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On the Krasnoselskii-type fixed point theorems for the sum of continuous and asymptotically nonexpansive mappings in Banach spaces

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Abstract

In this article, we prove some results concerning the Krasnoselskii theorem on fixed points for the sum $A + B$ of a weakly-strongly continuous mapping and an asymptotically nonexpansive mapping in Banach spaces. Our results encompass a number of previously known generalizations of the theorem.

Keywords: Krasnoselskii's fixed point theorem, asymptotically nonexpansive mapping, weakly-strongly continuous mapping, uniformly asymptotically regular, measure of weak noncompactness

1 Introduction

As is well known, Krasnoselskii's fixed point theorem has a wide range of applications to nonlinear integral equations of mixed type (see [1]). It has also been extensively employed to address differential and functional differential equations. His theorem actually combines both the Banach contraction principle and the Schauder fixed point theorem, and is useful in establishing existence theorems for perturbed operator equations. Since then, there have appeared a large number of articles contributing generalizations or modifications of the Krasnoselskii fixed point theorem and their applications (see [2]-[21]).

The study of asymptotically nonexpansive mappings concerning the existence of fixed points have become attractive to the authors working in nonlinear analysis. Goebel and Kirk [22] introduced the concept of asymptotically nonexpansive mappings in Banach spaces and proved a theorem on the existence of fixed points for such mappings in uniformly convex Banach spaces. In 1971, Cain and Nashed [23] generalized to locally convex spaces a well known fixed point theorem of Krasnoselskii for a sum of contraction and compact mappings in Banach spaces. The class of asymptotically nonexpansive mappings includes properly the class of nonexpansive mappings as well as the class of contraction mappings. Recently, Vijayaraju [21] proved by using the same method some results concerning the existence of fixed points for a sum of nonexpansive and continuous mappings and also a sum of asymptotically nonexpansive and continuous mappings in locally convex spaces. Very recently, Agarwal et al. [1] proved some existence theorems of a fixed point for the sum of a weakly-strongly continuous mapping and a nonexpansive mapping on a Banach space and under Krasnoselskii-, Leray Schauder-, and Furi-Pera-type conditions.

Motivated and inspired by Agarwal et al. [1] and Vijayaraju [21], in this article we will prove some new generalized forms of the Krasnoselskii theorem on fixed points for the sum $A + B$ of a weakly-strongly continuous mapping and an asymptotically nonexpansive mapping in Banach spaces. These results encompass a number of previously known generalizations of the theorem.

2 Preliminaries

Let M be a nonempty subset of a Banach space X and $T : M \to X$ be a mapping. We say that T is *weakly-strongly continuous* if for each sequence $\{x_n\}$ in M which converges weakly to x in M, the sequence $\{Tx_n\}$ converges strongly to Tx. The mapping T is *nonexpansive* if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in M$, and T is *asymptotically nonexpansive* (see [22]) if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ such that $||T^n x - T^n y|| \leq k_n||x - y||$ for all $n \geq 1$ and $x, y \in M$.

Definition 2.1. [21] If B and A map M into X, then B is called a *uniformly asymptotically regular* with respect to A if, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that

$$||B^n(x) - B^{n-1}(x) + A(x)|| \leq \varepsilon$$

for all $n \geq n_0$ and all $x \in M$.

Now, let us recall some definitions and results which will be needed in our further considerations. Let X be a Banach space, $\Omega(X)$ is the collection of all nonempty bounded subsets of X, and $\mathcal{W}(X)$ is the subset of $\Omega(X)$ consisting of all weak compact subsets of X. Let B_r denote the closed ball in X countered at 0 with radius $r > 0$. In [24], De Blasi introduced the following mapping $\omega : \Omega(X) \to [0, \infty)$ defined by

$$\omega(M) = \inf\{r > 0 : \text{there exists a set } N \in \mathcal{W}(X) \text{ such that } M \subseteq N + B_r\},$$

for all $M \in \Omega(X)$. For completeness, we recall some properties of $\omega(\cdot)$ needed below (for the proofs we refer the reader to [24]).

Lemma 2.2. [24]*Let M_1 and $M_2 \in \Omega(X)$, then we have*

(*i*) *If $M_1 \subseteq M_2$, then $\omega(M_1) \leq \omega(M_2)$.*
(*ii*) *$\omega(M_1) = 0$ if and only if M_1 is relatively weakly compact.*
(*iii*) *$\omega(\overline{M_1^w}) = \omega(M_1)$, where $\overline{M_1^w}$ is the weak closure of M_1.*
(*iv*) *$\omega(\lambda M_1) = |\lambda|\omega(M_1)$ for all $\lambda \in \mathbb{R}$.*
(*v*) *$\omega(co(M_1)) = w(M_1)$.*
(*vi*) *$\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.*
(*vii*) *If $(M_n)_{n\geq 1}$ is a decreasing sequence of nonempty, bounded and weakly closed subsets of X with $\lim_{n\to\infty} \omega(M_n) = 0$, then $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$ and $\omega(\bigcap_{n=1}^{\infty} M_n) = 0$, i.e., $\bigcap_{n=1}^{\infty} M_n$ is relatively weakly compact.*

Throughout this article, a measure of weak noncompactness will be a mapping $\psi : \Omega(X) \to [0, \infty)$ which satisfies the assumptions (i)-(vii) cited in Lemma 2.2.

Definition 2.3. [25] Let M be a closed subset of X and I, $T : M \to M$ be two mappings. A mapping T is said to be *demiclosed* at the zero, if for each sequence $\{x_n\}$ in M, the conditions $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

Lemma 2.4. [26]-[29]*Let X be a uniformly convex Banach space, M be a nonempty closed convex subset of X, and let $T : M \to M$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., for each sequence $\{x_n\}$ in M, if $\{x_n\}$ converges weakly to $q \in M$ and $\{(I - T)x_n\}$ converges strongly to* 0, *then* $(I - T)q = 0$.

Definition 2.5. [1,13] Let X be a Banach space and let ψ be a measure of weak noncompactness on X. A mapping $B : D(B) \subseteq X \to X$ is said to be *ψ-contractive* if it maps bounded sets into bounded sets and there is a $\beta \in [0, 1)$ such that $\psi(B(S)) \leq \beta\psi(S)$ for all bounded sets $S \subseteq D(B)$. The mapping $B : D(B) \subseteq X \to X$ is said to be *ψ-condensing* if it maps bounded sets into bounded sets and $\psi(B(S)) < \psi(S)$ whenever S is a bounded subset of $D(B)$ such that $\psi(S) > 0$.

Let $\mathcal{J}$ be a nonlinear operator from $D(\mathcal{J}) \subseteq X$ into X. In the next section, we will use the following two conditions:

$(\mathcal{H}1)$ If $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $D(\mathcal{J})$, then $(\mathcal{J}x_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

$(\mathcal{H}2)$ If $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $D(\mathcal{J})$, then $(\mathcal{J}x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

Remark 2.6. 1. Operators satisfying $(\mathcal{H}1)$ or $(\mathcal{H}2)$ are not necessarily weakly continuous (see [12,19,30]).

2. Every w-contractive mapping satisfies $(\mathcal{H}2)$.

3. A mapping $\mathcal{J}$ satisfies $(\mathcal{H}2)$ if and only if it maps relatively weakly compact sets into relatively weakly compact ones (use the Eberlein-Šmulian theorem [31]).

4. A mapping $\mathcal{J}$ satisfies $(\mathcal{H}1)$ if and only if it maps relatively weakly compact sets into relatively compact ones.

5. The condition $(\mathcal{H}2)$ holds true for every bounded linear operator.

The following fixed point theorems are crucial for our purposes.

Lemma 2.7. [12]*Let M be a nonempty closed bounded convex subset of a Banach space X. Suppose that $A : M \to X$ and $B : X \to X$ satisfying:*

(*i*) *A is continuous, AM is relatively weakly compact and A satisfies* $(\mathcal{H}1)$,
(*ii*) *B is a strict contraction satisfying* $(\mathcal{H}2)$,
(*iii*) *$Ax + By \in M$ for all $x, y \in M$.*

Then, there is an $x \in M$ such that $Ax + Bx = x$.

Lemma 2.8. [20]*Let M be a nonempty closed bounded convex subset of a Banach space X. Suppose that $A : M \to X$ and $B : M \to X$ are sequentially weakly continuous such that:*

(*i*) *AM is relatively weakly compact,*
(*ii*) *B is a strict contraction,*
(*iii*) *$Ax + By \in M$ for all $x, y \in M$.*

Then, there is an $x \in M$ such that $Ax + Bx = x$.

Lemma 2.9. [1]*Let X be a Banach space and let ψ be measure of weak noncompactness on X. Let Q and C be closed, bounded, convex subset of X with $Q \subseteq C$. In addition,* *let U be a weakly open subset of Q with $0 \in U$, and $F : \overline{U^w} \to C$ a weakly sequentially continuous and ψ-condensing mapping. Then either*

$$F \text{ has a fixed point,} \tag{2.1}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and, } \lambda \in (0,1) \text{ with } u = \lambda F u \tag{2.2}$$

here $\partial_Q U$ is the weak boundary of U in Q.

Lemma 2.10. [1]*Let X be a Banach space and $B : X \to X$ a k-Lipschitzian mapping, that is*

$$\forall x, y \in X, \|Bx - By\| \leq k\|x - y\|.$$

In addition, suppose that B verifies $(\mathcal{H}2)$. Then for each bounded subset S of X, we have $\psi(BS) \leq k\psi(S)$;

here, ψ is the De Blasi measure of weak noncompactness.

Lemma 2.11. [15,32]*Let X be a Banach space with $C \subseteq X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$, $F(\overline{U})$ bounded and $F : \overline{U} \to C$ a condensing mapping. Then, either F has a fixed point in $\overline{U}$ or there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$; here $\overline{U}$ and ∂U denote the closure of U in C and the boundary of U in C, respectively.*

Lemma 2.12. [15,32]*Let X be a Banach space and Q a closed convex bounded subset of X with $0 \in Q$. In addition, assume $F : Q \to X$ a condensing mapping with if $\{(x_j, \lambda_j)\}_{j=1}^{+\infty}$ is a sequence in $\partial Q \times [0, 1]$ converging to (x, λ) with $X = \lambda F(x)$ and $0 < \lambda < 1$, then $\lambda_j F(x_j) \in Q$ for j sufficiently large, holding. Then F has a fixed point.*

3 Main results

Now, we are ready to state and prove the main result of this section.

Theorem 3.1. *Let M be a nonempty bounded closed convex subset of a Banach space X. Let $A : M \to X$ and $B : M \to M$ satisfy the following:*

(i) A is weakly-strongly continuous, and AM is relatively weakly compact,
(ii) B is an asymptotically nonexpansive mapping with a sequence $(k_n) \subset [1, \infty)$ satisfying $(\mathcal{H}2)$,
(iii) if (x_n) is a sequence of M such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,
(iv) $I - B$ is demiclosed,
(v) $B^n x + Ay \in M$ for all $x, y \in M$ and $n = 1, 2,...,$
(vi) B is uniformly asymptotically regular with respect to A.

Then, there is an $x \in M$ such that $Ax + Bx = x$.

Proof. Suppose first that $0 \in M$ and let $a_n := (1 - \frac{1}{n})/k_n$ for all $n \in \mathbb{N}$. By hypothesis (v), we have

$$a_n B^n x + a_n Ay \in M \text{ for all } n \in \mathbb{N} \text{ and } x, y \in M.$$

Since B is asymptotically nonexpansive, it follows that

$$\begin{aligned} ||a_nB^nx - a_nB^ny|| &= a_n||B^nx - B^ny|| \\ &\le a_nk_n||x - y|| \\ &= (1 - \frac{1}{n})||x - y|| forall\ x, y \in M. \end{aligned} \tag{3.1}$$

Hence, a_nB^n is contraction on M. Therefore, by Lemma 2.7, there is an $x_n \in M$ such that

$$a_n(B^nx_n + Ax_n) = x_n, \tag{3.2}$$

for all $n \in \mathbb{N}$. This implies that

$$x_n - (B^nx_n + Ax_n) = (a_n - 1)(B^nx_n + Ax_n) \to 0 \text{ as } n \to \infty \tag{3.3}$$

since $a_n \to 1$ as $n \to \infty$ and M is bounded and $B^nx + Ay \in M$ for all $x, y \in M$. Since B is uniformly asymptotically regular with respect to A, it follows that

$$B^nx_n - B^{n-1}x_n + Ax_n \to 0 \text{ as } n \to \infty. \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$x_n - B^{n-1}x_n \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Now, it is noted that

$$\begin{aligned} ||x_n - Bx_n - Ax_n|| &= ||x_n - (B + A)x_n|| \\ &\le ||x_n - (B^n + A)x_n|| + ||(B^n + A)x_n - (B + A)x_n|| \\ &= ||x_n - (B^n + A)x_n|| + ||B^nx_n - Bx_n|| \\ &\le ||x_n - (B^n + A)x_n|| + k_1||B^{n-1}x_n - x_n||. \end{aligned} \tag{3.6}$$

Using (3.3) and (3.5) in (3.6), we get

$$x_n - Bx_n - Ax_n \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Using the fact that AM is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in M$. By (3.7), we have

$$(I - B)x_n \rightharpoonup y. \tag{3.8}$$

By hypothesis (*iii*), the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in M$. Since A is weakly-strongly continuous, $\{Ax_{n_k}\}$ converges strongly to Ax. Hence, we observe that

$$x_{n_k} - Bx_{n_k} = (I - B)x_{n_k} \to Ax \text{ as } k \to \infty. \tag{3.9}$$

Hence, by the demiclosedness of $I - B$, we have $Ax + Bx = x$.

To complete the proof, it remains to consider the case $0 \notin M$. In such a case, let us fix any element $x_0 \in M$ and let $M_0 = \{x - x_0, x \in M\}$. Define two mappings $A_0 : M_0 \to X$ and $B_0 : M_0 \to M$ by $A_0(x - x_0) = Ax - \frac{1}{2}x_0$ and $B_0(x - x_0) = Bx - \frac{1}{2}x_0$, for $x \in M$. By the result of the first case for A_0 and B_0, we have an $x \in M$ such that $A_0(x - x_0) + B_0(x - x_0) = x - x_0$. Hence $Ax + Bx = x$. □

Corollary 3.2. *Let M be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let $A : M \to X$ and $B : M \to M$ satisfy the following:*

(*i*) *A is weakly-strongly continuous,*
(*ii*) *B is an asymptotically nonexpansive mapping with a sequence* $(k_n) \subset [1, \infty)$,
(*iii*) $B^n x + Ay \in M$ *for all* $x, y \in M$, *and* $n = 1, 2,...,$
(*iv*) *B is uniformly asymptotically regular with respect to A.*

Then, there is an $x \in M$ *such that* $Ax + Bx = x$.

Our next result is the following:

Theorem 3.3. *Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that* $A : M \to X$ *and* $B : M \to M$ *are two weakly sequentially continuous mappings that satisfy the following:*

(*i*) *AM is relatively weakly compact,*
(*ii*) *B is an asymptotically nonexpansive mapping with a sequence* $(k_n) \subset [1, \infty)$,
(*iii*) *if* (x_n) *is a sequence of M such that* $((I - B)x_n)$ *is weakly convergent, then the sequence* (x_n) *has a weakly convergent subsequence,*
(*iv*) $B^n x + Ay \in M$ *for all* $x, y \in M$, *and* $n = 1, 2,...,$
(*v*) *B is uniformly asymptotically regular with respect to A.*

Then, there is an $x \in M$ *such that* $Ax + Bx = x$.

Proof. Without loss of generality, we may assume that $0 \in M$. Let $a_n := (1 - \frac{1}{n})/k_n \in (0, 1)$ for all $n \in \mathbb{N}$. By hypothesis (*iv*), we have

$$a_n B^n x + a_n A y \in M \text{ for all } n \in \mathbb{N} \text{ and } x, y \in M.$$

Since B is asymptotically nonexpansive, it follows that

$$\begin{aligned} ||a_n B^n x - a_n B^n y|| &= a_n ||B^n x - B^n y|| \\ &\le a_n k_n ||x - y|| \\ &= (1 - \frac{1}{n})||x - y||, \textit{for all } x, y \in M. \end{aligned} \tag{3.10}$$

Hence, $a_n B^n$ is a contraction on M. By Lemma 2.8, there is a $x_n \in M$ such that

$$a_n(B^n x_n + Ax_n) = x_n, \tag{3.11}$$

for all $n \in \mathbb{N}$. This implies that

$$x_n - (B^n x_n + Ax_n) = (a_n - 1)(B^n x_n + Ax_n) \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Since B is uniformly asymptotically regular with respect to A, it follows that

$$B^n x_n - B^{n-1} x_n + Ax_n \to 0 \text{ as } n \to \infty. \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$x_n - B^{n-1} x_n \to 0 \text{ as } n \to \infty. \tag{3.14}$$

Now, it is noted that

$$\begin{aligned} ||x_n - Bx_n - Ax_n|| &= ||x_n - (B + A)x_n|| \\ &\le ||x_n - (B^n + A)x_n|| + ||(B^n + A)x_n - (B + A)x_n|| \\ &= ||x_n - (B^n + A)x_n|| + ||B^n x_n - Bx_n|| \\ &\le ||x_n - (B^n + A)x_n|| + k_1 ||B^{n-1} x_n - x_n||. \end{aligned} \tag{3.15}$$

Using (3.12) and (3.14) in (3.15), we get

$$x_n - Bx_n - Ax_n \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Using the fact that AM is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in M$. Hence, by (3.16)

$$(I - B)x_n \rightharpoonup y. \tag{3.17}$$

By hypothesis (*iii*), the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in M$. Since A and B are weakly sequentially continuous, $\{Ax_{n_k}\}$ converges weakly to Ax, and $\{Bx_{n_k}\}$ converges weakly to Bx. Hence, $Ax + Bx = x$. □

Theorem 3.4. *Let Q and C be closed bounded convex subset of a Banach space X with* $Q \subseteq C$. *In addition, let U be a weakly open subset of Q with* $0 \in U$, $A : \overline{U^w} \to X$ *and* $B : X \to X$ *are two weakly sequentially continuous mappings satisfying the following:*

(*i*) $A(\overline{U^w})$ *is a relatively weakly compact,*
(*ii*) *B is an asymptotically nonexpansive mapping with a sequence* $(k_n) \subset [1, \infty)$,
(*iii*) *if* (x_n) *is a sequence of M such that* $((I - B)x_n)$ *is weakly convergent, then the sequence* (x_n) *has a weakly convergent subsequence,*
(*iv*) $B^n x + Ay \in C$ *for all* $x, y \in \overline{U^w}$, *and* $n = 1, 2, \ldots$,
(*v*) *B is uniformly asymptotically regular with respect to A.*

Then, either

$$A + B \text{ has a fixed point,} \tag{3.18}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0, 1) \text{ with } u = \lambda(A + B^n)u \tag{3.19}$$

here, $\partial_Q U$ *is the weak boundary of U in Q.*

Proof. Let $a_n := (1 - \frac{1}{n})/k_n \in (0, 1)$ for all $n \in \mathbb{N}$. We first show that the mapping $F_n = a_n A + a_n B^n$ is ψ-contractive with constant a_n. To see that, let S be a bounded subset of $\overline{U^w}$. Using the homogeneity and the subadditivity of the De Blasi measure of weak noncompactness, we obtain

$$\psi(F_n(S)) \leq \psi(a_n AS + a_n B^n S) \leq a_n \psi(AS) + a_n \psi(B^n S).$$

Keeping in mind that A is weakly compact and using Lemma 2.10, we deduce that

$$\psi(F_n(S)) \leq a_n k_n \psi(S).$$

This proves that F_n is ψ-contractive with constant a_n. Moreover, taking into account that $0 \in U$ and using assumption (*iv*), we infer that F_n map $\overline{U^w}$ into C. Next, we suppose that (3.19) does not occur, and F_n does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (3.18) occurs). If there exists a $u \in \partial_Q U$, and $\lambda \in (0, 1)$ with $u = \lambda F_n u$ then $u = \lambda a_n Au + \lambda a_n B^n u$. It is impossible since $\lambda a_n \in (0, 1)$. By Lemma 2.9, there exists $x_n \in \overline{U^w}$ such that

$$x_n = F_n x_n = a_n Ax_n + a_n B^n x_n,$$

for all $n \in \mathbb{N}$. This implies that

$$x_n - (B^n x_n + Ax_n) = (a_n - 1)(B^n x_n + Ax_n) \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Since B is uniformly asymptotically regular with respect to A, it follows that

$$B^n x_n - B^{n-1} x_n + Ax_n \to 0 \text{ as } n \to \infty. \tag{3.21}$$

From (3.20) and (3.21), we obtain

$$x_n - B^{n-1} x_n \to 0 \text{ as } n \to \infty. \tag{3.22}$$

Now, it is noted that

$$\begin{aligned} ||x_n - Bx_n - Ax_n|| &= ||x_n - (B + A)x_n|| \\ &\le ||x_n - (B^n + A)x_n|| + ||(B^n + A)x_n - (B + A)x_n|| \\ &= ||x_n - (B^n + A)x_n|| + ||B^n x_n - Bx_n|| \\ &\le ||x_n - (B^n + A)x_n|| + k_1 ||B^{n-1} x_n - x_n||. \end{aligned} \tag{3.23}$$

Using (3.20) and (3.22) in (3.23), we get

$$x_n - Bx_n - Ax_n \to 0 \text{ as } n \to \infty. \tag{3.24}$$

Since AM is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in \overline{U}$. Thus, we have

$$(I - B)x_n \rightharpoonup y. \tag{3.25}$$

By hypothesis (*iii*), the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in \overline{U}$. Since A and B are weakly sequentially continuous, $\{Ax_{n_k}\}$ converges weakly to Ax, and $\{Bx_{n_k}\}$ converges weakly to Bx. Hence, $Ax + Bx = x$. □

Theorem 3.5. *Let U be a bounded open convex set in a Banach space X with $0 \in U$. Suppose $A : \overline{U} \to X$ and $B : X \to X$ are continuous mappings satisfying the following:*

(*i*) $A(\overline{U})$ *is compact, and A is weakly-strongly continuous,*
(*ii*) *B is an asymptotically nonexpansive mapping with a sequence $(k_n) \subset [1, \infty)$, and I - B is demiclosed,*
(*iii*) *if (x_n) is a sequence of $\overline{U}$ such that $((I - B)x_n)$ is weakly convergent, then the sequence (x_n) has a weakly convergent subsequence,*
(*iv*) *B is uniformly asymptotically regular with respect to A.*

Then, either

$$A + B \text{ has a fixed point,} \tag{3.26}$$

or

$$\text{there is a point } u \in \partial U \text{ and } \lambda \in (0, 1) \text{ with } u = \lambda B^n u + \lambda Au. \tag{3.27}$$

Proof. Suppose (3.27) does not occur and let $a_n := (1 - \frac{1}{n})/k_n \in (0, 1)$ for all $n \in \mathbb{N}$. The mapping $F_n := a_n A + a_n B^n$ is the sum of a compact and a strict contraction. This implies that F_n is a condensing mapping (see [13]). By Lemma 2.11, we deduce that there is an $x_n \in \overline{U}$ such that

$$x_n = F_n x_n = a_n Ax_n + a_n B^n x_n,$$

for all $n \in \mathbb{N}$. This implies that

$$x_n - (B^n x_n + Ax_n) = (a_n - 1)(B^n x_n + Ax_n) \to 0 \text{ as } n \to \infty. \tag{3.28}$$

Since B is uniformly asymptotically regular with respect to A, it follows that

$$B^n x_n - B^{n-1} x_n + Ax_n \to 0 \text{ as } n \to \infty. \tag{3.29}$$

From (3.28) and (3.29), we obtain

$$x_n - B^{n-1} x_n \to 0 \text{ as } n \to \infty. \tag{3.30}$$

Now, it is noted that

$$\begin{aligned} ||x_n - Bx_n - Ax_n|| &= ||x_n - (B + A)x_n|| \\ &\le ||x_n - (B^n + A)x_n|| + ||(B^n + A)x_n - (B + A)x_n|| \\ &= ||x_n - (B^n + A)x_n|| + ||B^n x_n - Bx_n|| \\ &\le ||x_n - (B^n + A)x_n|| + k_1 ||B^{n-1} x_n - x_n||. \end{aligned} \tag{3.31}$$

Using (3.28) and (3.30) in (3.31), we get

$$x_n - Bx_n - Ax_n \to 0 \text{ as } n \to \infty. \tag{3.32}$$

Since AM is weakly compact and passing eventually to a subsequence, we may assume that $\{Ax_n\}$ converges weakly to some $y \in \overline{U}$. This implies that

$$(I - B)x_n \rightharpoonup y. \tag{3.33}$$

By hypothesis (*iii*), the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in \overline{U}$. Since A is weakly-strongly continuous, $\{Ax_{n_k}\}$ converges strongly to Ax. Consequently

$$x_{n_k} - Bx_{n_k} = (I - B)x_{n_k} \to Ax \text{ as } k \to \infty. \tag{3.34}$$

By the demiclosedness of I - B, we have $Ax + Bx = x$. □

Corollary 3.6. *Let U be a bounded open convex set in a uniformly convex Banach space X with $0 \in U$. Suppose $A : \overline{U} \to X$ and $B : X \to X$ are continuous mappings satisfying the following.*

(*i*) *$A(\overline{U})$ is compact, and A is weakly-strongly continuous,*
(*ii*) *B is an asymptotically nonexpansive mapping with a sequence $(k_n) \subset [1, \infty)$,*
(*iii*) *B is uniformly asymptotically regular with respect to A.*

Then, either

$$A + B \text{ has a fixed point,} \tag{3.35}$$

or

$$\text{there is a point } u \in \partial U \text{ and } \lambda \in (0, 1) \text{ with } u = \lambda B^n u + \lambda Au. \tag{3.36}$$

Theorem 3.7. *Let Q be a closed convex bounded set in a Banach space X with $0 \in Q$. Suppose $A : Q \to X$ and $B : X \to X$ are continuous mappings satisfying the following:*

(*i*) *$A(Q)$ is compact, and A is weakly-strongly continuous,* (*ii*) *B is an asymptotically nonexpansive mapping with a sequence* $(k_n) \subset [1, \infty)$, *and* $I - B$ *is demiclosed,*

(*iii*) *if* (x_n) *is a sequence of* $\overline{U}$ *such that* $((I - B)x_n)$ *is weakly convergent, then the sequence* (x_n) *has a weakly convergent subsequence,*

(*iv*) *if* $\{(x_j, \lambda_j)\}_{j=1}^{+\infty}$ *is a sequence of* $\partial Q \times [0, 1]$ *converging to* (x, λ) *with* $X = \lambda Ax + \lambda B^n x$ *and* $0 \le \lambda < 1$, *then* $\lambda_j Ax_j + \lambda_j B^n x_j \in Q$ *for j sufficiently large,*

(*v*) *B is uniformly asymptotically regular with respect to A.*

Then, A + B has a fixed point in Q.

Proof. We first define $F_n := a_n A + a_n B^n$, where $a_n := (1 - \frac{1}{n})/k_n \in (0, 1)$ for all $n \in \mathbb{N}$. Since F_n is the sum of a compact mapping and a strict contraction mapping, it follows that F_n is a condensing mapping. For any let fixed n, we have $\{(y_j, \lambda_j)\}_{j=1}^{+\infty}$ is a sequence of $\partial Q \times [0, 1]$ converging to (y, λ) with $y = \lambda F_n(y)$ and $0 \le \lambda < 1$. Then $y = a_n \lambda Ay + a_n \lambda B^n y$. From assumption (*iv*), it follows that $a_n \lambda_j Ay_j + a_n \lambda_j B^n y_j \in Q$ for j sufficiently large. Applying Lemma 2.12 to F_n, we deduce that there is an $x_n \in Q$ such that

$$x_n = F_n x_n = a_n Ax_n + a_n B^n x_n.$$

As in Theorem 3.5 this implies that

$$(I - B)x_n \rightharpoonup y. \tag{3.37}$$

By hypothesis (*iii*), the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in Q$. Since A is weakly-strongly continuous, $\{Ax_{n_k}\}$ converges strongly to Ax. It follows that

$$x_{n_k} - Bx_{n_k} = (I - B)x_{n_k} \to Ax \text{ as } k \to \infty. \tag{3.38}$$

Hence, by the demiclosedness of $I - B$, we have $Ax + Bx = x$. □

Corollary 3.8. *Let Q be a closed convex bounded set in a uniformly convex Banach space X with* $0 \in Q$. *Suppose* $A : Q \to X$ *and* $B : X \to X$ *are continuous mappings satisfying the following:*

(*i*) $A(Q)$ *is compact and A is weakly-strongly continuous,*

(*ii*) *B is an asymptotically nonexpansive mapping with a sequence* $(k_n) \subset [1, \infty)$,

(*iii*) *if* $\{(x_j, \lambda_j)\}_{j=1}^{+\infty}$ *is a sequence of* $\partial Q \times [0, 1]$ *converging to* (x, λ) *with* $X = \lambda Ax + \lambda B^n x$ *and* $0 \le \lambda < 1$, *then* $\lambda_j Ax_j + \lambda_j B^n x_j \in Q$ *for j sufficiently large,*

(*iv*) *B is uniformly asymptotically regular with respect to A.*

Then, A + B has a fixed point in Q.

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Authors' contributions
The work presented here was carried out in collaboration between all authors. SP and AA defined the research theme. SP designed theorems and methods of proof and interpreted the results. AA proved the theorems, interpreted the results and wrote the paper. All authors have contributed to, seen and approved the manuscript.

Competing interests

The authors declare that they have no competing interests.

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