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Integral mean estimates for polynomials whose zeros are within a circle

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Abstract

Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \le K \le 1$, then for each $\delta > 0$, $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, Aziz and Ahmad (Glas Mat Ser III 31:229-237, 1996) proved that

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^{\delta} d\theta\right\}^{\frac{1}{\delta}} \le \left\{\int_0^{2\pi} |1 + Ke^{i\theta}|^{q\delta} d\theta\right\}^{\frac{1}{q\delta}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{p\delta} d\theta\right\}^{\frac{1}{p\delta}}.$$

In this paper, we extend the above inequality to the class of polynomials $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, having all its zeros in $|z| \le K \le 1$, and obtain a generalization as well as refinement of the above result.

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1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. If $P(z)$ has all its zeros in $|z| \le 1$, then it was shown by Turan [1] that

$$Max_{|z|=1}|P'(z)| \ge \frac{n}{2} Max_{|z|=1}|P(z)|. \tag{1}$$

Inequality (1) is best possible with equality for $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. As an extension of (1), Malik [2] proved that if $P(z)$ has all its zeros in $|z| \le K$, where $K \le 1$, then

$$Max_{|z|=1}|P'(z)| \ge \frac{n}{1+K} Max_{|z|=1}|P(z)|. \tag{2}$$

Malik [3] also obtained a generalization of (1) in the sense that the right-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved the following:

Theorem A. *If $P(z)$ has all its zeros in $|z| \le 1$, then for each $\delta > 0$*

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^{\delta}d\theta\right\}^{\frac{1}{\delta}} \le \left\{\int_0^{2\pi}|1+e^{i\theta}|^{\delta}d\theta\right\}^{\frac{1}{\delta}} Max_{|z|=1}|P'(z)|. \tag{3}$$

The result is sharp, and equality in (3) holds for $P(z) = (z + 1)^n$. If we let $\delta \to \infty$ in (3), we get (1).

As a generalization of Theorem A, Aziz and Shah [4] proved the following:

Theorem B. *If $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$ is a polynomial of degree n having all its zeros in the disk $|z| \le K$, $K \le 1$, then for each $\delta > 0$,*

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^{\delta}d\theta\right\}^{\frac{1}{\delta}} \le \left\{\int_0^{2\pi}|1+K^{\mu}e^{i\theta}|^{\delta}d\theta\right\}^{\frac{1}{\delta}} Max_{|z|=1}|P'(z)|. \tag{4}$$

Aziz and Ahmad [5] generalized (3) in the sense that $Max_{|z|=1}|P'(z)|$ on $|z| = 1$ on the right-hand side of (3) is replaced by a factor involving the integral mean of $|P'(z)|$ on $|z| = 1$ and proved the following:

Theorem C. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le K \le 1$, then for $\delta > 0$, $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^{\delta}d\theta\right\}^{\frac{1}{\delta}} \le \left\{\int_0^{2\pi}|1+Ke^{i\theta}|^{q\delta}d\theta\right\}^{\frac{1}{q\delta}} \left\{\int_0^{2\pi}|P'(e^{i\theta})|^{p\delta}d\theta\right\}^{\frac{1}{p\delta}}. \tag{5}$$

If we let $p \to \infty$ (so that $q \to 1$) in (5), we get (3).

In this paper, we consider a class of polynomials $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$, having all the zeros in $|z| \le K \le 1$, and thereby obtain a more general result by proving the following:

Theorem 1. *If $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$ is a polynomial of degree n having all its zeros in the disk $|z| \le K$, $K \le 1$, then for each $\delta > 0$, $q > 1$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$and for every complex number λ with $|\lambda| < 1$*

$$\begin{aligned} &n\left\{\int_0^{2\pi}|P(e^{i\theta})+\lambda m|^{\delta}d\theta\right\}^{\frac{1}{\delta}} \\ &\le \left\{\int_0^{2\pi}\left|1+\left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]e^{i\theta}\right|^{q\delta}d\theta\right\}^{\frac{1}{q\delta}} \left\{\int_0^{2\pi}|P'(e^{i\theta})|^{p\delta}d\theta\right\}^{\frac{1}{p\delta}}, \end{aligned} \tag{6}$$

where $m = Min_{|z|=K}|P(z)|$.

If we take $\lambda = 0$ in Theorem 1, we get the following:

Corollary 1. *If $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$ is a polynomial of degree n having all its zeros in the disk $|z| \le K$, $K \le 1$, then for each $\delta > 0$, $q > 1$, $p > 1$ with*

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^{\delta} d\theta\right\}^{\frac{1}{\delta}} \leq \left\{\int_0^{2\pi} \left|1+\left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]e^{i\theta}\right|^{q\delta} d\theta\right\}^{\frac{1}{q\delta}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{p\delta} d\theta\right\}^{\frac{1}{p\delta}},$$

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^{\delta} d\theta\right\}^{\frac{1}{\delta}} \leq \left\{\int_0^{2\pi} \left|1+\left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]e^{i\theta}\right|^{q\delta} d\theta\right\}^{\frac{1}{q\delta}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{p\delta} d\theta\right\}^{\frac{1}{p\delta}}, \tag{7}$$

For $\mu = 1$ in Theorem 1, we have the following:

Corollary 2. *If* $P(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n having all its zeros in the disk* $|z| \leq K$, $K \leq 1$, *then for each* $\delta > 0$, $q > 1$, $p > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})+\lambda m|^{\delta} d\theta\right\}^{\frac{1}{\delta}} \leq \left\{\int_0^{2\pi} \left|1+\left[\frac{n|a_n|K^{2}+|a_{n-1}|}{n|a_n|+|a_{n-1}|}\right]e^{i\theta}\right|^{q\delta} d\theta\right\}^{\frac{1}{q\delta}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{p\delta} d\theta\right\}^{\frac{1}{p\delta}}, \tag{8}$$

where $m = Min_{|z|=K}|P(z)|$.

Remark 1: Since all the zeros of $P(z)$ lie in $|z| \leq K$, therefore, $\frac{1}{n}\left|\frac{a_{n-1}}{a_n}\right| \leq K K \leq 1$, it can be easily verified that

$$\frac{n|a_n|K^2+|a_{n-1}|}{n|a_n|+|a_{n-1}|} \leq K.$$

It shows that for $\lambda = 0$, Corollary 2 provides a refinement of the result of Aziz and Ahmad [5].

The next result immediately follows from Theorem 1, if we let $p \to \infty$ (so that $q \to 1$)

Corollary 3. *If* $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ *is a polynomial of degree n having all its zeros in the disk* $|z| \leq K$, $K \leq 1$, *then for each* $\delta > 0$ *and for every complex number* λ *with* $|\lambda| < 1$

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})+\lambda m|^{\delta} d\theta\right\}^{\frac{1}{\delta}} \leq \left\{\int_0^{2\pi} \left|1+\left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]e^{i\theta}\right|^{\delta} d\theta\right\}^{\frac{1}{\delta}} Max_{|z|=1}|P'(z)|. \tag{9}$$

Also if we let $\delta \to \infty$ in the Corollary 3 and note that

$$\lim_{\delta\to\infty}\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^\delta d\theta\}^{\frac{1}{\delta}} = Max_{|z|=1}|P(z)|,$$

we get from (9)

$$nMax_{|z|=1}|P(z)+\lambda m| \le \frac{n|a_n|(K^{2\mu}+K^{\mu-1})+\mu|a_{n-\mu}|(1+K^{\mu-1})}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}Max_{|z|=1}|P'(z)| \quad for \quad |z|=1. \tag{10}$$

If z_0 be such that $Max_{|z|=1}|P(z)| = |P(z_0)|$, then from (10), we have

$$n|P(z_0)+\lambda m| \le \frac{n|a_n|(K^{2\mu}+K^{\mu-1})+\mu|a_{n-\mu}|(1+K^{\mu-1})}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}Max_{|z|=1}|P'(z)| \quad for \quad |z|=1.$$

Choosing an argument of λ such that

$$|P(z_0)+\lambda m| = |P(z_0)|+|\lambda|m,$$

we get

$$n(|P(z_0)|+|\lambda|m) \le \frac{n|a_n|(K^{2\mu}+K^{\mu-1})+\mu|a_{n-\mu}|(1+K^{\mu-1})}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}Max_{|z|=1}|P'(z)|. \tag{11}$$

From inequality (11), we conclude the following:

Corollary 4. *If* $P(z) := a_nz^n + \sum_{j=\mu}^n a_{n-j}z^{n-j}$, $1 \le \mu \le n$ *is a polynomial of degree n having all its zeros in the disk* $|z| \le K$, $K \le 1$, *then for* $0 \le t \le 1$, *we have*

$$Max_{|z|=1}|P'(z)| \ge n\frac{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}{n|a_n|(K^{2\mu}+K^{\mu-1})+\mu|a_{n-\mu}|(1+K^{\mu-1})}\{Maz_{|z|=1}|P(z)|+tMin_{|z|=K}|P(z)|\}.$$

Further, if we take $K = t = \mu = 1$ in the Corollary 4, we get a result of Aziz and Dawood [6].

2. Lemmas

For the proof of this theorem, we need the following lemmas.

The first lemma is due to Qazi [7].

Lemma 1. *If* $P(z) := a_0 + \sum_{j=\mu}^n a_jz^j$, $1 \le \mu \le n$ *is a polynomial of degree n having no zeros in the disk* $|z| < K$, $K \ge 1$, *then*

$$\left[\frac{n|a_0|K^{\mu+1}+\mu|a_\mu|K^{2\mu}}{n|a_0|+\mu|a_\mu|K^{\mu+1}}\right]|P'(z)| \le |Q'(z)| \quad for \quad |z|=1,$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ *and* $\frac{\mu}{n}|\frac{a_\mu}{a_0}|K^\mu \le 1$.

Lemma 2. *If* $P(z) := a_nz^n + \sum_{j=\mu}^n a_{n-j}z^{n-j}$ *is a polynomial of degree n having all its zeros in the disk* $|z| \le K \le 1$, *then*

$$|Q'(z)| \le \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]|P'(z)| \quad for\ |z|=1,\ 1 \le \mu \le n,$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

Proof of Lemma 2

Since all the zeros of $P(z)$ lie in $|z| \leq K \leq 1$, therefore all the zeros of $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ lie in $|z| \geq \frac{1}{K} \geq 1$. Hence, applying Lemma 1 to the polynomial $Q(z) := \bar{a}_n + \sum_{j=\mu}^{n} \bar{a}_{n-j}z^j$, we get

$$\left[\frac{n|a_n|\frac{1}{K^{\mu+1}}+\mu|a_{n-\mu}|\frac{1}{K^{2\mu}}}{n|a_n|+\mu|a_{n-\mu}|\frac{1}{K^{\mu+1}}}\right]|Q'(z)| \leq |P'(z)|.$$

Or, equivalently

$$|Q'(z)| \leq \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]|P'(z)|, \quad |z| = 1.$$

This proves Lemma 2.

Remark 1: Lemma 3 of Govil and Mc Tume [8] is a special case of this lemma when $\mu = 1$.

Proof of Theorem 1

Let $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}z$, we have $P(z) = z^n\overline{Q(\frac{1}{\bar{z}})}$. This gives

$$P'(z) = nz^{n-1}\overline{Q(\frac{1}{\bar{z}})} - z^{n-2}\overline{Q'(\frac{1}{\bar{z}})}. \tag{12}$$

Equivalently,

$$zP'(z) = nz^n\overline{Q(\frac{1}{\bar{z}})} - z^{n-1}\overline{Q'(\frac{1}{\bar{z}})}. \tag{13}$$

This implies

$$|P'(z)| = |nQ(z) - zQ'(z)| \qquad for \qquad |z| = 1. \tag{14}$$

Let $m = Min_{|z|=K}|P(z)|$, so that $m \leq |P(z)|$ for $|z| = K$. Therefore, for every complex number λ with $|\lambda| < 1$, we have $|m\lambda| < |P(z)|$ on $|z| = K$. Since $P(z)$ has all its zeros in $|z| \leq K \leq 1$, by Rouche's theorem, it follows that all the zeros of the polynomial $G(z) = P(z) + \lambda m$ lie in $|z| \leq K \leq 1$.

If $H(z) = z^n\overline{G(\frac{1}{\bar{z}})} = Q(z) + m\bar{\lambda}z^n$, then by applying Lemma 2 to the polynomial $G(z) = P(z) + \lambda m$, we have for $|z| = 1$

$$|H'(z)| \leq \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]|G'(z)|, \qquad 1 \leq \mu \leq n.$$

This gives

$$|Q'(z) + nm\bar{\lambda}z^{n-1}| \leq \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]|P'(z)|, \qquad 1 \leq \mu \leq n. \tag{15}$$

Using (14) in (15), we get

$$|Q'(z) + nm\bar{\lambda}z^{n-1}| \leq \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right]|nQ(z) - zQ'(z)| \quad for \quad |z| = 1, \quad 1 \leq \mu \leq n. \tag{16}$$

Since $P(z)$ has all its zeros in $|z| \leq K \leq 1$, by Gauss{Lucas theorem so does $P'(z)$. It follows that $nQ(z)$ - $zQ'(z)$, which is simply (see (12))

$$z^{n-1}\overline{P'\left(\frac{1}{\bar{z}}\right)},$$

has all its zeros in $|z| \geq \frac{1}{K} \geq 1$. Hence,

$$W(z) = \left[\frac{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}\right] \cdot \frac{z(Q'(z) + nm\bar{\lambda}z^{n-1})}{(nQ(z) - zQ'(z))} \tag{17}$$

is analytic for $|z| <1$, $|W(z)| \leq 1$ for $|z| = 1$ and W(0)=0. Thus, the function

$$1 + \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right] . W(z)$$

is subordinate to the function

$$1 + \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right] z$$

for $|z| <1$. Hence, by a property of subordination (for reference see [[9], p. 36, Theorem 1.6.17] or [[10], p. 454] or [11]), we have for each $\delta >0$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}&\int_0^{2\pi} \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}\right] W(e^{i\theta})\right|^{\delta} d\theta \\ &\leq \int_0^{2\pi} \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}\right] e^{i\theta}\right|^{\delta} d\theta.\end{aligned} \tag{18}$$

Also from (17), we have

$$1 + \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right] W(z) \quad = \frac{n(Q(z)+m\bar{\lambda}z^n)}{nQ(z)-zQ'(z)}.$$

Therefore,

$$n|Q(z) + m\bar{\lambda}z^n| = \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}\right] W(z)\right| |nQ(z) - zQ'(z)|, \tag{19}$$

which implies

$$n|H(z)| = \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}\right] W(z)\right| |nQ(z) - zQ'(z)|. \tag{20}$$

Using (14) and the fact that $|H(z)| = |G(z)| = |P(z) + \lambda m|$ for $|z| = 1$, we get from (20)

$$n|P(z) + \lambda m| = \left|1 + \left[\frac{n|a_n|K^{2\mu}+\mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1}+\mu|a_{n-\mu}|}\right] W(z)\right| |P'(z)| \quad \mathit{for} \quad |z| = 1.$$

Hence, for each $\delta > 0$ and $0 \le \theta < 2\pi$, we have

$$\begin{aligned} n^{\delta} \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^{\delta} d\theta \\ = \int_0^{2\pi} \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-v}|}\right] W(e^{i\theta})\right|^{\delta} |P'(e^{i\theta})|^{\delta} d\theta. \end{aligned} \tag{21}$$

This gives with the help of Hölder's inequality for $p > 1$, $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$and for every $\delta > 0$,

$$\begin{aligned} n^{\delta} \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^{\delta} d\theta \\ \le \left\{ \int_0^{2\pi} \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}\right] W(e^{i\theta})\right|^{q\delta} d\theta \right\}^{\frac{1}{q}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{p\delta} d\theta \right\}^{\frac{1}{p}} \end{aligned} \tag{22}$$

Combining (18) and (22), we get for $\delta > 0$ and $0 \le \theta < 2\pi$,

$$\begin{aligned} n^{\delta} \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^{\delta} d\theta \\ \le \left\{ \int_0^{2\pi} \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}\right] e^{i\theta}\right|^{q\delta} d\theta \right\}^{\frac{1}{q}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{p\delta} d\theta \right\}^{\frac{1}{p}} \end{aligned} \tag{23}$$

This is equivalent to

$$\begin{aligned} n\left\{ \int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^{\delta} d\theta \right\}^{\frac{1}{\delta}} \\ \le \left\{ \int_0^{2\pi} \left|1 + \left[\frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}\right] e^{i\theta}\right|^{q\delta} d\theta \right\}^{\frac{1}{q\delta}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{p\delta} d\theta \right\}^{\frac{1}{p\delta}} \end{aligned} \tag{24}$$

which proves the desired result.

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Authors' contributions
GS studied the related literature under the supervision of WMS and jointly developed the idea and drafted the manuscript. GS made the text _le and communicated the manuscript. GS also revised it as per the directions of the referee under the guidance of WMS. Both the authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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