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Any two-dimensional Normed space is a generalized Day-James space

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Abstract
It is proved that any two-dimensional normed space is isometrically isomorphic to a generalized Day-James space ℓ_ψ-ℓ_ϕ, introduced by W. Nilsrakoo and S. Saejung.

Keywords: Normed space, Day-James space, Birkhoff orthogonality

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The Day-James space ℓ_p-ℓ_q is defined for $1 \le p, q \le \infty$ as the space $\mathbb{R}^2$ endowed with the norm

$$||x||_{p,q} = \begin{cases} ||x||_p \text{ if } x_1x_2 \geq 0, \\ ||x||_q \text{ if } x_1x_2 \leq 0, \end{cases}$$

where $x = (x_1, x_2)$. James [1] considered the space ℓ_p - ℓ_q with $1/p + 1/q = 1$ as an example of a two-dimensional normed space where Birkhoff orthogonality is symmetric. Recall that if x and y are vectors in a normed space then x is said to be Birkhoff orthogonal to y, ($x \perp_B y$), if $||x + \lambda y|| \geq ||x||$ for every scalar λ [2]. Birkhoff orthogonality coincides with usual orthogonality in inner product spaces. In arbitrary normed spaces Birkhoff orthogonality is in general not symmetric (e.g., in $\mathbb{R}^2$ with $||\cdot||_\infty$), and it is symmetric in a normed space of three or more dimension if and only if the norm is induced by an inner product. This last significant property was obtained in gradual stages by Birkhoff [2], James [1,3], and Day [4]. The first reference related to the symmetry of Birkhoff orthogonality in two-dimensional spaces seems to be Radon [5] in 1916. He considered plane convex curves with conjugate diameters (as in ellipses) in order to solve certain variational problems.

The procedure that James used to get two-dimensional normed spaces where Birkhoff orthogonality is symmetric was extended by Day [4] in the following way. Let $(X, ||\cdot||_X)$ be a two-dimensional normed space and let $u, v \in X$ be such that $||u||_X = ||v||_X = 1$, $u \perp_B v$, and $v \perp_B u$ (see Lemma below). Then, taking a coordinate system where $u = (1, 0)$ and $v = (0, 1)$ and defining

$$||(x_1, x_2)||_{X,X^*} = \begin{cases} ||(x_1, x_2)||_X \ \text{ if } x_1x_2 \geq 0, \\ ||(x_1, x_2)||_{X^*} \text{ if } x_1x_2 \leq 0, \end{cases}$$

one gets that in the space $(X, ||\cdot||_{X,X^*})$ Birkhoff orthogonality is symmetric. Moreover, Day also proved that surprisingly the norm of any two-dimensional space where Birkhoff orthogonality is symmetric can be constructed in the above way.

A norm on $\mathbb{R}^2$ is called absolute if $||(x_1, x_2)|| = ||(|x_1|, |x_2|)||$ for any $(x_1, x_2) \in \mathbb{R}^2$. Following Nilsrakoo and Saejung [6] let AN_2 be the family of all absolute and normalized (i.e., $||(1, 0)|| = ||(0, 1)|| = 1$) norms on $\mathbb{R}^2$. Examples of norms in $AN2$ are ℓ_p norms. Bonsall and Duncan [7] showed that there is a one-to-one correspondence between AN_2 and the family Ψ_2 of all continuous and convex functions $\psi : [0, 1] \to \mathbb{R}$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1\text{-}t, t\} \leq \psi(t) \leq 1$ $(0 \leq t \leq 1)$. The correspondence is given by $\psi(t) = ||(1\text{-}t, t)||$ for $||\cdot||$ in AN_2, and by

$$||(x_1, x_2)||_\psi = \begin{cases} (|x_1| + |x_2|)\, \psi\left(\dfrac{|x_2|}{|x_1| + |x_2|}\right) & \text{if } (x_1, x_2) \neq (0,0), \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

for ψ in Ψ_2.

In [6] the family of norms $||\cdot||_{p,q}$ of Day-James spaces ℓ_p - ℓ_q is extended to the family N_2 of norms defined in $\mathbb{R}^2$ as

$$||(x_1, x_2)||_{\psi,\varphi} = \begin{cases} ||(x_1 + x_2)||_\psi & \text{if } x_1, x_2 \geq 0, \\ ||(x_1 + x_2)||_\varphi & \text{if } x_1, x_2 \leq 0, \end{cases}$$

for ψ, $\phi \in \Psi_2$. The space $\mathbb{R}_2$ endowed with the above norm is called an ℓ_ψ-ℓ_ϕ space.

The purpose of this paper is to show that any two-dimensional normed space is isometrically isomorphic to an ℓ_ψ-ℓ_ϕ space. To this end we shall use the following lemma due to Day [8]. The nice proof we reproduce here is taken from the PhD Thesis of del Río [9], and is based on explicitly developing the idea underlying one of the two proofs given by Day.

Lemma 1 [8]. *Let* $(X, ||\cdot||)$ *be a two-dimensional normed space. Then, there exist* $u, v \in X$ *such that* $||u|| = ||v|| = 1$, $u \perp_B v$, *and* $v \perp_B u$.

Proof. Let $e, \hat{e} \in X$ be linearly independent, and for $x \in X$ let $(x_1, x_2) \in \mathbb{R}^2$ be the coordinates of x in the basis $\{e, \hat{e}\}$. Let $S = \{x \in X : ||x|| = 1\}$, and for $x \in S$ consider the linear functional f_x: $y \in X \mapsto f_x(y) = x_2y_1 - x_1y_2$. Then it is immediate to see that f_x attains the norm in $y \in S$ (i.e., $|x_2y_1 - x_1y_2| \geq |x_2z_1 - x_1z_2|$, for all $z_1e + z_2\hat{e} \in S$) if and only if $y \perp_B x$. Therefore if $u, v \in S$ are such that $|u_2v_1 - u_1v_2| = \max_{(x,\ y)\in S\times S} |x_2y_1 - x_1y_2|$ then $u \perp_B v$ and $v \perp_B u$. □

Theorem 2 *For any two-dimensional normed space* $(X, ||\cdot||_X)$ *there exist* $\psi, \phi \in \Psi_2$ *such that* $(X, ||\cdot||_X)$ *is isometrically isomorphic to* $(\mathbb{R}^2, ||\cdot||_{\psi,\ \phi})$.

Proof. By Lemma 1 we can take $u, v \in X$ such that $||u|| = ||v|| = 1$, $u \perp_B v$, and $v \perp_B u$. Then u and v are linearly independent and $(X, ||\cdot||_X)$ is isometrically isomorphic to $(\mathbb{R}^2, ||\cdot||_{\mathbb{R}2})$, where $||(x_1, x_2)||_{\mathbb{R}2} := ||x_1u + x_2v||_X$. Defining $\psi(t) = ||(1 - t)u + tv||_X$, $\phi(t) = ||(1 - t)u - tv||_X$, $(0 \leq t \leq 1)$, one trivially has that $\psi, \phi \in \Psi_2$ and $||(x_1, x_2)||_{\mathbb{R}2} = ||(x_1, x_2)||_{\psi,\ \phi}$ for all $(x_1, x_2) \in \mathbb{R}^2$. □

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Competing interests

The author declares that they have no competing interests.

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