Horváth *Journal of Inequalities and Applications* 2011, **2011**:26
http://www.journalofinequalitiesandapplications.com/content/2011/1/26

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A parameter-dependent refinement of the discrete Jensen's inequality for convex and mid-convex functions

László Horváth

Correspondence: lhorvath@almos.vein.hu
Department of Mathematics, University of Pannonia, 8200 Veszprém, Egyetem u. 10., Hungary

Abstract

In this paper, a new parameter-dependent refinement of the discrete Jensen's inequality is given for convex and mid-convex functions. The convergence of the introduced sequences is also studied. One of the proofs requires an interesting convergence theorem with probability theoretical background. We apply the results to define some new quasi-arithmetic and mixed symmetric means and study their monotonicity and convergence.

1 Introduction and the main results

The considerations of this paper concern

(A_1) an arbitrarily given real vector space X, a convex subset C of X, and a finite subset $\{x_1,\ldots, x_n\}$ of C, where $n \geq 1$ is fixed;

(A_2) a convex function $f : C \to \mathbb{R}$, and a discrete distribution $p_1,\ldots, p_n$, which means that $p_j \geq 0$ with $\sum_{j=1}^{n} p_j = 1$;

(A_3) a mid-convex function $f : C \to \mathbb{R}$, and a discrete distribution $p_1,\ldots, p_n$ with rational p_j $(1 \leq j \leq n)$.

The function $f : C \to \mathbb{R}$ is called convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \quad x,\ y \in C, \quad 0 \leq \alpha \leq 1, \tag{1}$$

and mid-convex if

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y), \quad x,\ y \in C.$$

For a variety of applications, the discrete Jensen's inequalities are important:

Theorem A. (see [1]) (a) *If (A_1) and (A_2) are satisfied, then*

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leq \sum_{j=1}^{n} p_j f(x_j). \tag{2}$$

(b) *If (A_1) and (A_3) are satisfied, then (2) also holds.*

Let $\mathbb{N} := \{0, 1, 2,\ldots\}$ and let $\mathbb{N}_+ := \{1, 2,\ldots\}$.

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Various attempts have been made to refine inequality (2) in the following ways: Assume either (A_1) and (A_2) or (A_1) and (A_3). Let $m \geq 2$ be an integer, and let I denote either the set $\{1,..., m\}$ or the set $\mathbb{N}_+$.

(B) Create a decreasing real sequence $(B_k)_{k\in I}$ such that $B_k = B_k(f, x_i, p_i)$ $(k \in I)$ is a sum whose index set is a subset of $\{1,..., n\}^k$ and

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leq \cdots \leq B_k \leq \cdots \leq B_1 = \sum_{j=1}^{n} p_j f(x_j), \quad k \in I. \tag{3}$$

(C) Create an increasing real sequence $(C_k)_{k\in I}$ such that $C_k = C_k(f, x_i, p_i)$ $(k \in I)$ is a sum whose index set is a subset of $\{1,..., k\}^n$ and

$$f\left(\sum_{j=1}^{n} p_j x_j\right) = C_1 \leq \cdots \leq C_k \leq \cdots \leq \sum_{j=1}^{n} p_j f(x_j), \quad k \in I. \tag{4}$$

The next two typical results belong to the group of refinements of type (B).

These examples use $p_1 = \cdots = p_n = \frac{1}{n}$. In [2], Pečarić and Volenec have constructed the sequence

$$f_k := \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<\cdots<i_k\leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right), \quad 1 \leq k \leq n, \tag{5}$$

while the other sequence

$$\bar{f}_k := \frac{1}{\binom{n+k-1}{k}} \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right), \quad k \in \mathbb{N}_+ \tag{6}$$

is due to Pečarić and Svrtan [3]. In a recent work, [4] Horváth and Pečarić define a lot of new sequences, they generalize and give a uniform treatment a number of well-known results from this area, especially (5) and (6) are extended. Horváth develops a method in [5] to construct decreasing real sequences satisfying (3). His paper contains some improvements of the results in [4] and gives a new approach of the topic. The description of the sequences in [4,5] requires some work, so we do not go into the details. The problem (B) has been considered for the classical Jensen's inequality by Horváth [6].

We turn now to the group of refinements of type (C). In contrast to the previous problem, it is not easy to find such results. Recently, Xiao et al. [7] have obtained the sequence

$$F_k := \frac{1}{\binom{n+k-2}{k-1}} \sum_{\substack{i_1+\cdots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+\,(1\leq j\leq n)}} f\left(\frac{1}{n+k-1}\sum_{j=1}^{n} i_j x_j\right), \quad k \in \mathbb{N}_+, \tag{7}$$

which satisfies (4) with $p_1 = \cdots = p_n = \frac{1}{n}$.

In this paper, we establish a new solution of the problem (C). The constructed sequence $(C_k(\lambda))_{k\geq 0}$ depends on a parameter λ belonging to $[1, \infty[$, and we can use arbitrary discrete distribution $p_1,..., p_n$, not just the appropriate discrete uniform distribution. We give the limit of the sequence under fixed parameter. We also study the convergence of the sequence when the parameter varies and $k \in \mathbb{N}$ is fixed. Finally, some applications are given which concern the theme of means.

The next theorems are the main results of this paper. We need some further hypotheses:

(A_4) Let $\lambda \geq 1$.

(A_5) Let $\lambda \geq 1$ be rational.

First, we give a refinement of the discrete Jensen's inequality (2).

Theorem 1 *Suppose either (A_1), (A_2), and (A_4) or (A_1), (A_3), and (A_5). Introduce the sets*

$$S_k := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N},$$

and for $k \in \mathbb{N}$ define the numbers

$$\begin{aligned} C_k(\lambda) &= C_k(x_1, \ldots, x_n; p_1, \ldots, p_n; \lambda) \\ &:= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1! \ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) f\left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right). \end{aligned} \tag{8}$$

Then,

$$f\left(\sum_{j=1}^{n} p_j x_j\right) = C_0(\lambda) \leq C_1(\lambda) \leq \cdots \leq C_k(\lambda) \leq \cdots \leq \sum_{j=1}^{n} p_j f(x_j), \quad k \in \mathbb{N}.$$

Remark 2 *(a) It follows from the definition of S_k that $S_k \subset \{0,\ldots, k\}^n$ ($k \in \mathbb{N}$).*

(b) It is easy to see that

$$C_k(1) = f\left(\sum_{j=1}^{n} p_j x_j\right), \quad k \in \mathbb{N}. \tag{9}$$

Finally, we establish two convergence theorems.

Theorem 3 *Suppose (A_1), (A_2), and (A_4). Suppose × is a normed space and f is continuous. Then,*

(a) For every fixed $\lambda > 1$

$$\lim_{k\to\infty} C_k(\lambda) = \sum_{j=1}^{n} p_j f(x_j).$$

(b) The function $\lambda \to C_k(\lambda)$ ($\lambda \geq 1$) is continuous for every $k \in \mathbb{N}$.

The proof of Theorem 3(a) requires a lemma (see Lemma 15), which is interesting in its own right. Probability theoretical technique will be used to handle this problem.

Remark 4 *In the previous theorem, it suffices to consider the case when* (A_1), (A_2), *and* (A_4) *are satisfied. Really, if f is mid-convex and continuous, then convex.*

By (9)

$$\lim_{k\to\infty} C_k(1) = f\left(\sum_{j=1}^{n} p_j x_j\right).$$

We come now to the second convergence theorem.

Theorem 5 *Suppose either* (A_1), (A_2), *and* (A_4) *or* (A_1), (A_3), *and* (A_5). *For each fixed* $k \in \mathbb{N}_+$

$$\lim_{\lambda\to\infty} C_k(\lambda) = \sum_{j=1}^{n} p_j f(x_j).$$

2 Discussion and applications

Suppose either (A_1), (A_2), and (A_4) or (A_1), (A_3), and (A_5). First, we give three special cases of (8).

(a) $k = 1$, $n \in \mathbb{N}_+$:

$$C_1(\lambda) = \frac{1}{n+\lambda-1} \sum_{i=1}^{n} (1+(\lambda-1)p_i) f\left(\frac{\sum_{j=1}^{n} p_j x_j + (\lambda-1)p_i x_i}{1+(\lambda-1)p_i}\right).$$

(b) $k \in \mathbb{N}$, $n = 2$:

$$C_k(\lambda) = \frac{1}{(\lambda+1)^k} \sum_{i=0}^{k} \binom{k}{i} \left(\lambda^i p_1 + \lambda^{k-i} p_2\right) f\left(\frac{\lambda^i p_1 x_1 + \lambda^{k-i} p_2 x_2}{\lambda^i p_1 + \lambda^{k-i} p_2}\right).$$

(c) $p_1 = \cdots = p_n := \frac{1}{n}$:

$$C_k(\lambda) = \frac{1}{n(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j}\right) f\left(\frac{\sum_{j=1}^{n} \lambda^{i_j} x_j}{\sum_{j=1}^{n} \lambda^{i_j}}\right).$$

Assume further that f is strictly convex (strictly mid-convex) that is strict inequality holds in (1) whenever $x \neq y$ and $0 < \alpha < 1$. In this case, equality is satisfied in (2) if and only if $x_1 = \cdots = x_n$, and therefore, it comes from the third part of the proof of Theorem 1 that

$$C_k(\lambda) < \sum_{j=1}^{n} p_j f(x_j), \quad k \in \mathbb{N}, \tag{10}$$

if not all x_i are equal.

If $p_1 = \cdots = p_n := \frac{1}{n}$ and f is strictly convex (strictly mid-convex), then the analysis of the proof of Theorem 1 shows that

$$f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) = C_0(\lambda) < C_1(\lambda) < \cdots < C_k(\lambda) < \cdots < \frac{1}{n}\sum_{j=1}^{n} f(x_j), \quad k \in \mathbb{N},$$

whenever not all x_i are equal.

If the inequality (10) holds, X is a normed space and f is continuous (see Remark 4), then Theorem 3(b) and Theorem 5 insure that the range of the function $\lambda \to C_k(\lambda)$ $(k \in \mathbb{N}_+)$ is the interval

$$\left[f\left(\sum_{j=1}^{n} p_j x_j\right), \quad \sum_{j=1}^{n} p_j f(x_j) \right[.$$

Conjecture 6 *Suppose either* (A_1), (A_2), *and* (A_4) *or* (A_1), (A_3), *and* (A_5).

The function $\lambda \to C_k(\lambda)$ $(\lambda \ge 1)$ *is increasing for every* $k \in \mathbb{N}$.

Next, we define some new quasi-arithmetic means and study their monotonicity and convergence. About means see [8].

Definition 7 *Let* $I \subset \mathbb{R}$ *be an interval, let* $x_j \in I$ $(1 \le j \le n)$, *let* $p_1,\ldots, p_n$ *be a discrete distribution, and let* g, $h : I \to \mathbb{R}$ *be continuous and strictly monotone functions. Let* $\lambda \ge 1$. *We define the quasi-arithmetic means with respect to (8) by*

$$\begin{aligned} M_{h,g}(k, \lambda) := h^{-1}\Bigg(&\frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right) \\ &\cdot (h\circ g^{-1})\left(\frac{\sum_{j=1}^{n}\lambda^{i_j}p_j g(x_j)}{\sum_{j=1}^{n}\lambda^{i_j}p_j}\right)\Bigg), \quad k \in \mathbb{N}. \end{aligned} \tag{11}$$

Some other means are also needed.

Definition 8 *Let* $I \subset \mathbb{R}$ *be an interval, let* $x_j \in I$ $(1 \le j \le n)$, *and let* $p_1,\ldots, p_n$ *be a discrete distribution. For a continuous and strictly monotone function* $z : I \to \mathbb{R}$, *we introduce the following mean*

$$M_z := z^{-1}\left(\sum_{j=1}^{n} p_j z(x_j)\right). \tag{12}$$

We now prove the monotonicity of the means (11) and give limit formulas.

Proposition 9 *Let* $I \subset \mathbb{R}$ *be an interval, let* $x_j \in I$ $(1 \le j \le n)$, *let* $p_1,\ldots, p_n$ *be a discrete distribution, and let* g, $h : I \to \mathbb{R}$ *be continuous and strictly monotone functions. Let* $\lambda \ge 1$. *Then,*

(a)

$$M_g = M_{h,g}(0, \lambda) \le \cdots \le M_{h,g}(k, \lambda) \le \cdots \le M_h, \quad k \in \mathbb{N},$$

if either $h \circ g^{-1}$ is convex and h is increasing or $h \circ g^{-1}$ is concave and h is decreasing.

(b)

$$M_g = M_{h,g}(0, \lambda) \geq \cdots \geq M_{h,g}(k, \lambda) \geq \cdots \geq M_h, \quad k \in \mathbb{N},$$

if either $h \circ g^{-1}$ is convex and h is decreasing or $h \circ g^{-1}$ is concave and h is increasing.

(c) Moreover, in both cases

$$\lim_{k\to\infty} M_{h,g}(k, \lambda) = M_h$$

for each fixed $\lambda > 1$, and

$$\lim_{\lambda\to\infty} M_{h,g}(k, \lambda) = M_h$$

for each fixed $k \in \mathbb{N}_+$.

Proof. Theorem 1 can be applied to the function $h \circ g^{-1}$, if it is convex ($-h \circ g^{1}$, if it is concave) and the n-tuples $(g(x_1),..., g(x_n))$, then upon taking h^{-1}, we get (a) and (b). (c) comes from Theorems 3(a) and 5. ■

As a special case, we consider the following example.

Example 10 *If $I :=]0, \infty[$, $h := \ln$, and $g(x) := x$ $(x \in]0, \infty[)$, then by Proposition 9(b), we have the following inequality. for every $x_j > 0$ $(1 \leq j \leq n)$, $\lambda \geq 1$, and $k \in \mathbb{N}_+$*

$$\sum_{j=1}^{n} p_j x_j \geq \prod_{(i_1,\ldots,i_n)\in S_k} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right)^{\frac{1}{(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \geq \prod_{j=1}^{n} x_j^{p_j},$$

which gives a sharpened version of the arithmetic mean - geometric mean inequality

$$\frac{1}{n}\sum_{j=1}^{n} x_j \geq \prod_{(i_1,\ldots,i_n)\in S_k} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j}} \right)^{\frac{1}{n(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \sum\limits_{j=1}^{n} \lambda^{i_j}} \geq \prod_{j=1}^{n} x_j^{\frac{1}{n}}.$$

Finally, we investigate some mixed symmetric means.

The power means of order $r \in \mathbb{R}$ are defined as follows:

Definition 11 *Let $x_j \in]0, \infty[$ $(1 \leq j \leq n)$, and let $p_1,..., p_n$ be a discrete distribution with $p_j > 0$ $(1 \leq j \leq n)$.*

$$M_r = M_r(x_1, \ldots, x_n; p_1, \ldots, p_n) := \begin{cases} \left(\sum\limits_{j=1}^{n} p_j x_j^r\right)^{\frac{1}{r}}, & r \neq 0 \\ \left(\prod\limits_{j=1}^{n} x_j^{p_j}\right), & r \neq 0 \end{cases}.$$

If $r \neq 0$, then the power means of order r belong to the means (12) ($z :]0, \infty[\to \mathbb{R}$, $z(x) := x^r$), while we get the power means of order 0 by taking limit. Supported by the power means, we can introduce mixed symmetric means corresponding to (8):

Definition 12 *Let $x_j \in]0, \infty[$ $(1 \le j \le n)$, and let $p_1,..., p_n$ be a discrete distribution with $p_j > 0$ $(1 \le j \le n)$. Let $\lambda \ge 1$, and $k \in \mathbb{N}$. We define the mixed symmetric means with respect to (8) by*

$$M_{s,t}(k, \lambda) := \left(\frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) \cdot M_t^s \left(x_1, \ldots, x_n; \frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right) \right)^{\frac{1}{s}},$$

if $s, t \in \mathbb{R}$ and $s \ne 0$, and

$$M_{0,t}(k, \lambda) := \prod_{(i_1,\ldots,i_n)\in S_k} \left(M_t \left(x_1, \ldots, x_n; \frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right) \right)^{\frac{1}{(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right)},$$

where $t \in \mathbb{R}$.

The monotonicity and the convergence of the previous means are studied in the next result.

Proposition 13 *Let $x_j \in]0, \infty[$ $(1 \le j \le n)$, and let $p_1,..., p_n$ be a discrete distribution with $p_j > 0$ $(1 \le j \le n)$. Let $\lambda \ge 1$, and $k \in \mathbb{N}$. Suppose $s, t \in \mathbb{R}$ such that $s \le t$. Then,*

(a)

$$M_t = M_{s,t}(0, \lambda) \ge \cdots \ge M_{s,t}(k, \lambda) \ge \cdots \ge M_s, \quad k \in \mathbb{N}.$$

(b) In case of $s, t \ne 0$

$$\lim_{k\to\infty} M_{s,t}(k, \lambda) = M_s$$

for each fixed $\lambda > 1$, and

$$\lim_{\lambda\to\infty} M_{s,t}(k, \lambda) = M_s$$

for each fixed $k \in \mathbb{N}_+$.

Proof. Assume $s, t \ne 0$. Then, Proposition 9 (b) can be applied with $g, h :]0, \infty[\to \mathbb{R}$, $g(x) := x^t$, and $h(x) := x^s$. If $s = 0$ or $t = 0$, the result follows by taking limit. ■

3 Some lemmas and the proofs of the main results

Lemma 14 *Let $k \in \mathbb{N}$ and $(i_1,..., i_n) \in S_{k+1}$ be fixed. If we set*

$$z(i_1, \ldots, i_n) := \{j \in \{1, \ldots, n\} | i_j \ne 0\},$$

then

$$\sum_{j\in z(i_1,\dots,i_n)} \frac{k!}{i_1!\dots i_{j-1}!(i_j-1)!i_{j+1}!\dots i_n!} = \frac{(k+1)!}{i_1!\dots i_n!}.$$

Proof. The lowest common denominator is $i_1!\dots\, i_n!$. Combined with $\sum_{j=1}^{n} i_j = k+1$, the result follows. ■

The proof of Theorem 1.

Proof. (a) We separate the proof of this part of the theorem into three steps.

Let $\lambda \geq 1$ be fixed.

I. Since $S_0 = \{(0,\dots, 0)\}$

$$C_0(\lambda) = \left(\sum_{j=1}^{n} \lambda^0 p_j\right) f\left(\frac{\sum_{j=1}^{n} \lambda^0 p_j x_j}{\sum_{j=1}^{n} \lambda^0 p_j}\right) = f\left(\sum_{j=1}^{n} p_j x_j\right).$$

II. Next, we prove that $C_k(\lambda) \leq C_{k+1}(\lambda)$ $(k \in \mathbb{N})$.

It is easy to check that for every $(i_1,\dots, i_n) \in S_k$

$$\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j} = \frac{1}{n+\lambda-1}$$

$$\cdot \sum_{l=1}^{n} \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j + (\lambda-1)\lambda^{i_l} p_l x_l}{\sum_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l} \cdot \frac{\sum_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right).$$

With the help of Theorem A, this yields that

$$f\left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right) \leq \frac{1}{n+\lambda-1} \sum_{l=1}^{n} \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right.$$

$$\left. \cdot f\left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j + (\lambda-1)\lambda^{i_l} p_l x_l}{\sum_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l}\right)\right).$$

Consequently,

$$C_k(\lambda) \le \frac{1}{(n+\lambda-1)^{k+1}} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!}$$
$$\cdot \sum_{l=1}^{n} \left(\left(\sum_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l \right) f \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j + (\lambda-1)\lambda^{i_l} p_l x_l}{\sum_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l} \right) \right). \tag{13}$$

By Lemma 14, it is easy to see that the right-hand side of (13) can be written in the form

$$\frac{1}{(n+\lambda-1)^{k+1}} \sum_{(i_1,\ldots,i_n)\in S_{k+1}} \frac{(k+1)!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) f \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right)$$

which is just $C_{k+1}(\lambda)$.

III. Finally, we prove that

$$C_k(\lambda) \le \sum_{j=1}^{n} p_j f(x_j), \quad k \in \mathbb{N}_+. \tag{14}$$

It follows from Theorem A that

$$\begin{aligned} C_k(\lambda) &\le \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \left(\frac{k!}{i_1!\ldots i_n!} \sum_{j=1}^{n} \lambda^{i_j} p_j f(x_j) \right) \\ &= \frac{1}{(n+\lambda-1)^k} \sum_{j=1}^{n} \left(\sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \lambda^{i_j} \right) p_j f(x_j), \quad k \in \mathbb{N}_+. \end{aligned} \tag{15}$$

The multinomial theorem shows that

$$\sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \lambda^{i_j} = (n+\lambda-1)^k, \quad 1 \le j \le n,$$

hence (15) implies (14). ■

The proof of Theorem 3 (a) is based on the following interesting result. The σ-algebra of Borel subsets of $\mathbb{R}^n$ is denoted by $\mathcal{B}^n$.

Lemma 15 *Let $p_1,\ldots, p_n$ be a discrete distribution with $n \ge 2$, and let $\lambda > 1$. Let $l \in \{1,\ldots, n\}$ be fixed. e_l denotes the vector in $\mathbb{R}^n$ that has 0s in all coordinate positions except the lth, where it has a 1. Let $q_1,\ldots, q_n$ be also a discrete distribution such that $q_j > 0$ $(1 \le j \le n)$ and*

$$q_l > \max(q_1, \ldots q_{l-1}, q_{l+1}, \ldots, q_n). \tag{16}$$

If

$$g : \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n | t_j > 0\,(1 \le j \le n), \quad \sum_{j=1}^{n} t_j = 1 \right\} \to \mathbb{R}$$

is a bounded function for which

$$\tau_l := \lim_{e_l} g$$

exists in $\mathbb{R}$ *and* $p_l > 0$, *then*

$$\lim_{k\to\infty} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} q_1^{i_1}\ldots q_n^{i_n} g\left(\frac{\lambda^{i_1}p_1}{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j}, \ldots, \frac{\lambda^{i_n}p_n}{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j}\right) = \tau_l. \tag{17}$$

Proof. To prove the result, we can obviously suppose that $l = 1$.

For the sake of clarity, we shall denote the element $(i_1,\ldots, i_n)$ of S_k by $(i_{1k},\ldots, i_{nk})$ $(k \in \mathbb{N}_+)$.

Let $\xi_k := (\xi_{1k},\ldots, \xi_{nk})$ $(k \in \mathbb{N}_+)$ be a $(\mathbb{R}^n, \mathcal{B}^n)$-random variable on a probability space $(\Omega, \mathcal{A}, P)$ such that ξ_k has multinomial distribution of order k and with parameters $q_1,\ldots, q_n$. A fundamental theorem of the statistics (see [9], Theorem 5.4.13), which is based on the multidimensional central limit theorem and the Cochran-Fisher theorem, implies that

$$\lim_{k\to\infty} P\left(\sum_{j=1}^{n}\frac{(\xi_{jk}-kq_j)^2}{kq_j} < t\right) = F_{n-1}(t), \quad t\in\mathbb{R}, \tag{18}$$

where F_{n-1} means the distribution function of the Chi-squared distribution (χ^2-distribution) with n - 1 degrees of freedom.

Choose $0 < \varepsilon < 1$. Since F_{n-1} is continuous and strictly increasing on $]0, \infty[$, there exists a unique $t_\varepsilon > 0$ such that

$$F_{n-1}(t_\varepsilon) = 1 - \varepsilon.$$

Define

$$S_k^1 := \left\{ (i_{1k}, \ldots, i_{nk}) \in S_k \mid \sum_{j=1}^{n} k\frac{\left(\frac{i_{jk}}{k} - q_j\right)^2}{q_j} < t_\varepsilon \right\}.$$

The definition of the set S_k^1 shows that

$$\sum_{(i_{1k},\ldots,i_{nk})\in S_k^1} \frac{k!}{i_{1k}!\ldots i_{nk}!} q_1^{i_{1k}}\ldots q_n^{i_{nk}} = P\left((\xi_{1k}, \ldots, \xi_{nk}) \in S_k^1\right) \tag{19}$$

$$\begin{aligned} &= P\left(\sum_{j=1}^{n} k\frac{\left(\frac{\xi_{jk}}{k} - q_j\right)^2}{q_j} < t_\varepsilon\right) = P\left(\sum_{j=1}^{n}\frac{(\xi_{jk}-kq_j)^2}{kq_j} < t_\varepsilon\right) \\ &= F_{n-1}(t_\varepsilon) + \left(P\left(\sum_{j=1}^{n}\frac{(\xi_{jk}-kq_j)^2}{kq_j} < t_\varepsilon\right) - F_{n-1}(t_\varepsilon)\right) \\ &= 1 - \varepsilon + \delta_\varepsilon(k), \quad k\in\mathbb{N}_+, \end{aligned} \tag{20}$$

where by (18)

$$\lim_{k\to\infty} \delta_\varepsilon(k) = 0. \tag{21}$$

For $j = 1,..., n$ construct the sequences $(I_k^j)_{k\geq 1}$ by

$$I_k^j := i_{jk}^*, \text{ if } \left|\frac{i_{jk}^*}{k} - q_j\right| = \max\left\{\left|\frac{i_{jk}}{k} - q_j\right| \mid (i_{1k}, \ldots, i_{nk}) \in S_k^1\right\}, \quad k \in \mathbb{N}_+. \tag{22}$$

We claim that

$$\lim_{k\to\infty} \frac{I_k^j}{k} = q_j, \quad 1 \leq j \leq n. \tag{23}$$

Fix $1 \leq j \leq n$. If (23) is false, then (22) yields that we can find a positive number ρ, a strictly increasing sequence $(k_u)_{u\geq 1}$ and points

$$(i_{1k_u}, \ldots, i_{nk_u}) \in S_{k_u}^1, \quad u \in \mathbb{N}_+ \tag{24}$$

such that

$$\left|\frac{i_{jk_u}}{k_u} - q_j\right| \geq \rho, \quad u \in \mathbb{N}_+,$$

and therefore,

$$k_u \frac{\left(\dfrac{i_{jk_u}}{k_u} - q_j\right)^2}{q_j} \geq k_u \frac{\rho^2}{q_j} \to \infty \text{ as } u \to \infty,$$

contrary to (24).

Let

$$q := \max(q_2, \ldots, q_n).$$

It follows from (16) that

$$\gamma := \frac{1}{3}(q_1 - q) > 0. \tag{25}$$

By (22) and (23), we can find an integer k_γ such that for each $k > k_\gamma$

$$\left|\frac{i_{jk}}{k} - q_j\right| \leq \left|\frac{I_k^j}{k} - q_j\right| < \gamma, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1, \quad 1 \leq j \leq n.$$

Thus, for every $k > k_\gamma$

$$\frac{i_{1k}}{k} > q_1 - \gamma \text{ and } \frac{i_{jk}}{k} < q_j + \gamma, \quad 2 \leq j \leq n, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1,$$

and hence, we get from (25) that

$$i_{1k} - i_{jk} > k\gamma \quad 2 \leq j \leq n, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1, \quad k > k_\gamma. \tag{26}$$

We can see that

$$i_{1k} - i_{jk} \to \infty \text{ as } k \to \infty, \quad 2 \leq j \leq n, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1. \tag{27}$$

Now, set $S_k^2 := S_k \backslash S_k^1 (k \in \mathbb{N}_+)$, and consider the sequences

$$a_k^1 := \sum_{(i_{1k},\dots,i_{nk})\in S_k^1} \frac{k!}{i_{1k}!\dots i_{nk}!} q_1^{i_{1k}} \dots q_n^{i_{nk}} g\left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}, \dots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}\right),$$

and

$$a_k^2 := \sum_{(i_{1k},\dots,i_{nk})\in S_k^2} \frac{k!}{i_{1k}!\dots i_{nk}!} q_1^{i_{1k}} \dots q_n^{i_{nk}} g\left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}, \dots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}\right),$$

where $k \in \mathbb{N}_+$. The sum of these sequences is just the studied sequence in (17). Since $p_1 > 0$, we obtain from (27) that

$$\lim_{k\to\infty} \frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j} = 1, \quad (i_{1k}, \dots, i_{nk}) \in S_k^1, \tag{28}$$

and

$$\lim_{k\to\infty} \frac{\lambda^{i_{lk}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j} = 0, \quad 2 \le l \le n, \quad (i_{1k}, \dots, i_{nk}) \in S_k^1. \tag{29}$$

According to (26), the convergence is uniform for all the possible sequences in (28) and (29); hence, for every $\varepsilon_1 > 0$, we can find an integer $k_{\varepsilon_1} > k_\gamma$ that for all $k > k_{\varepsilon_1}$

$$\tau_1 - \varepsilon_1 < g\left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}, \dots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}\right) < \tau_1 + \varepsilon_1, \quad (i_{1k}, \dots, i_{nk}) \in S_k^1. \tag{30}$$

Bringing in (19-20), we find that

$$P\left((\xi_{1k}, \dots, \xi_{nk}) \in S_k^2\right) = \varepsilon - \delta_\varepsilon(k), \quad k \in \mathbb{N}_+,$$

and therefore, thanks to (19-20), (30) and the boundedness of g ($|g| \le m$)

$$\begin{aligned} &(1-\varepsilon + \delta_\varepsilon(k))(\tau_1 - \varepsilon_1) - (\varepsilon - \delta_\varepsilon(k))m \le a_k^1 + a_k^2 \\ &\le (1-\varepsilon + \delta_\varepsilon(k))(\tau_1 + \varepsilon_1) + (\varepsilon - \delta_\varepsilon(k))m, \quad k > k_{\varepsilon_1}. \end{aligned}$$

Consequently, by (21)

$$\begin{aligned} (1-\varepsilon)(\tau_1 - \varepsilon_1) - \varepsilon m &\le \liminf_{k\to\infty}(a_k^1 + a_k^2) \le \limsup_{k\to\infty}(a_k^1 + a_k^2) \\ &\le (1-\varepsilon)(\tau_1 + \varepsilon_1) + \varepsilon m, \end{aligned}$$

and this proves the convergence claim (17).

The proof is now complete. ■

The proof of Theorem 3.

Proof. (a) We have only to observe that for every fixed $1 \le l \le n$

$$\lim_{k\to\infty} \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\dots,i_n)\in S_k} \frac{k!}{i_1!\dots i_n!} \lambda^{i_l} p_l f\left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right) = p_l f(x_l). \tag{31}$$

The case $p_l = 0$ is trivial.

To prove the case $p_l > 0$, define the function

$$g \, : \, \left\{ (t_1, \, \dots, \, t_n) \in \mathbb{R}^n | t_j > 0 \, (1 \le j \le n), \quad \sum_{j=1}^{n} t_j = 1 \right\} \to \mathbb{R}$$

by

$$g(t_1, \, \dots, \, t_n) := f\left(\sum_{j=1}^{n} t_j x_j\right).$$

Consequently, the limit in (31) can be written in the form

$$\lim_{k\to\infty} p_l \sum_{(i_1,\dots,i_n)\in S_k} \frac{k!}{i_1!\dots i_n!} \left(\frac{1}{n+\lambda-1}\right)^{i_1} \cdots \left(\frac{1}{n+\lambda-1}\right)^{i_{l-1}} \left(\frac{\lambda}{n+\lambda-1}\right)^{i_l}$$
$$\cdot \left(\frac{1}{n+\lambda-1}\right)^{i_{l+1}} \cdots \left(\frac{1}{n+\lambda-1}\right)^{i_n} g\left(\frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \dots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right).$$

Now, we can apply Lemma 15 with

$$q_j = \frac{1}{n+\lambda-1}, \quad 1 \le j \le n, \quad j \ne l, \text{ and } q_l = \frac{\lambda}{n+\lambda-1}$$

and

$$\lim_{e_l} g = f(x_l), \quad 1 \le l \le n.$$

(b) Elementary considerations show this part of the theorem.

The proof is complete. ■

The proof of Theorem 5.

Proof. Theorem A confirms that f is bounded on the set

$$G := \left\{ \sum_{j=1}^{n} t_j x_j \in C | t_j \ge 0 \quad (1 \le j \le n), \quad \sum_{j=1}^{n} t_j = 1 \right\},$$

where t_j $(1 \le j \le n)$ is also rational if f is mid-convex.

It is elementary that for every $(i_1,\dots, i_n) \in S_k$

$$\lim_{\lambda\to\infty} \frac{\lambda^{i_l}}{(n+\lambda-1)^k} = \begin{cases} 1, & \text{if } i_l = k \\ 0, & \text{if } i_l < k \end{cases}, \quad 1 \le l \le n.$$

By the definition of the set S_k, $(0,..., 0, k, 0,..., 0)$ (the vector has 0s in all coordinate positions except the lth) is the only element of S_k for which $i_l = k$ $(1 \le l \le n)$. By using the boundedness of f on G, the previous assumptions imply the result, bringing the proof to an end. ■

Acknowledgements
This study was supported by the Hungarian National Foundations for Scientific Research Grant No. K73274.

Competing interests
The author declares that he has no competing interests.

Received: 8 March 2011 Accepted: 25 July 2011 Published: 25 July 2011

References

1. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge Mathematical Library Series. Cambridge University Press (1967)
2. Pečarić, JE, Volenec, V: Interpolation of the Jensen inequality with some applications, Österreích. Akad Wiss Math-Natur Kl Sitzungsber II. **197**, 463–467 (1988)
3. Pečarić, JE, Svrtan, D: Unified approach to refinements of Jensen's inequalities. Math Inequal Appl. **5**, 45–47 (2002)
4. Horváth, L, Pečarić, JE: A refinement of the discrete Jensen's inequality. Math Inequal Appl. (to appear)
5. Horváth, L: A method to refine the discrete Jensen's inequality for convex and mid-convex functions. doi:10.1016/j.mcm.2011.05.060
6. Horváth, L: Inequalities corresponding to the classical Jensen's inequality. J Math Inequal. **3**, 189–200 (2009)
7. Xiao, Z-G, Srivastava, HM, Zhang, Z-H: Further refinements of the Jensen inequalities based upon samples with repetitions. Math Comput Mod. **51**, 592–600 (2010)
8. Mitrinović, DS, Pečarić, JE, Fink, AM: Classical and New Inequalities in Analysis, vol. 61 of Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht (1993)
9. Dacunha-Castello, D, Duflo, M: Probability and Statistics I. Springer-Verlag, New York (1986)

doi:10.1186/1029-242X-2011-26
Cite this article as: Horváth: A parameter-dependent refinement of the discrete Jensen's inequality for convex and mid-convex functions. *Journal of Inequalities and Applications* 2011 **2011**:26.