Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A note on the Königs domain of compact composition operators on the Bloch space

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Abstract

Let $\mathbb{D}$ be the unit disk in the complex plane. We define $\mathcal{B}_0$ to be the little Bloch space of functions f analytic in $\mathbb{D}$ which satisfy $\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0$. If $\varphi : \mathbb{D} \to \mathbb{D}$is analytic then the composition operator $C_\phi : f \mapsto f \circ \phi$ is a continuous operator that maps $\mathcal{B}_0$ into itself. In this paper, we show that the compactness of C_ϕ , as an operator on $\mathcal{B}_0$, can be modelled geometrically by its principal eigenfunction. In particular, under certain necessary conditions, we relate the compactness of C_ϕ to the geometry of $\Omega = \sigma(\mathbb{D})$, where σ satisfies Schöder's functional equation $\sigma \circ \phi = \phi'(0)\sigma$.

2000 Mathematics Subject Classification: Primary 30D05; 47B33 Secondary 30D45.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane and $\mathbb{T}$ its boundary. We define the Bloch space $\mathcal{B}$ to be the Banach space of functions, f, analytic in $\mathbb{D}$ with

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

This space has many important applications in complex function theory, see [1] for an overview of many of them. We denote by $\mathcal{B}_0$ the little Bloch space of functions in $\mathcal{B}$ that satisfy $\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0$. This space coincides with the closure of the polynomials in $\mathcal{B}$.

Suppose now that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then we may define the operator, C_ϕ , acting on $\mathcal{B}_0$ as $f \mapsto f \circ \phi$. It was shown in [2] that every such operator maps $\mathcal{B}_0$ continuously into itself. Moreover, it was proved that C_ϕ is compact on $\mathcal{B}_0$ if and only if ϕ satisfies

$$\lim_{|z|\to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| = 0. \quad (1)$$

Recall that the hyperbolic geometry on $\mathbb{D}$ is defined by the distance

$$\text{disk}(z, w) = \inf \int_\Gamma \lambda_{\mathbb{D}}(\eta)|d\eta|$$

where the infimum is taken over all sufficiently smooth arcs that have endpoints z and w.

Here, $\lambda_{\mathbb{D}}(\eta) = (1 - |\eta|^2)^{-1}$ is the Poincaré density of $\mathbb{D}$. The hyperbolic derivative of ϕ is given by $\phi'(z)/(1 - |\phi(z)|^2)$ and functions that satisfy (1) are called little hyperbolic Bloch functions or written $\varphi \in \mathcal{B}_0^{\mathcal{H}}$.

The Schröder functional equation is the equation

$$\sigma \circ \varphi = \gamma\sigma. \tag{2}$$

Note that this is just the eigenfunction equation for C_ϕ. Kœnigs' theorem states that if ϕ has fixed point at the origin then (2) has a unique solution for $\gamma = \phi'(0)$ which we call the *Kœnigs function* and denote by σ from here on. In the study of the geometric properties of ϕ in relation to the operator theoretic properties of C_ϕ, it has become evident that the Kœnigs function is much more fruitful to study than ϕ itself. In particular, see [3] for a discussion of the Kœnigs function in relation to compact composition operators on the Hardy spaces.

If we let $\Omega = \sigma(\mathbb{D})$ be the *Kœnigs domain* of ϕ, then (2) may be interpreted as implying that the action of ϕ on $\mathbb{D}$ is equivalent to multiplication by γ on Ω. It is due to this that the pair (Ω, γ) is often called the geometric model for ϕ.

In this paper, we study the geometry of Ω when $\varphi \in \mathcal{B}_0^{\mathcal{H}}$. In order to do this, we will use the hyperbolic geometry of Ω. If $f : \mathbb{D} \to \Omega$ is a universal covering map and Ω is a hyperbolic domain in $\mathbb{C}$, then the Poincaré density on Ω is derived from the equation

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z),$$

which is independent of the choice of f. Since this equation, in terms of differentials, is $\lambda_\Omega(w)|dw| = \lambda_{\mathbb{D}}(z)|dz|$ (for $w = f(z)$), we see that the hyperbolic distance on $\mathbb{D}$ defined above carries over to a hyperbolic distance on Ω. For a more thorough treatment of the hyperbolic metric, see [4].

In [5], the Königs domain of a compact composition operator on the Hardy space was studied and the following result was proved.

Theorem A. *Let ϕ be a univalent self-map of $\mathbb{D}$ with a fixed point in $\mathbb{D}$. Suppose that for some positive integer n_0 there are at most finitely many points of $\mathbb{T}$ at which φ_{n_0} has an angular derivative. Then the following are equivalent.*

1. *Some power of C_ϕ is compact on the Hardy space H^2;*
2. *σ lies in H^p for every $p < \infty$;*
3. *$\Omega = \sigma(\mathbb{D})$ does not contain a twisted sector.*

Here, Ω is said to contain a *twisted sector* if there is an unbounded curve $\Gamma \in \Omega$ with

$$\delta_\Omega(w) \geq \varepsilon|w|$$

for some $\varepsilon > 0$ and all $w \in \Gamma$, where δ_Ω is the distance from w to the boundary of Ω as defined below. The purpose of this paper is to provide a similar result to this in the context of the Bloch space.

2 Simply connected domains

Throughout this section, we assume that Ω is an unbounded simply connected domain in $\mathbb{C}$ with $0 \in \Omega$. As in the previous section, σ represents the Riemann mapping of $\mathbb{D}$

onto Ω with $\sigma(0) = 0$ and $\sigma'(0) > 0$. We will also define ϕ via the Schröder functional equation. Throughout we let

$$\delta_\Omega(w) = \inf_{\zeta \notin \Omega} |w - \zeta|,$$

so that $\delta_\Omega(w)$ is the Euclidean distance from w to the boundary of Ω.

Theorem 1. *Let ϕ be a univalent function mapping $\mathbb{D}$into $\mathbb{D}$, $\phi(0) = 0$. Suppose that the closure of $\varphi(\mathbb{D})$intersects $\mathbb{T}$only at finitely many fixed points and is contained in a Stolz angle of opening no greater than $\alpha\pi$ there.*

If $|\phi'(0)| > 16 \tan(\alpha\pi/2)$ then the following are equivalent

1. *C_ϕ is compact on $\mathcal{B}$;*
2. $\lim\limits_{\substack{w \to \infty \\ w \in \gamma\Omega}} \dfrac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} = 0$;
3. *For every $n > 0$, $\sigma^n \in \mathcal{B}_0$.*

Remark: It has recently been shown by Smith [6] that compactness of C_ϕ on $\mathcal{B}$ is equivalent to compactness of C_ϕ on $\mathcal{B}_0$, *BMOA* and *VMOA* when ϕ is univalent and so in the above theorem, the first condition could read: C_ϕ is compact on $\mathcal{B}$, $\mathcal{B}_0$, *BMOA* and *VMOA* Before proceeding, we prove the following lemma.

Lemma 1. *Under the hypotheses of the theorem, w and γw tend to the same prime end at ∞, and $\partial\gamma\Omega \subset \Omega$.*

Proof. The first assertion follows from the fact that the closure of $\varphi(\mathbb{D})$ touches $\mathbb{T}$ only at fixed points. Suppose now that the second assertion is false and there are distinct prime ends ρ_1 and ρ_2 with $\rho_1 = \gamma\rho_2$. Then under the boundary correspondence given by σ there are distinct points $\eta, \zeta \in \mathbb{T}$ with

$$\sigma(\eta) = \gamma\sigma(\zeta) = \sigma(\varphi(\zeta)).$$

It follows that $\varphi(\zeta) \in \mathbb{T}$ and therefore ζ is a fixed point of ϕ. Hence, we have the contradiction $\rho_1 = \rho_2$. □

Proof. We first prove that 1 is equivalent to 2.

By the results of Madigan and Matheson [2], and Smith [6] cited above C_ϕ is compact on $\mathcal{B}$ if and only if

$$\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| = 0.$$

However, by Schröder's equation

$$\begin{aligned}\frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| &= \frac{\lambda_{\mathbb{D}}(\varphi(z))}{\lambda_{\mathbb{D}}(z)} |\varphi'(z)| \\ &= \frac{\lambda_\Omega(\sigma \circ \varphi(z))}{\lambda_\Omega(\sigma(z))} \frac{|\sigma' \circ \varphi(z)\varphi'(z)|}{|\sigma'(z)|} \\ &= |\gamma| \frac{\lambda_\Omega(\gamma w)}{\lambda_\Omega(w)}\end{aligned}$$

Since Ω is simply connected, $\lambda_\Omega(w) \boxtimes 1/\delta_\Omega(w)$ and so C_ϕ is compact on $\mathcal{B}$ if and only if

$$\lim_{w \to \partial\Omega} \frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} = 0. \tag{3}$$

Since $\gamma\Omega \subset \Omega$, $\gamma w \to \partial\Omega$ implies that $w \to \partial\Omega$. Therefore, (3) holds if and only if

$$\lim_{\gamma w \to \partial\Omega} \frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} = 0.$$

By the Lemma, we see that $\gamma w \to \partial\Omega$ means $w \to \infty$ and $w \in \gamma\Omega$, and we have shown that 1 and 2 are equivalent.

Suppose that 2 holds and let $\varepsilon > 0$ be given. Then we can find a $R > 0$ so that $\delta_\Omega(w) < \varepsilon\delta_\Omega(\gamma w)$ for all $|w| > R$, since there are only a finite number of prime ends at ∞. Choose $w \in \Omega$ arbitrarily with modulus greater than R and let n satisfy $|\gamma|^{-n}R < |w| \le |\gamma|^{-n-1} R$.

Then we have that $\delta_\Omega(w) < \varepsilon^n \delta_\Omega(\gamma^n w)$ and hence

$$\frac{-\log\delta_\Omega(w)}{\log|w|} > \frac{-n\log\varepsilon - \log\delta_\Omega(\gamma^n w)}{-(n+1)\log|\gamma| + \log R}.$$

Now as $w \to \infty$ in $\gamma\Omega$, $\gamma^n w$ lies in a closed set properly contained in Ω and therefore $\delta_\Omega(\gamma^n w)$ is bounded below by a constant independent of w. We thus have that

$$\liminf_{w \to \infty} \frac{-\log\delta_\Omega(w)}{\log|w|} > \frac{-\log\varepsilon}{-\log|\gamma|}$$

and since ε was arbitrary, the left-hand side of the above inequality must tend to ∞. Hence, we have shown that $\lim_{w\to\infty} |w|^\beta \delta_\Omega(w) = 0$ for every $\beta > 0$.

Now $\sigma^n \in \mathcal{B}_0$ may be interpreted geometrically as $\lim_{w\to\partial\Omega} n|w|^{n-1}\delta_\Omega(w) = 0$ and this follows from the above argument. Therefore, 2 implies 3.

To show that 3 implies 2, we need to show that if

$$\lim_{w\to\infty} f(w) = \lim_{w\to\infty} \frac{-\log\delta_\Omega(w)}{\log|w|} = \infty$$

then 2 holds.

To complete the proof, we require the following lemma whose proof we merely sketch.

Lemma 2. *Under the hypotheses of the theorem,*

$$\limsup_{w\to\infty} \frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} \le K < 1.$$

Sketch of Proof. First note that

$$\limsup_{|w|\to 1} \frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} \le \frac{16}{|\varphi'(0)|} \limsup_{|z|\to 1} \frac{\delta_{\varphi(\mathbb{D})}(z)}{\delta_{\mathbb{D}}(z)}.$$

Now if $\varphi(\mathbb{D})$ lies in a non-tangential angle of opening $\alpha\pi$ at ζ, then a short calculation shows that

$$\limsup_{z\to\zeta} \frac{\delta_{\varphi(\mathbb{D})}(z)}{\delta_{\mathbb{D}}(z)} \le \tan\frac{\alpha\pi}{2}$$

and the assertion follows. □

Now with f defined above, we have

$$f(\gamma w) - f(w) = \frac{-\log \delta_\Omega(\gamma w)}{\log|\gamma w|} - \frac{-\log \delta_\Omega(w)}{\log|w|}$$
$$\sim \frac{\log \delta_\Omega(w)/\delta_\Omega(\gamma w)}{\log|w|} < 0$$

for large enough w. Hence,

$$\frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} = \frac{|\gamma|^{f(\gamma w)}}{|w|^{f(w)-f(\gamma w)}} \leq |\gamma|^{f(\gamma w)} \to 0$$

as $w \to \infty$ and so 2 holds. □

It is of interest to consider the growth of σ since condition 3 would imply that it has very slow growth. The following corollary follows from 3 and the fact that functions in $\mathcal{B}_0$ grow at most of order log 1/(1 - |z|).

Corollary 1. *Suppose that ϕ satisfies the hypotheses of the Theorem and that any of the equivalent conditions holds, then for $r = |z|$.*

$$\log|\sigma(z)| = o\left(\log\log\frac{1}{1-r}\right).$$

We also provide the following restatement of the hypotheses of Theorem 1 to illustrate the main properties of the Königs domain.

Corollary 2. *Let Ω be an unbounded domain in $\mathbb{C}$ with $\gamma\Omega \subset \Omega$ and $0 \in \Omega$. Suppose that has Ω only finitely many prime ends at ∞ and*

$$\limsup_{w\to\infty} \frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} < 1.$$

In addition, suppose that $\partial\gamma\Omega \subset \Omega$. If $\sigma : \mathbb{D} \to \Omega$, $\sigma(0) = 0$, $\sigma'(0) > 0$, and ϕ is defined by Schröder's equation, then the following are equivalent.

1. *C_ϕ is compact on $\mathcal{B}$;*
2. $\displaystyle\lim_{\substack{w\to\infty \\ w\in\gamma\Omega}} \frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} = 0$;
3. *For every $n > 0$, $\sigma^n \in \mathcal{B}_0$.*

The hypothesis on the boundary of Ω is vital. If we do not assume that $\partial\gamma\Omega \subset \Omega$, then we deduce from the proof of the Theorem that $\varphi \in \mathcal{B}_0^{\mathcal{H}}$ is equivalent to

$$\lim_{\gamma w\to\partial\Omega} \frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} = 0. \tag{4}$$

In this situation, the finite part of the boundary of Ω plays a complicated role in the behaviour of ϕ. We conclude this section by constructing a domain that displays very bad boundary properties. This answers a question of Madigan and Matheson in [2].

In [2] it was shown that if $\partial\phi(D)$ touches $\mathbb{T} = \partial\mathbb{D}$ in a cusp, then $\varphi \in \mathcal{B}_0^{\mathcal{H}}$. However, it is not sufficient that $\partial\phi(D)$ touches $\mathbb{T}$ at an angle greater that 0. The question was raised of whether or not it is possible that $\overline{\varphi(D)} \cap \mathbb{T}$ can be infinite.

With the hypothesis that $\partial\gamma\Omega \subset \Omega$ the prime ends at ∞ correspond to points of $\overline{\varphi(D)}$ that touch $\mathbb{T}$. Therefore, $\overline{\varphi(D)} \cap \mathbb{T}$ is at most countable. A natural question to ask is whether or not $\Lambda(\overline{\varphi(D)} \cap \mathbb{T})$ can ever be positive, where Λ represents linear measure.

This example is well known in the setting of the unit disk, see [7, Corollary 5.3]. We describe here the construction in terms of the Königs domain.

Theorem 2. *There is a univalent function* $\varphi \in \mathcal{B}_0^{\mathcal{H}}$*such that* $\overline{\varphi(D)} \cap \mathbb{T} = \mathbb{T}$.

Proof. We construct the domain Ω so that it satisfies (4). Let $0 < \gamma < 1$ be given. We will define a nested sequence $\Theta_n \subset \mathbb{T}$, $n = 1, 2, \ldots$ so that

$$\partial\Omega = \cup_{n\geq 1} \left\{ re^{i\theta} : \gamma^{-n} \leq r < \infty, \theta \in \Theta_n \right\}, \tag{5}$$

where $\Theta_n \subset \Theta_{n+1}$ for all $n = 1, 2, \ldots$.

First let $N > 2$ be chosen arbitrarily and let $\Theta_1 = \{2\pi k/N : k = 0, \ldots, N - 1\}$.

Suppose now that Θ_n has been defined, then let Θ_{n+1} be such that $\Theta_n \subset \Theta_{n+1}$ and whenever $\theta \in \Theta_n$ is isolated, we define a sequence $\theta_k \in \Theta_{n+1}$, $k = 1, 2, \ldots$, so that $\theta_k \to \theta$ as $k \to \infty$ and for each k there is a j so that $\theta - \theta_k = \theta_{j\ -\theta}$. Moreover, assume that

$$\lim_{k\to\infty} \frac{\theta_{k+1} - \theta_k}{(\theta - \theta_k)^2} = 0. \tag{6}$$

In this way, we define the sequence of sets Θ_n, $n = 1, 2, \ldots$. We will, furthermore, assume that for each $e^{i\theta} \in \mathbb{T}$, there is a sequence $\theta_n \in \Theta_n$, $n = 1, 2, \ldots$, such that $\theta_n \to \theta$.

We claim that this gives the desired domain Ω with boundary defined by (5).

To see this, let $\gamma w \in \Omega$ be arbitrary, then by construction, we may find a $\zeta \in \partial\Omega$ so that $\delta_\Omega(\gamma w) = |\zeta - \gamma w|$. It is readily seen that for such ζ, there is an n so that $\zeta \in \{re^{i\theta} : r \geq \gamma^{-n}\}$ for some $\theta \in \Theta_n$ and moreover, θ is isolated in Θ_n.

If we now consider w, we may find a sequence $\theta_k \to \theta$ as $k \to \infty$ so that $\{re^{i\theta_k} : r \geq \gamma^{-n-1}\} \in \partial\Omega$ for all k hence we may fix a k so that $\delta_\Omega(w) = |w - n|$ for $\eta = re^{i\theta_k}$.

By estimating the line segment $[w, \eta]$ by the arc of $r\mathbb{T}$ joining w to η, we see that $\delta_\Omega(w)$ ⊠ $|w||\alpha - \theta_k|$ where $w = re^{i\alpha}$.Therefore, we have the estimate $\delta_\Omega(w) \leq |w||\theta_{k+1} - \theta_k|$. By a similar argument, we deduce the estimate $\delta_\Omega(\gamma w)$ ⊠ $|\gamma w||\theta - \theta_k|$ and so

$$\frac{\delta_\Omega(w)}{\delta_\Omega(\gamma w)} \leq \gamma^{-1} \left| \frac{\theta_{k+1} - \theta_k}{\theta - \theta_k} \right| \leq \gamma^{-1} |\theta - \theta_k|$$

by (6) and so the construction is complete.

We claim that if $\sigma : \mathbb{D} \to \Omega$ is defined as usual and ϕ is given by Schröder's equation, then $\overline{\varphi(\mathbb{D})} \cap \mathbb{T} = \mathbb{T}$.

In fact, if $\theta \in \Theta_n$ is isolated, then the ray $R = \{re^{i\theta} : r \geq \gamma^{-n-1}\}$ is contained in a single prime end of Ω. Therefore, to each such ray, there exists a point $\zeta \in \mathbb{T}$ that corresponds to R under σ. Since $\gamma R \subset \partial\Omega$, we thus have that ζ corresponds to a prime end p under ϕ with $p \cap \mathbb{T} \neq \emptyset$.

On the other hand, if $\theta \in \Theta_n$ is isolated, then $R' = \{re^{i\theta} : \gamma^{-n} \leq r < \gamma^{-n-1}\}$ satisfies $\gamma R' \cap \partial\Omega = \emptyset$, and so there is an arc $\rho_\theta \subset \mathbb{D}$ such that $\sigma(\rho_\theta) = R'$ and ρ_θ has an end-point in $\mathbb{T}$.

Hence, each $\eta \in \mathbb{T}$ is contained in a prime end of $\varphi(\mathbb{D})$ and

$$\varphi(\mathbb{D}) = \mathbb{D} \setminus \bigcup_{\theta \in \Theta_n \text{ isolated}} \rho_\theta .$$

The result follows. □

3 Multiply connected domains

The geometric arguments of the previous section potentially lend themselves to multiply connected domains in the following way. Suppose that Ω is a domain in $\mathbb{C}$ with $0 \in \Omega$ and $\gamma\Omega \subset \Omega$ for some $\gamma \in \mathbb{D}\setminus\{0\}$. Let σ be a universal covering map of $\mathbb{D}$ onto Ω with $\sigma(0) = 0$. Then $\sigma'(0) \neq 0$ and we may define ϕ via (2). Now we have

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = |\gamma|\frac{\lambda_\Omega(\gamma w)}{\lambda_\Omega(w)}.$$

However, if Ω is not simply connected, then σ is an infinitely sheeted covering of Ω and therefore the equation $\sigma(z) = 0$ has infinitely many distinct solutions, z_n, $n = 0, 1, \ldots$.

Now, since

$$\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|\varphi'(z_n)| = |\gamma| > 0$$

for all $n \geq 0$, we see that $\varphi \notin \mathcal{B}_0^{\mathcal{H}}$. Thus, we have proved the following result.

Proposition 1. *Suppose that* $\Omega \subset \mathbb{C}$ *is a domain satisfying* $0 \in \Omega$ *and* $\gamma\Omega \subset \Omega$, *and let* $\sigma : \mathbb{D} \to \Omega$ *be a universal covering map with* $\sigma(0) = 0$.

If ϕ, *as defined by (2) is in* $\mathcal{B}_0^{\mathcal{H}}$ *then* Ω *is simply connected.*

4 Competing interests

The author declares that they have no competing interests.

Received: 31 January 2011 Accepted: 10 August 2011 Published: 10 August 2011

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doi:10.1186/1029-242X-2011-31
Cite this article as: Jones: A note on the Königs domain of compact composition operators on the Bloch space. *Journal of Inequalities and Applications* 2011 **2011**:31.