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The compactness of the sum of weighted composition operators on the ball algebra

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Abstract

In this paper, we investigate the compactness of the sum of weighted composition operators on the unit ball algebra, and give the characterization of compact differences of two weighted composition operators on the ball algebra. The connectness of the topological space consisting of non-zero weighted composition operators on the unit ball algebra is also studied.

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1. Introduction

Let $H(B_N)$ be the class of all holomorphic functions on B_N and $S(B_N)$, the collection of all holomorphic self mappings of B_N, where B_N is the unit ball in the N-dimensional complex space $\mathbb{C}^N$. The closure of B_N will be written as $\overline{B_N}$. Denote by $A = A(B_N)$, the unit ball algebra of all continuous functions on $\overline{B_N}$ that are holomorphic on B_N. Then A is the Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in B_N\}.$$

And let $H^\infty = H^\infty(B_N)$ be the set of all bounded holomorphic functions on B_N. We denote by $B(A)$ ($B(H^\infty)$ resp.) the unit ball of A ($H^\infty(B_N)$ resp.).

For $u, \varphi \in A$ with $\varphi \in S(B_N)$, recall that the *composition operator* C_φ induced by φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z));$$

the *multiplication operator* induced by u is defined by

$$M_u f(z) = u(z)f(z);$$

and the *weighted composition operator* uC_φ induced by φ and u is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z))$$

for $z \in B_N$ and $f \in H(B_N)$. If let $u \equiv 1$, then $uC_\varphi = C_\varphi$; if let $\varphi = id$, then $uC_\varphi = M_u$. It is clear that these operators are linear and uC_φ is bounded on A. For some results in this topic see, for example, [1-6], and so on.

Let X be a Banach space of analytic functions, we write $C(X)$, for the space of composition operators on X under the operator norm topology.

Moorhouse [7] considered the compactness of finite sum of composition operators on weighted Bergman spaces in the disk and gave a partial answer to the component structure of $C(A_\alpha^2)$. Kriete and Moorhouse [8] continued to study the finite linear combination of composition operators acting on the Hardy space or weighted Bergman spaces on the disk. Hosokawa and Ohno, [9], and [10], discussed the topological structures of the sets of composition operators and gave a characterization of compact difference on Bloch space in the unit disk. Fang and Zhou [11] then gave a characterization of compact difference between the Bloch space and the set of all bounded analytic functions on the unit polydisk. Fang and Zhou [12] also studied the compact differences of composition operators on the space of bounded analytic functions in polydisk. Hosokawa and co-workers [13] studied the topological components of the topological space of weighted composition operators on the space of bounded analytic functions on the open unit disk in the uniform operator topology. These results were extended to the setting of $H^\infty(B_N)$ by Toews [14], and independently by Gorkin and co-workers [15], and the setting of $H^\infty(D^N)$ by Fang and Zhou [12], where D^N is the unit polydisk. Bonet and co-workers [16] discussed the same problem for the composition operator on the weighted Banach spaces of holomorphic functions in the unit disk, which was also extended to the unit polydisc by Wolf in [17]. The case of weighted composition operators on the above spaces was treated by Lindström and Wolf [18]. Ohno [19] studied the differences of weighted composition operators on the disk algebra.

Building on these foundations, we study the compactness of the sum of certain class of weighted composition operators on the unit ball algebra, and give the characterization of compact differences of two weighted composition operators on the ball algebra. The connectness of the topological space consisted of non-zero weighted composition operators on the unit ball algebra is also studied.

2. The sum of weighted composition operators on *A*

In this section, we will give necessary and sufficient conditions for the sum of several weighted composition operators to be compact on A.

Let T be a bounded linear operator on a Banach space. Recall that T is said to be *compact* if T maps every bounded set into relatively compact one. And if T is a compact operator, T must map every weakly convergent sequence into norm convergent one, and a linear operator with that property is called to be *completely continuous*. In general, a completely continuous operator may not be always compact.

If $\varphi \in S(B_N) \cap A$, denote by $\Gamma_\varphi = \{\zeta \in \partial B_N : |\varphi(\zeta)| = 1\}$. For $z, w \in B_N$, the *involution automorphism* between z and w is given by

$$\Phi_w(z) = \frac{w - P_w(z) - s_w Q_w(z)}{1 - < z, w >},$$

where $P_w(z) = \frac{<z,w>}{<w,w>} w$, $Q_w(z) = z - \frac{<z,w>}{<w,w>} w$, and $s_w = \sqrt{1 - |w|^2}$. The *induced distance* between z and w is defined as

$$d_\infty(z, w) = \sup_{f \in B(H^\infty)} |f(z) - f(w)|.$$

Pseudo-hyperbolic metric is defined by

$$\rho(z,w) = \sup\{|f(z)| : \|f\|_\infty = 1, f(w) = 0\},$$

and a theorem of Bear [20] gives that

$$d_\infty(z,w) = \frac{2 - 2\sqrt{1-\rho(z,w)^2}}{\rho(z,w)}.$$

Since the pseudo-hyperbolic metric $\rho(z, w) \le 1$ we have that

$$\begin{aligned} & 1-\rho(z,w)^2 \le \sqrt{1-\rho(z,w)^2} \\ \Rightarrow\ & 1-\sqrt{1-\rho(z,w)^2} \le \rho(z,w)^2 \\ \Rightarrow\ & d_\infty(z,w) \le 2\rho(z,w). \end{aligned}$$

Toews [14] proved the following lemma:

Lemma 2.1. *For any $z \in B_N$, we have*

(a): $\rho(z; w) = |\Phi_w(z)|$ *for any $w \in B_N$, and*
(b): $\{z : \rho(z, w) < \lambda\} = \Phi_w(\lambda B_N)$.

For $\zeta \in \partial B_N$ and $\alpha > 1$, we define a *Koranyi approach* region by

$$\Delta(\zeta,\alpha) = \left\{ z \in B_N : |1 - \langle z, \zeta\rangle| < \frac{\alpha}{2}\left(1-|z|^2\right)\right\}.$$

Since $U\Delta(\zeta, \alpha) = \Delta(U\zeta, \alpha)$ for any unitary transformation U in $\mathbb{C}^N$, we can concentrate on understanding $\Delta(e_1, \alpha)$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^N$. Next lemma is crucial.

Lemma 2.2. *For a real number $1 < \alpha < 2$ and $e_1 = (1, 0, ..., 0)$ in ∂B_N, if $\{w_n\}$ is an arbitrary sequence in the Koranyi approach region $\Delta(e_1, \alpha)$, then*

$$\left|\frac{|w_n|^2 - \langle e_1, w_n\rangle}{1-\langle e_1, w_n\rangle}\right| > \frac{2}{\alpha} - 1 > 0.$$

On the other hand, for every $\alpha > 2$, we can find a sequence $\{w_n\} \subset \Delta(e_1, \alpha)$ and $w_n \to e_1$ as $n \to \infty$ satisfying

$$\lim_{n\to\infty}\left|\frac{|w_n|^2 - \langle e_1, w_n\rangle}{1-\langle e_1, w_n\rangle}\right| = 0.$$

Proof. First, let $1 < \alpha < 2$. We have that

$$\begin{aligned} \left|\frac{|w_n|^2 - \langle e_1, w_n\rangle}{1-\langle e_1, w_n\rangle}\right| &> \frac{2}{\alpha}\frac{\left||w_n|^2 - \langle e_1, w_n\rangle\right|}{1-|w_n|^2} \\ &\ge \frac{2}{\alpha}\left(\frac{\left|1-|w_n|^2\right| - |1-\langle e_1, w_n\rangle|}{1-|w_n|^2}\right) \\ &> \frac{2}{\alpha}\left(1 - \frac{\frac{\alpha}{2}(1-|w_n|^2)}{1-|w_n|^2}\right) \\ &= \frac{2}{\alpha} - 1 > 0. \end{aligned}$$

Suppose $\alpha > 2$. Let $w_n = (r_n, w'_n)$, where r_n is real with $r_n \to 1^-$ as $n \to \infty$ and $|w'_n| = \sqrt{1 - r_n}$. It is clear that for sufficient large n

$$|1 - \langle e_1, w_n \rangle| = 1 - r_n < \frac{\alpha}{2}\left(r_n - r_n^2\right)$$
$$= \frac{\alpha}{2}\left(1 - r_n^2 - 1 + r_n\right) = \frac{\alpha}{2}\left(1 - |w_n|^2\right).$$

This means that $\{w_n\} \subset \Delta(e_1, \alpha)$ for $\alpha > 2$. Now we compute

$$\lim_{n\to\infty}\left|\frac{|w_n|^2 - \langle e_1, w_n\rangle}{1 - \langle e_1, w_n\rangle}\right| = \lim_{n\to\infty}\frac{r_n^2 + 1 - r_n - r_n}{1 - r_n} = \lim_{n\to\infty}(1 - r_n) = 0.$$

□

For $\varphi_1, \varphi_2, ..., \varphi_p \in S(B_N) \cap A$, we simplify the notations Γ_{φ_j} and $C_{\varphi_j}$by Γ_j and C_j respectively. It is obvious that each weighted composition operator is a bounded operator on the ball algebra.

For any $\zeta \in \bigcup_{j=1}^{p} \Gamma_j$, denote by $I(\zeta) = \{k \in \{1, 2, ..., p\}: \zeta \in \Gamma_k\}$. If $k, j \in I(\zeta)$, define $k \sim j$ whenever $\rho(\varphi_j(z_n), \varphi_k(z_n)) \to 0$ holds for every sequence $\{z_n\} \subset B_N$ converging to ζ. It is clear that the relation $\sim$ is an equivalent relation on $I(\zeta)$. Denote by $I(\zeta)/\sim$ as the quotient set, and $I_s(\zeta)$'s, which is called *index sets* of ζ, as the elements of $I(\zeta)/\sim$. The lower index $s \in \{1, ..., p\}$ is given by the smallest number in $Is(\zeta)$.

Next, we are going to study the compactness of $\sum_{j=1}^{p} u_j C_j$ acting on the ball algebra. In fact, it is difficult to characterize the compactness of the sum of several arbitrary u_jC_j's. Here, we just consider the sum whose symbols are in a certain class of self mappings of B_N.

Our admissible symbols of self mappings of B_N are constrained by the following condition: Let $\varphi \in S(B_N) \cap A$ and Γ_φ as defined above. For every $\zeta \in \Gamma_\varphi$ and any arbitrary small $\varepsilon > 0$, if there exists a Koranyi region $\Delta(\varphi(\zeta), \alpha)$ with $1 < \alpha < 2$ such that

(*): $(N(\varphi(\zeta), \varepsilon) \cap \Delta(\varphi(\zeta), \alpha)) \cap \varphi(B_N) \neq \emptyset$

where $N(\varphi(\zeta), \varepsilon)$ is the neighborhood of $\varphi(\zeta)$ with radii ε. We collect all self mappings of B_N satisfied **(*)** and denote it by A^*.

Remark 2.3. If $\varphi \in S(B_N) \cap A$ and the closure of $\varphi(B_N)$ has no boundary contact with the unit sphere, we say that φ satisfies condition **(*)** trivially, that is to say $\varphi \in A^*$.

The following theorem is our main theorem.

Theorem 2.4. *Let $u_j \in A$ and $\varphi_j \in A^*$ for $j = 1, ..., p$. Then $\sum_{j=1}^{p} u_j C_j$is compact on A if and only if and only if $\sum_{k \in I_s(\zeta)} u_k(\zeta) = 0$for every $\zeta \in \bigcup_{j=1}^{p} \Gamma_j$and every index set $I_s(\zeta)$ of ζ.*

Proof. To prove the necessity, let ζ be an arbitrary point in $\bigcup_{j=1}^{p} \Gamma_j$, $I(\zeta)$ and $I_s(\zeta)$'s are defined as above. We can find out a sequence $\{w_n\} \subset B_N$ satisfying the following properties: as $n \to \infty$,

(1): $w_n \to \zeta$;
(2): $\varphi_k(w_n) \to \varphi_k(\zeta)$ with $k \in I_s(\zeta)$ and $k \neq s$;
(3): $\varphi_l(w_n) \to \varphi_l(\zeta)$ with $l \notin I_s(\zeta)$;

(4): $\langle \Phi_{\varphi_s(w_n)}(\varphi_s(\zeta)), \varphi_s(w_n)\rangle \to \lambda \neq 0.$

The item (4) holds since every $\varphi_j \in A^*$ and Lemma 2.2 guarantees the limit is non zero. From the item (3), we can choose the subsequences (also denoted by $\{w_n\}$) satisfying

(5): $\langle \Phi_{\varphi_s(w_n)}(\varphi_l(w_n)), \varphi_s(w_n)\rangle \to \sigma_{ls} \neq 0.$

For such a sequence $\{w_n\}$, define the functions

$$\begin{aligned} f_n(z) =& (\langle \Phi_{\varphi_s(w_n)}(z), \varphi_s(w_n)\rangle - 1)(\langle \Phi_{\varphi_s(w_n)}(z), \varphi_s(w_n)\rangle - \lambda) \\ &\cdot \prod_{l \notin I_s(\zeta)} (\langle \Phi_{\varphi_s(w_n)}(z), \varphi_s(w_n)\rangle - \sigma_{ls}). \end{aligned}$$

Then $f_n \in A$ and $||f_n||_\infty \leq 2^{p+1}$. We claim that $f_n(z) \to 0$ for all $z \in \overline{B_N}$ as $n \to \infty$. Indeed, if $z = \varphi_s(\zeta)$, the item (4) guarantees $f_n(\varphi_s(\zeta)) \to 0$; and if $z \neq \varphi_s(\zeta)$, we have

$$\begin{aligned} &\lim_{n\to\infty} \langle \Phi_{\varphi_s(w_n)}(z), \varphi_s(w_n)\rangle \\ &= \lim_{n\to\infty} \frac{\langle \varphi_s(w_n), \varphi_s(w_n)\rangle - \frac{\langle z, \varphi_s(w_n)\rangle}{\langle \varphi_s(w_n), \varphi_s(w_n)\rangle}\langle \varphi_s(w_n), \varphi_s(w_n)\rangle}{1 - \langle z, \varphi_s(w_n)\rangle} \\ &= \lim_{n\to\infty} \frac{|\varphi_s(w_n)|^2 - \langle z, \varphi_s(w_n)\rangle}{1 - \langle z, \varphi_s(w_n)\rangle} \\ &= 1. \end{aligned}$$

So f_n converges weakly to 0 in A, thus $\| \sum_{j=1}^p u_j C_j f_n\|_\infty \to 0$ as $n \to \infty$.

On the other hand,

$$\begin{aligned} &\left\| \sum_{j=1}^p u_j C_j f_n \right\|_\infty \geq \left| \sum_{j=1}^p u_j(w_n) f_n(\varphi_j(w_n)) \right| \\ &\geq \left| \sum_{k\in I_s(\zeta)} u_k(w_n) \right| |\lambda| \prod_{l\notin I_s(\zeta)} \sigma_{ls} - \sum_{l\notin I_s(\zeta)} |u_l(w_n)||f_n(\varphi_l(w_n))| \\ &\quad - \sum_{k\in I_s(\zeta)\setminus\{s\}} |u_k(w_n)||f_n(\varphi_s(w_n)) - f_n(\varphi_k(w_n))| \\ &\geq \left| \sum_{k\in I_s(\zeta)} u_k(w_n) \right| |\lambda| \prod_{l\notin I_s(\zeta)} \sigma_{ls} - \sum_{l\notin I_s(\zeta)} |u_l(w_n)||f_n(\varphi_l(w_n))| \\ &\quad - 2 \sum_{k\in I_s(\zeta)\setminus\{s\}} \| u_k\|_\infty \| f_n\|_\infty \rho(\varphi_s(w_n), \varphi_k(w_n)). \end{aligned}$$

We have that $\rho(\varphi_s(w_n), \varphi_k(w_n)) \to 0$ for every $k \in I_s(\zeta) \setminus \{s\}$. And from the item (5), we know that $|f_n(\varphi_l(w_n))| \to 0$ for every $l \notin I_s(\zeta)$. Hence we have

$$\sum_{k\in I_s(\zeta)} u_k(\zeta) = 0.$$

To verify the sufficiency, we cannot use the "weak convergence theorem" (Proposition 3.11 of [21]), because the unit ball algebra is not closed in the compact open topology. Here, we use the definition of compact operator to illustrate that the sum of weighted composition operators is compact. Let $f_n \in A$ and $||f_n||_\infty = 1$. By the normal family argument, there exists a subsequence $\{f_{n_k}\}$of $\{f_n\}$ and a function g analytic on B_N such that f_{n_k} converges to g uniformly on compact subsets of B_N. Here, we have

$$\sup_{z\in B_N}|g(z)| \le 1.$$

Now define a function G on $\overline{B_N}$ by setting

$$G(z) = \begin{cases} \sum_{l\notin I(z)} u_l C_l g(z) & z \in \bigcup_{j=1}^p \Gamma_j; \\ \sum_{j=1}^p u_j C_j g(z) & \text{otherwise.} \end{cases}$$

Next, we just need to show that G is continuous on $\overline{B_N}$, and $\sum_{j=1}^p u_j C_j f_{n_k}$ converges uniformly to G on $\overline{B_N}$. In the following proof, we simplify the subsequence f_{n_k} by f_n.

We first prove that G is continuous on $\overline{B_N}$. Indeed, it is obvious that G is continuous on $\overline{B_N}\setminus \bigcup_{j=1}^p \Gamma_j$.

For $\zeta \in \bigcup_{j=1}^p \Gamma_j$, let $I_s(\zeta)$'s be its index sets and $I(\zeta) = \bigcup_s I_s(\zeta)$. Suppose that $\{z_n\}$ be a sequence in B_N converging to ζ such that $\varphi_j(z_n) \to \varphi_j(\zeta)$ $j = 1, ..., p$. So for each $I_s(\zeta)$.

- $\rho(\varphi_k(z_n), \varphi_s(z_n)) \to 0$ where $k \in I_s(\zeta)$, and
- $\rho(\varphi_l(z_n), \varphi_s(z_n)) \nrightarrow 0$ where $l \notin I(\zeta)$.

Then, we compute

$$\begin{aligned} &\left|\sum_{j=1}^p u_j C_j g(z_n) - \sum_{l\notin\in I(\zeta)} u_l C_l g(\zeta)\right| \\ &\le \sum_s \left(\| g\|_\infty \left|\sum_{k\in I_s(\zeta)} u_k(z_n)\right| + 2\sum_{k\in I_s(\zeta)\setminus\{s\}} \| u_k\|_\infty \| g\|_\infty \rho(\varphi_s(z_n), \varphi_k(z_n))\right) \\ &\quad + \left|\sum_{l\notin I(\zeta)} (u_l C_l g(z_n) - u_l C_l g(\zeta))\right|. \end{aligned}$$

So, we have $\lim_{z_n\to\zeta} \sum_{j=1}^p u_j C_j g(z_n) = G(\zeta)$, and this holds for every $\zeta \in \bigcup_{j=1}^p \Gamma_j$. Thus G is continuous on $\overline{B_N}$.

Next, we shall show that $\sum_{j=1}^p u_j C_j f_n$ converges uniformly to G on $\overline{B_N}$. Suppose not, and we may assume that, for $\varepsilon > 0$, $\| \sum_{j=1}^p u_j C_j f_n - G\|_\infty > \varepsilon > 0$. Then there exists a sequence $\{z_n\} \subset B_N$ such that

$$|\sum_{j=1}^p u_j C_j f_n(z_n) - G(z_n)| > \varepsilon \text{ for every } n. \tag{2.1}$$

This implies that $\max_j\{|\varphi_j(z_n)|\} \to 1$ as $n \to \infty$. Here we may assume that $z_n \to \zeta \in \partial B_N$, and define $I(\zeta)$, $I_s(\zeta)$ as before. Similarly as the analysis above, we have that

- $\rho(\varphi_k(z_n), \varphi_s(z_n)) \to 0$ where $k \in I_s(\zeta)$,
- $\rho(\varphi_l(z_n); \varphi_s(z_n)) \nrightarrow 0$ where $l \notin I_s(\zeta)$.

Hence

$$
\begin{aligned}
&|\sum_{j=1}^{p}(u_jC_j)f_n(z_n) - G(z_n)| \\
&\leq \sum_{s}\left(\| f_n\|_\infty \left| \sum_{k\in I_s(\zeta)} u_k(z_n)\right| + 2 \sum_{k\in I_s(\zeta)\setminus\{s\}} \| u_k\|_\infty \ \| f_n\|_\infty \rho(\varphi_s(z_n), \varphi_k(z_n))\right) \\
&+ \left| \sum_{l\notin I(\zeta)} u_l(z_n)[f_n(\varphi_l(z_n)) - g(\varphi_l(z_n))]\right| .
\end{aligned}
$$

When $l \notin I(\zeta)$, note that the limit of $|\varphi_l(z_n)|$ is strictly less than 1, and f_n converges to g on compact subsets of B_N, thus $f_n(\varphi_l(z_n))$ - $g(\varphi_l(z_n)) \to 0$. By the preliminary conditions we know that $|\sum_{k\in I_s(\zeta)} u_k(\zeta)| = 0$ for each index sets $I_s(\zeta)$. Together with the above two items, we have $\sum_{j=1}^{p} u_jC_jf_n(z_n) - G(z_n) \to 0$. This fact contradicts (2.1). Thus, we have proved the sufficient condition. □

As a corollary of the sum theorem, we state the characterization of the compact differences of two weighted composition operators on A.

Theorem 2.5 *Let u, $v \in A$ and φ, $\psi \in A^*$. Then uC_φ - vC_ψ is compact on A if and only if the following three conditions hold:*

(a): *If $\zeta \in \Gamma_\varphi$ and $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) \neq 0$, then $u(\zeta) = 0$;*
(b): *If $\zeta \in \Gamma_\psi$ and $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) \neq 0$, then $v(\zeta) = 0$;*
(c): *If $\zeta \in \Gamma_\varphi \cap \Gamma_\psi$, then $u(\zeta) = v(\zeta)$.*

Corollary 2.6 *Let $u \in A$ and $\varphi \in A^*$, the associated weighted composition operator uC_φ act compactly on A if and only if $u(\zeta) = 0$ for every $\zeta \in \Gamma_\varphi$.*

Proof. Just set $v(z) = -u(z)$ and $\psi(z) = \varphi(z)$ in the proof of Theorem 2.4. □

Corollary 2.7 *Let φ, $\psi \in A^*$. Then the following conditions are equivalent:*

(i): *C_φ - C_ψ is compact on A;*
(ii): *C_φ - C_ψ is completely continuous on A*
(iii): $\Gamma_\varphi = \Gamma_\psi$.

Now, we illustrate Theorem 2.4 with the following example.

Example. If $\varphi_j(j = 1, ..., 5)$ are analytic self maps of B_N with following formulas:

$$
\begin{aligned}
\varphi_1(z_1, \ldots, z_N) &= \left(\frac{z_1+1}{2}, \frac{z_2}{2}, \ldots, \frac{z_N}{2}\right), \\
\varphi_2(z_1, \ldots, z_N) &= \left(\frac{z_2}{2}, \frac{z_1+1}{2}, \ldots, \frac{z_N}{2}\right), \\
\varphi_3(z_1, \ldots, z_N) &= \left(\frac{z_1+3}{4}, \frac{z_2}{4}, \ldots, \frac{z_N}{4}\right), \\
\varphi_4(z_1, \ldots, z_N) &= \left(\frac{z_1^2+1}{2}, 0, \ldots, 0\right), \\
\varphi_5(z_1, \ldots, z_N) &= \left(\frac{z_1}{2}, \frac{z_2}{2}, \ldots, \frac{z_N}{2}\right),
\end{aligned}
$$

and

$$u_1(z_1,\ldots,z_N) = 2z_1, \quad u_2(z_1,\ldots,z_2) = z_2 + \cdots + z_N,$$
$$u_3(z_1,\ldots,z_N) = -3z_1, \quad u_4(z_1,\ldots,z_N) = (z_1+1)/2, \quad u_5 \equiv 1.$$

Then, $\sum_{j=1}^{5} u_j C_j$ is a compact operator on A.

Proof. First, note that $\Gamma_1 = \{(1, 0, ..., 0)\} = \{e_1\}$, $\Gamma_2 = \{e_1\}$, $\Gamma_3 = \{e_1\}$, $\Gamma_4 = \{\pm e_1\}$, $\Gamma_2 = \varnothing$. And $\varphi_1(e_1) = \varphi_3(e_1) = e_1$, $\varphi_2(e_1) = (0, 1, 0, ..., 0) = (e_2)$, $\varphi_4(\pm e_1) = e_1$. Then, $\varphi_j \in A^*$ for each $j = 1, ..., 5$. Take φ_1 for example, the sequence $\{(1 - 1/n, 0, ..., 0)\}$ converges to $\varphi(e_1)$ radially, and they have pre-images $\{(1 - 2/n, 0, ..., 0)\}$ of φ_1 in B_N. Hence, φ_1 satisfies (*). Then, we have $I_1(e_1) = \{1, 3, 4\}$, $I_2(e_1) = \{2\}$ and $I(-e_1) = I_4(-e_1) = \{4\}$. Compute that

$I_1(e_1)$**:** $u_1(e_1) + u_3(e_1) + u_4(e_1) = 2 - 3 + 1 = 0$,
$I_2(e_1)$**:** $u_2(e_1) = 0$,
$I(-e_1)$**:** $u_4(-e_1) = 0$.

By Theorem 2.4, $\sum_{j=1}^{5} u_j C_j$ is compact on A. □

3. Connectness of weighted composition operators on *A*

Shapiro and Sundberg [22] first discovered the relationship between compact differences and topological structure of the collection of composition operators on Hardy space. In this section, we will investigate the connectness of the topological space consisted of all weighted composition operators act on the unit ball algebra, and we use the notion $\mathcal{C}(A)$ to represent that topological space. It is trivial that

$\mathcal{C}(A)$ is a connected topological space.

Because we can connect any uC_φ and 0 (the weighted composition operator uC_φ when $u \equiv 0$) by the continuous path $T_t : [0,1] \to \mathcal{C}(A)$ defined by $t \mapsto tuC_\varphi$. Indeed, for many function spaces, we denote by S generally, the weighted composition operators topological space $\mathcal{C}(S)$ is connected by the same path constructed above. So, we are interested in the connectness of the non-zero weighted composition operators topological space, and here we denote by $\mathcal{C}_w(S)$. In this setting, the connectness of $\mathcal{C}_w(S)$ depends on the function space S.

Theorem 3.1 $\mathcal{C}_w(A)$*is also a connected topological space.*

Proof. Let distinct uC_φ, $vC_\psi \in \mathcal{C}_w(A)$. We construct a path

$$T(t) : [0,1] \to \mathcal{C}_w(A) \text{ by } t \mapsto u_t C_{\varphi_t}$$

where $u_t(z) = tu(z) + (1 - t)v(z)$ and $\varphi_t(z) = t\varphi(z) + (1 - t)\psi(z)$. To avoid $u_t = 0$ and $\varphi_t = 0$ for some $t \in [0, 1]$, we first assume that

(1): $v \neq \theta u$ for every $\theta < 0$;
(2): $\psi \neq \kappa\varphi$ for every $\kappa < 0$ with $\kappa\varphi \in S(B_N)$, where $\psi = (\psi_1, ..., \psi_N)$ and $\varphi = (\varphi_1, ..., \varphi_N)$.

We are going to prove that $T(t)$ is a continuous path which connects uC_φ and vC_ψ, and here we simply denote by $C_t = C_{\varphi_t}$. For any s, $t \in [0, 1]$,

$$\| u_t C_t - v_t C_s \| \leq \| u_t(C_t - C_s) \| + \| (u_t - u_s)C_s \| .$$

The second term:

$$\| (u_t - u_s)C_s \| \leq |t - s| \cdot \| u + v \|_\infty \| C_s \| \leq |t - s| \cdot \| u + v \|_\infty .$$

So, it converges to 0 as $s \to t$.

The first term:

$$\begin{aligned} & \| u_t(C_t - C_s) \| \\ \leq & \| u_t \|_\infty \| C_t - C_s \| \\ = & \| u_t \|_\infty \sup_{f \in B(A)} \sup_{z \in B_N} |(C_t - C_s) f(z)| \\ \leq & \| u_t \|_\infty \sup_{f \in B(H^\infty)} \sup_{z \in B_N} |f(\varphi_t(z)) - f(\varphi_s(z))| \\ = & \| u_t \|_\infty \sup_{z \in B_N} \sup_{f \in B(H^\infty)} |f(\varphi_t(z)) - f(\varphi_s(z))| \\ \leq & 2 \| u_t \|_\infty \sup_{z \in B_N} \rho(\varphi_t(z), \varphi_s(z)). \end{aligned} \tag{3.1}$$

So, it converges to 0 as $s \to t$, since φ_t, $\varphi_s \in A$ (See Lemma 6 in [14]). Here, we obtain that $\{T(t)\}$ is a continuous path which connects any different uC_φ and vC_ψ in $\mathcal{C}_w(A)$.

Next, for any given $uC_\varphi \in \mathcal{C}_w(A)$, we need to find out continuous pathes from uC_φ to either θuC_φ or $uC_{\kappa\varphi}$ where $\theta < 0$ and $\kappa < 0$ with $\kappa\varphi \in S(B_N) \cap A$.

It is obvious that $-uC_\varphi$ and θuC_φ can be connected. Then, we construct

$$R(t) = e^{i\pi t} u \cdot C_\varphi : [0, 1] \to \mathcal{C}_w(A)$$

to connect uC_φ and $-uC_\varphi$ continuously: for s, $t \in [0, 1]$,

$$\left\| e^{i\pi t} uC_\varphi - e^{i\pi s} uC_\varphi \right\| \leq \left| e^{i\pi t} - e^{i\pi s} \right| \| u \|_\infty ||C_\varphi|| \leq \pi |t - s| \| u \|_\infty \| C_\varphi \| \to 0$$

as $|t - s| \to 0$. Thus, θuC_φ, with any $\theta < 0$, can be connected continuously with uC_φ.

Similarly as the analysis above, we note that $-\varphi$ cannot be equal to $\kappa\varphi$ multiplied by any negative real numbers. Thus, $uC_{-\varphi}$ and $uC_{\kappa\varphi}$ can be connected. To connect $uC_{-\varphi}$ and uC_φ, we put $\mu = i\varphi \in S(B_N)$, and

$$\begin{aligned} Y_1(t) &= uC_{t\varphi + (1-t)\mu} : [0, 1] \to \mathcal{C}_w(A); \\ Y_2(t) &= uC_{t(-\varphi) + (1-t)\mu} : [0, 1] \to \mathcal{C}_w(A) \end{aligned}$$

By the same method of computation as in (3.1), $Y_1(t)$ and $Y_2(t)$ can be shown to be two continuous pathes in $\mathcal{C}_w(A)$ from uC_φ and $uC_{-\varphi}$ to uC_μ.

Thus, the space $\mathcal{C}_w(A)$ is a connected topological space. □

Corollary 3.2 *The topological space consisting of composition operators acting on the unit ball algebra is connected.*

From the last result, we know that two weighted composition operators can be in the same component in $\mathcal{C}_w(A)$ without compact differences.

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Authors' contributions
All authors conceived and drafted the manuscript, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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