of the following statements imply the third:*

(1) $A \overset{-}{\leq} B$,

(2) $CA \overset{-}{\leq} CB$,

(3) $\dim (R(B - A) \cap N(C)) = \dim (R(B) \cap N(C)) - \dim (R(A) \cap N(C))$.

Proof. Applying Lemma 1, we have

$$\begin{aligned} r(CB - CA) = r(C(B - A)) &= r(B - A) - \dim (R(B - A) \cap N(C)), \\ r(CB) &= r(B) - \dim (R(B) \cap N(C)), \\ r(CA) &= r(A) - \dim (R(A) \cap N(C)). \end{aligned}$$

Hence,

$$\begin{aligned} &(r(B - A) - r(B) + r(A)) - (r(CB - CA) - r(CB) + r(CA)) \\ &\quad = \dim (R(B - A) \cap N(C)) + \dim (R(A) \cap N(C)) - \dim (R(B) \cap N(C)). \end{aligned}$$

On account of (1.6) this theorem can be easily obtained. □

Similarly, we can prove the following results.

Corollary 4 *Let A, $B \in C^{m\times n}$, $C \in C^{n\times k}$. Then any two of the following statements imply the third:*

(1) $A \overset{-}{\leq} B$,

(2) $AC \overset{-}{\leq} BC$,

(3) $\dim (R(B^* - A^*) \cap N(C^*)) = \dim (R(B^*) \cap N(C^*)) - \dim (R(A^*) \cap N(C^*))$.

Summarizing Theorem 6, Corollary 4 and $N(C) = R^{\perp}(C^*)$, the following results are obtained immediately.

Corollary 5 *Let A, $B \in C^{m\times n}$. Then the following statements are equivalent:*

(1) $A \overset{-}{\leq} B$,

(2) $B^{\dagger}A \overset{-}{\leq} B^{\dagger}B$ *and* $R(A) \subseteq R(B)$,

(3) $AB^{\dagger} \overset{-}{\leq} BB^{\dagger}$ *and* $R(A^*) \subseteq R(B^*)$.

Furthermore,

$$\begin{aligned} &AB^{\dagger} \overset{-}{\leq} BB^{\dagger} \text{ and } R(A) \subseteq R(B) \Leftrightarrow B^{\dagger}AB^{\dagger} \overset{-}{\leq} B^{\dagger} \text{ and } R(A) \subseteq R(B), \\ &B^{\dagger}A \overset{-}{\leq} B^{\dagger}B \text{ and } R(A^*) \subseteq R(B^*) \Leftrightarrow B^{\dagger}AB^{\dagger} \overset{-}{\leq} B^{\dagger} \text{ and } R(A^*) \subseteq R(B^*), \end{aligned}$$

and

$$A \overset{-}{\leq} B \Leftrightarrow B^{\dagger}AB^{\dagger} \overset{-}{\leq} B^{\dagger},\ R(A) \subseteq R(B) \text{ and } R(A^*) \subseteq R(B^*).$$

In the previous section, we study the star ordering of block matrix. A similar consequence on the minus ordering is established as below.

Theorem 7 *Let A, $C \in C^{m\times n}$, and B, $D \in C^{m\times k}$ be minus ordered as $A \overset{-}{\leq} C$, $B \overset{-}{\leq} D$. If $R(C) \cap R(D) = \{0\}$, then $\begin{pmatrix} A & B \end{pmatrix} \overset{-}{\leq} \begin{pmatrix} C & D \end{pmatrix}$.*

Proof. From $A \overset{-}{\leq} C$ and $B \overset{-}{\leq} D$, in view of (1.7), it follows that

$$AC^{\dagger}C = A,\ CC^{\dagger}A = A\ (or\ R(A) \subseteq R(C)),\ \ AC^{\dagger}A = A; \tag{3.1}$$

and

$$BD^{\dagger}D = B,\ DD^{\dagger}B = B\ (or\ R(B) \subseteq R(D)),\quad BD^{\dagger}B = B; \tag{3.2}$$

The conditions of the middle part of (3.1) and (3.2) show that

$$R\left(A\ B\right) \subseteq R\left(C\ D\right) or \left(C\ D\right)\ \left(C\ D\right)^{\dagger}\left(A\ B\right) = \left(A\ B\right). \tag{3.3}$$

According to Lemma 2 and the assumption $R(C) \cap R(D) = \{0\}$, we have

$$\left(C\ D\right)^{\dagger} = \begin{pmatrix} C^{\dagger} - C^{\dagger}D(Q_C D)^{\dagger} \\ D^{\dagger} - D^{\dagger}C(Q_D C)^{\dagger} \end{pmatrix},$$

where $Q_C = I_m - CC^{\dagger}$ and $QD = I_m - DD^{\dagger}$.

From (3.1) and (3.2), we can verify the following equalities

$$\left(A\ B\right)\left(C\ D\right)^{\dagger}\left(C\ D\right) = \left(A\ B\right), \tag{3.4}$$

$$\left(A\ B\right)\left(C\ D\right)^{\dagger}\left(A\ B\right) = \left(A\ B\right). \tag{3.5}$$

On account of (1.7), combining (3.3), (3.4) and (3.5) shows that $\left(A\ B\right) \overset{-}{\leq} \left(C\ D\right)$ □

Note that, $A\overset{-}{\leq}C$ and $B\overset{-}{\leq}D$ lead to $R(A) \subseteq R(C)$ and $R(B) \subseteq R(D)$, hence, the condition $R(C) \cap R(D) = \{0\}$ implies that $R(A) \cap R(B) = \{0\}$. Therefore, this theorem can also be proved by Definition (1.6).

Since

$$\begin{aligned} r\left[\left(C\ D\right) - \left(A\ B\right)\right] &= r\left(C - A\ D - B\ \right) \\ &= r(C - A) + r(D - B) \\ &= r(C) + r(D) - r(A) - r(B) \\ &= r\left(C\ D\right) - r\left(A\ B\right)\ , \end{aligned}$$

hence, $\left(A\ B\right) \overset{-}{\leq} \left(C\ D\right)$.

The following statement can be deduced from Lemma 3.

Theorem 8 *Let A, C ∈* $C^{m\times n}$ *be minus ordered as* $A\overset{-}{\leq}C$, *and B, D ∈* $C^{m\times k}$. *If* $R(D) \subseteq R(C)$, *then* $\left(A\ B\right) \overset{-}{\leq} \left(C\ D\right)$ *if and only if* $B = AC^{\dagger}D$.

Corollary 6 *Let A, C ∈* $C^{m\times n}$ *be minus ordered as,* $A\overset{-}{\leq}C$, *and B, D ∈* $C^{k\times n}$.

(1) *If* $B\overset{-}{\leq}D$ *and* $R(C^*) \cap R(D^*) = \{0\}$, *then* $\begin{pmatrix} A \\ B \end{pmatrix} \overset{-}{\leq} \begin{pmatrix} C \\ D \end{pmatrix}$.

(2) *If* $R(D^*) \subseteq R(C^*)$, *then* $\begin{pmatrix} A \\ B \end{pmatrix} \overset{-}{\leq} \begin{pmatrix} C \\ D \end{pmatrix}$ *if and only if* $B = DC^{\dagger}A$.

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Authors' contributions
XL carried out the main part of this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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