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Regularization of ill-posed mixed variational inequalities with non-monotone perturbations

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Abstract

In this paper, we study a regularization method for ill-posed mixed variational inequalities with non-monotone perturbations in Banach spaces. The convergence and convergence rates of regularized solutions are established by using a priori and a posteriori regularization parameter choice that is based upon the generalized discrepancy principle.

Keywords: monotone mixed variational inequality, non-monotone perturbations, regularization, convergence rate

1 Introduction

Variational inequality problems in finite-dimensional and infinite-dimensional spaces appear in many fields of applied mathematics such as convex programming, nonlinear equations, equilibrium models in economics, and engineering (see [1-3]). Therefore, methods for solving variational inequalities and related problems have wide applicability. In this paper, we consider the mixed variational inequality: for a given $f \in X^*$, find an element $x_0 \in X$ such that

$$\langle Ax_0 - f, x - x_0 \rangle + \varphi(x) - \varphi(x_0) \geq 0, \quad \forall x \in X, \tag{1}$$

where $A : X \to X^*$ is a monotone-bounded hemicontinuous operator with domain $D(A) = X$, $\phi : X \to \mathbb{R}$ is a proper convex lower semicontinuous functional and X is a real reflexive Banach space with its dual space X^*. For the sake of simplicity, the norms of X and X^* are denoted by the same symbol $||\cdot||$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$.

By S_0 we denote the solution set of the problem (1). It is easy to see that S_0 is closed and convex whenever it is not empty. For the existence of a solution to (1), we have the following well-known result (see [4]):

Theorem 1.1. *If there exists $u \in$ dom ϕ satisfying the coercive condition*

$$\lim_{||x|| \to \infty} \frac{\langle Ax, x - u\rangle + \varphi(x)}{||x||} = \infty, \tag{2}$$

then (1) *has at least one solution.*

Many standard extremal problems can be considered as special cases of (1). Denote ϕ by the indicator function of a closed convex set K in X,

$$\varphi(x) \equiv I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the problem (1) is equivalent to that of finding $x_0 \in K$ such that

$$\langle Ax_0 - f, x - x_0 \rangle \geq 0, \quad \forall x \in K. \tag{3}$$

In the case K is the whole space X, the later variational inequality is of the form of the following operator equation:

$$Ax_0 = f. \tag{4}$$

When A is the Gâteaux derivative of a finite-valued convex function F defined on X, the problem (1) becomes the nondifferentiable convex optimization problem (see [4]):

$$\min_{x \in X} \{F(x) + \varphi(x)\}. \tag{5}$$

Some methods have been proposed for solving problem (1), for example, the proximal point method (see [5]), and the auxiliary subproblem principle (see [6]). However, the problem (1) is in general ill-posed, as its solutions do not depend continuously on the data (A, f, ϕ), we used stable methods for solving it. A widely used and efficient method is the regularization method introduced by Liskovets [7] using the perturbative mixed variational inequality:

$$\langle A_h x_\alpha^\tau + \alpha U(x_\alpha^\tau - x_*) - f_\delta, x - x_\alpha^\tau \rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x_\alpha^\tau) \geq 0, \quad \forall x \in X, \tag{6}$$

where A_h is a monotone operator, α is a regularization parameter, U is the duality mapping of X, $x_* \in X$ and $(A_h, f_\delta, \phi_\varepsilon)$ are approximations of (A, f, ϕ), $\tau = (h, \delta, \varepsilon)$. The convergence rates of the regularized solutions defined by (6) are considered by Buong and Thuy [8].

In this paper, we do not require $A_h : x_* \in X$ to be monotone. In this case, the regularized variational inequality (6) may be unsolvable. In order to avoid this fact, we introduce the regularized problem of finding $x_\alpha^\tau \in X$ such that

$$\begin{aligned} &\langle A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x - x_\alpha^\tau \rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x_\alpha^\tau) \\ &\qquad \geq -\mu g(\|x_\alpha^\tau\|)\|x - x_\alpha^\tau\|, \quad \forall x \in X, \ \mu \geq h, \end{aligned} \tag{7}$$

where μ is positive small enough, U^s is the generalized duality mapping of X (see Definition 1.3) and x_* is in X which plays the role of a criterion of selection, g is defined below.

Assume that the solution set S_0 of the inequality (1) is non-empty, and its data A, f, ϕ are given by A_h, f_δ, ϕ_ε satisfying the conditions:

(1) $\| f - f_\delta \| \leq \delta$, $\delta \to 0$;

(2) $A_h : X \to X^*$ is not necessarily monotone, $D(A_h) = D(A) = X$, and

$$\|A_h x - Ax\| \leq hg(\|x\|), \quad \forall x \in X, \ h \to 0, \tag{8}$$

with a non-negative function $g(t)$ satisfying the condition

$$g(t) \leq g_0 + g_1 t^\nu, \quad \nu = s - 1, \ g_0, \ g_1 \geq 0;$$

(3) $\phi_\varepsilon : X \to \mathbb{R}$ is a proper convex lower semicontinuous functional for which there exist positive numbers c_ε and r_ε such that

$$\varphi_\varepsilon(x) \geq -c_\varepsilon ||x|| \quad \text{as } ||x|| > r_\varepsilon$$

and

$$|\varphi_\varepsilon(x) - \varphi(x)| \leq \varepsilon d(||x||), \quad \forall x \in X, \ \varepsilon \to 0, \tag{9}$$

$$|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| \leq C_0 ||x - y||, \quad \forall x, y \in X, \tag{10}$$

where C_0 is some positive constant, $d(t)$ has the same properties as $g(t)$.

In the next section we consider the existence and uniqueness of solutions x_α^τ of (7), for every $\alpha > 0$. Then, we show that the regularized solutions x_α^τ converge to $x_0 \in S_0$, the x_*-minimal norm solution defined by

$$||x_0 - x_*|| = \arg\min_{x \in S_0} ||x - x_*||.$$

The convergence rate of the regularized solutions x_α^τ to x_0 will be established under the condition of inverse-strongly monotonicity for A and the regularization parameter choice based on the generalized discrepancy principle.

We now recall some known definitions (see [9-11]).

Definition 1.1. An operator $A : D(A) = X \to X^*$ is said to be

(a) hemicontinuous if $A(x + t_n y) \rightharpoonup Ax$ as $t_n \to 0^+$, $x, y \in X$, and demicontinuous if $x_n \to x$ implies $Ax_n \rightharpoonup Ax$;

(b) monotone if $\langle Ax - Ay, x - y\rangle \geq 0$, $\forall x, y \in X$;

(c) inverse-strongly monotone if

$$\langle Ax - Ay, x - y\rangle \geq m_A ||Ax - Ay||^2, \quad \forall x, y \in X, \tag{11}$$

where m_A is a positive constant.

It is well-known that a monotone and hemicontinuous operator is demicontinuous and a convex and lower semicontinuous functional is weakly lower semicontinuous (see [9]). And an inverse-strongly monotone operator is not strongly monotone (see [10]).

Definition 1.2. It is said that an operator $A : X \to X^*$ has S-property if the weak convergence $x_n \rightharpoonup x$ and $\langle Ax_n - Ax, x_n - x\rangle \to 0$ imply the strong convergence $x_n \to x$ as $n \to \infty$.

Definition 1.3. The operator $U^s : X \to X^*$ is called the generalized duality mapping of X if

$$U^s(x) = \{x^* \in X^* : \langle x^*, x\rangle = ||x^*|| \, ||x|| \, ; \, ||x^*|| = ||x||^{s-1}\}, \quad s \geq 2. \tag{12}$$

When $s = 2$, we have the duality mapping U. If X and X^* are strictly convex spaces, U^s is single-valued, strictly monotone, coercive, and demicontinuous (see [9]).

Let $X = L^p(\Omega)$ with $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^m$ measurable, we have

$$U(\varphi) = ||\varphi||^{2-p}_{L^p(\Omega)} |\varphi(t)|^{p-2}\varphi(t), \quad t \in \Omega.$$

Assume that the generalized duality mapping U^s satisfies the following condition:

$$\langle U^s(x) - U^s(y), x - y\rangle \geq m_s ||x - y||^s, \quad \forall x, \ y \in X, \tag{13}$$

where m_s is a positive constant. It is well-known that when X is a Hilbert space, then $U^s = I$, $s = 2$ and $m_s = 1$, where I denotes the identity operator in the setting space (see [12]).

2 Main result

Lemma 2.1. *Let X^* be a strictly convex Banach space. Assume that A is a monotone-bounded hemicontinuous operator with $D(A) = X$ and conditions* **(2)** *and* **(3)** *are satisfied. Then, the inequality* (7) *has a non-empty solution set S_ε for each $\alpha > 0$ and $f_\delta \in X^*$.*

Proof. Let $x_\varepsilon \in \operatorname{dom} \phi_\varepsilon$. The monotonicity of A and assumption (**3**) imply the following inequality:

$$\frac{\langle Ax + \alpha U^s(x - x_*), x - x_\varepsilon\rangle + \varphi_\varepsilon(x)}{||x||} \geq \frac{\alpha ||x - x_*||^{s-1}(||x - x_*|| - ||x_* - x_\varepsilon||)}{||x||} - ||Ax_\varepsilon|| \left(1 + \frac{||x_\varepsilon||}{||x||}\right) - c_\varepsilon, \quad s \geq 2,$$

for $||x|| > r_\varepsilon$. Consequently, (2) is fulfilled for the pair $(A + \alpha U^s, \phi_\varepsilon)$. Thus, for each $\alpha > 0$ and $f_\delta \in X^*$, there exists a solution of the following inequality:

$$\langle Ax + \alpha U^s(x - x_*) - f_\delta, z - x\rangle + \varphi_\varepsilon(z) - \varphi_\varepsilon(x) \geq 0, \quad \forall z \in X,\ x \in X. \tag{14}$$

Observe that the unique solvability of this inequality follows from the monotonicity of A and the strict monotonicity of U^s. Indeed, let x_1 and x_2 be two different solutions of (14). Then,

$$\langle Ax_1 + \alpha U^s(x_1 - x_*) - f_\delta, z - x_1\rangle + \varphi_\varepsilon(z) - \varphi_\varepsilon(x_1) \geq 0, \quad \forall z \in X \tag{15}$$

and

$$\langle Ax_2 + \alpha U^s(x_2 - x_*) - f_\delta, z - x_2\rangle + \varphi_\varepsilon(z) - \varphi_\varepsilon(x_2) \geq 0, \quad \forall z \in X. \tag{16}$$

Putting $z = x_2$ in (15) and $z = x_1$ in (16) and add the obtained inequalities, we obtain

$$\langle Ax_1 - Ax_2, x_2 - x_1\rangle + \alpha\langle U^s(x_1 - x_*) - U^s(x_2 - x_*), x_2 - x_1\rangle \geq 0.$$

Due to the monotonicity of A and the strict monotonicity of U^s, the last inequality occurs only if $x_1 = x_2$.

Let $x_\alpha^{\delta,\varepsilon}$ be a solution of (14), that is,

$$\langle Ax_\alpha^{\delta,\varepsilon} + \alpha U^s(x_\alpha^{\delta,\varepsilon} - x_*) - f_\delta, z - x_\alpha^{\delta,\varepsilon}\rangle + \varphi_\varepsilon(z) - \varphi_\varepsilon(x_\alpha^{\delta,\varepsilon}) \geq 0, \quad \forall z \in X. \tag{17}$$

For all $h > 0$, making use of (8), from (17) one gets

$$\langle A_h x_\alpha^{\delta,\varepsilon} + \alpha U^s(x_\alpha^{\delta,\varepsilon} - x_*) - f_\delta, z - x_\alpha^{\delta,\varepsilon}\rangle + \varphi_\varepsilon(z) - \varphi_\varepsilon(x_\alpha^{\delta,\varepsilon}) \geq -hg(||x_\alpha^{\delta,\varepsilon}||)||z - x_\alpha^{\delta,\varepsilon}||, \quad \forall z \in X. \tag{18}$$

Since $\mu \geq h$, we can conclude that each $x_\alpha^{\delta,\varepsilon}$ is a solution of (7).
□

Let x_α^τ be a solution of (7). We have the following result.

Theorem 2.1. *Let X and X^* be strictly convex Banach spaces and A be a monotone-bounded hemicontinuous operator with $D(A) = X$. Assume that conditions* **(1)**-**(3)** *are* *satisfied, the operator U^s satisfies condition* (13) *and, in addition, the operator A has the S-property. Let*

$$\lim_{\alpha\to 0}\frac{\mu+\delta+\varepsilon}{\alpha}=0. \tag{19}$$

Then $\{x_\alpha^\tau\}$*converges strongly to the* x_**-minimal norm solution* $x_0 \in S_0$.

Proof. By (1) and (7), we obtain

$$\begin{aligned}&\langle A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x_0 - x_\alpha^\tau\rangle + \varphi_\varepsilon(x_0) - \varphi_\varepsilon(x_\alpha^\tau)\\ &\quad + \langle Ax_0 - f, x_\alpha^\tau - x_0\rangle + \varphi(x_\alpha^\tau) - \varphi(x_0) \geq -\mu g(||x_\alpha^\tau||)||x_0 - x_\alpha^\tau||.\end{aligned}$$

This inequality is equivalent to the following

$$\begin{aligned}\alpha\langle U^s(x_\alpha^\tau - x_*) - U^s(x_0 - x_*), x_\alpha^\tau - x_0\rangle &\leq \alpha\langle U^s(x_0 - x_*), x_0 - x_\alpha^\tau\rangle\\ &+ \langle A_h x_\alpha^\tau - Ax_\alpha^\tau, x_0 - x_\alpha^\tau\rangle\\ &+ \langle Ax_0 - Ax_\alpha^\tau, x_\alpha^\tau - x_0\rangle + \langle f - f_\delta, x_0 - x_\alpha^\tau\rangle\\ &+ \varphi_\varepsilon(x_0) - \varphi(x_0) + \varphi(x_\alpha^\tau) - \varphi_\varepsilon(x_\alpha^\tau)\\ &+ \mu g(||x_\alpha^\tau||)||x_0 - x_\alpha^\tau||.\end{aligned} \tag{20}$$

The monotonicity of A, assumption **(1)**, and the inequalities (8), (9), (13) and (20) yield the relation

$$\begin{aligned}m_s||x_\alpha^\tau - x_0||^s &\leq \left[\frac{h+\mu}{\alpha}g(||x_\alpha^\tau||) + \frac{\delta}{\alpha}\right]||x_0 - x_\alpha^\tau||\\ &\quad + \frac{\varepsilon}{\alpha}[d(||x_0||) + d(||x_\alpha^\tau||)] + \langle U^s(x_0 - x_*), x_0 - x_\alpha^\tau\rangle.\end{aligned} \tag{21}$$

Since $\mu/\alpha \to 0$ as $\alpha \to 0$ (and consequently, $h/\alpha \to 0$), it follows from (19) and the last inequality that the set x_α^τ are bounded. Therefore, there exists a subsequence of which we denote by the same x_α^τ weakly converges to $\bar{x} \in X$.

We now prove the strong convergence of $\{x_\alpha^\tau\}$ to $\bar{x}$. The monotonicity of A and U^s implies that

$$\begin{aligned}0 &\leq \langle Ax_\alpha^\tau - A\bar{x}, x_\alpha^\tau - \bar{x}\rangle\\ &\leq \langle Ax_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*) - A\bar{x} - \alpha U^s(\bar{x} - x_*), x_\alpha^\tau - \bar{x}\rangle\\ &= \langle Ax_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - \bar{x}\rangle - \langle A\bar{x} + \alpha U^s(\bar{x} - x_*), x_\alpha^\tau - \bar{x}\rangle.\end{aligned} \tag{22}$$

In view of the weak convergence of $\{x_\alpha^\tau\}$ to $\bar{x}$, we have

$$\lim_{\alpha\to 0}\langle A\bar{x} + \alpha U^s(\bar{x} - x_*), x_\alpha^\tau - \bar{x}\rangle = 0. \tag{23}$$

By virtue of (8),

$$\begin{aligned}&\langle Ax_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - \bar{x}\rangle\\ &\quad = \langle Ax_\alpha^\tau - A_h x_\alpha^\tau + A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - \bar{x}\rangle\\ &\quad \leq \langle A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - \bar{x}\rangle + hg(||x_\alpha^\tau||)||x_\alpha^\tau - \bar{x}||.\end{aligned} \tag{24}$$

Using further (7), we deduce

$$\begin{aligned}&\langle A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - \bar{x}\rangle\\ &\quad = \langle A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x_\alpha^\tau - \bar{x}\rangle + \langle f_\delta, x_\alpha^\tau - \bar{x}\rangle\\ &\quad \leq \langle f_\delta, x_\alpha^\tau - \bar{x}\rangle + \varphi_\varepsilon(\bar{x}) - \varphi_\varepsilon(x_\alpha^\tau) + \mu g(||x_\alpha^\tau||)||\bar{x} - x_\alpha^\tau||.\end{aligned} \tag{25}$$

Since $x_\alpha^\tau \rightharpoonup \bar{x}$ and ϕ_ε is proper convex weakly lower semicontinuous, we have from (25) that

$$\lim_{\alpha\to 0}\langle A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - \bar{x}\rangle \le 0. \tag{26}$$

By (22)-(24) and (26), it results that

$$\lim_{\alpha\to 0}\langle A x_\alpha^\tau - A\bar{x}, x_\alpha^\tau - \bar{x}\rangle = 0.$$

Finally, the S property of A implies the strong convergence of $\{x_\alpha^\tau\}$ to $\bar{x} \in X$.

We show that $\bar{x} \in S_0$. By (8) and take into account (7) we obtain

$$\begin{aligned}\langle A x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x - x_\alpha^\tau\rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x_\alpha^\tau)\\ \ge -(h+\mu)g(\|x_\alpha^\tau\|)\|x - x_\alpha^\tau\|, \quad \forall x \in X.\end{aligned} \tag{27}$$

Since the functional ϕ is weakly lower semicontinuous,

$$\varphi(\bar{x}) \le \lim_{\alpha\to 0}\inf \varphi(x_\alpha^\tau). \tag{28}$$

Since $\{x_\alpha^\tau\}$ is bounded, by (9), there exists a positive constant c_2 such that

$$\varphi(x_\alpha^\tau) \le \varphi_\varepsilon(x_\alpha^\tau) + c_2\varepsilon. \tag{29}$$

By letting $\alpha \to 0$ in the inequality (7), provided that A is demicontinuous, from (8), (9), (28), (29) and condition **(1)** imply that

$$\langle A\bar{x} - f, x - \bar{x}\rangle + \varphi(x) - \varphi(\bar{x}) \ge 0, \quad \forall x \in X.$$

This means that $\bar{x} \in S_0$.

We show that $\bar{x} = x_0$. Applying the monotonicity of U^s and the inequalities (8), (9) and (13), we can rewrite (17) as

$$\begin{aligned}\langle U^s(x - x_*), x_\alpha^\tau - x\rangle \le \left[\frac{h+\mu}{\alpha}g(\|x_\alpha^\tau\|) + \frac{\delta}{\alpha}\right]\|x - x_\alpha^\tau\|\\ + \frac{\varepsilon}{\alpha}[d(\|x\|) + d(\|x_\alpha^\tau\|)], \quad \forall x \in S_0.\end{aligned}$$

Since $\alpha \to 0$, ε/α, δ/α, $\mu/\alpha \to 0$ (and $h/\alpha \to 0$), the last inequality becomes

$$\langle U^s(x - x_*), \bar{x} - x\rangle \le 0, \quad \forall x \in S_0.$$

Replacing x by $t\bar{x} + (1-t)x$, $t \in (0, 1)$ in the last inequality, dividing by $(1 - t)$ and then letting t to 1, we get

$$\langle U^s(\bar{x} - x_*), \bar{x} - x\rangle \le 0, \quad \forall x \in S_0$$

or

$$\langle U^s(\bar{x} - x_*), \bar{x} - x_*\rangle \le \langle U^s(\bar{x} - x_*), x - x_*\rangle, \quad \forall x \in S_0.$$

Using the property of U^s, we have that $\|\bar{x} - x_*\| \le \|x - x_*\|$, $\forall x \in S_0$. Because of the convexity and the closedness of S_0, and the strictly convexity of X, we can conclude that $\bar{x} = x_0$. The proof is complete.

□

Now, we consider the problem of choosing posteriori regularization parameter $\tilde{\alpha} = \alpha(\mu, \delta, \varepsilon)$ such that

$$\lim_{\mu,\delta,\varepsilon\to 0}\alpha(\mu,\delta,\varepsilon)=0 \text{ and } \lim_{\mu,\delta,\varepsilon\to 0}\frac{\mu+\delta+\varepsilon}{\alpha(\mu,\delta,\varepsilon)}=0.$$

To solve this problem, we use the function for selecting $\tilde{\alpha}=\alpha(\mu,\delta,\varepsilon)$ by generalized discrepancy principle, i.e. the relation $\tilde{\alpha}=\alpha(\mu,\delta,\varepsilon)$ is constructed on the basis of the following equation:

$$\rho(\tilde{\alpha})=(\mu+\delta+\varepsilon)^p\tilde{\alpha}^{-q},\quad p,q>0, \tag{30}$$

with $\rho(\tilde{\alpha})=\tilde{\alpha}\left(c+||x^\tau_{\tilde{\alpha}}-x_*||^{s-1}\right)$, where $x^\tau_{\tilde{\alpha}}$ is the solution of (7) with $\alpha=\tilde{\alpha}$, c is some positive constant.

Lemma 2.2. *Let X and X^* be strictly convex Banach spaces and $A: X\to X^*$ be a monotone-bounded hemicontinuous operator with $D(A)=X$. Assume that conditions* **(1)**, **(2)** *are satisfied, the operator U^s satisfies condition* (13). *Then, the function $\rho(\alpha)=\alpha\left(c+||x^\tau_\alpha-x_*||^{s-1}\right)$ is single-valued and continuous for $\alpha\ge\alpha_0>0$, where $x^\tau_\alpha$is the solution of* (7).

Proof. Single-valued solvability of the inequality (7) implies the continuity property of the function $\rho(\alpha)$. Let α_1, $\alpha_2\ge\alpha_0$ be arbitrary ($\alpha_0>0$). It follows from (7) that

$$\begin{aligned}\alpha_1\langle U^s(x^\tau_{\alpha_1}-x_*),x^\tau_{\alpha_2}-x^\tau_{\alpha_1}\rangle+\ &\alpha_2\langle U^s(x^\tau_{\alpha_2}-x_*),x^\tau_{\alpha_1}-x^\tau_{\alpha_2}\rangle\\ &+\ \langle A_hx^\tau_{\alpha_1}-A_hx^\tau_{\alpha_2},x^\tau_{\alpha_2}-x^\tau_{\alpha_1}\rangle\\ &\ge\ -\mu\left(g(||x^\tau_{\alpha_1}||)+g(||x^\tau_{\alpha_2}||)\right)||x^\tau_{\alpha_1}-x^\tau_{\alpha_2}||,\end{aligned} \tag{31}$$

where $x^\tau_{\alpha_1}$ and $x^\tau_{\alpha_2}$ are solutions of (7) with $\alpha=\alpha_1$ and $\alpha=\alpha_2$. Using the condition **(2)** and the monotonicity of A, we have

$$\begin{aligned}\alpha_1\langle U^s(x^\tau_{\alpha_1}-x_*)-U^s(x^\tau_{\alpha_2}-x_*),x^\tau_{\alpha_1}-x^\tau_{\alpha_2}\rangle&\\ \le(\alpha_2-\alpha_1)\langle U^s(x^\tau_{\alpha_2}-x_*),x^\tau_{\alpha_1}-x^\tau_{\alpha_2}\rangle&\\ +(h+\mu)\left(g(||x^\tau_{\alpha_1}||)+g(||x^\tau_{\alpha_2}||)\right)||x^\tau_{\alpha_1}-x^\tau_{\alpha_2}||.&\end{aligned}$$

It follows from (13) and the last inequality that

$$m_s||x^\tau_{\alpha_1}-x^\tau_{\alpha_2}||^s\ \le\frac{|\alpha_1-\alpha_2|}{\alpha_0}||x^\tau_{\alpha_2}-x_*||^{s-1}+(h+\mu)\left(g(||x^\tau_{\alpha_1}||)+g(||x^\tau_{\alpha_2}||)\right).$$

Obviously, $x^\tau_{\alpha_1}\to x^\tau_{\alpha_2}$ as $\mu\to 0$ and $\alpha_1\to\alpha_2$. It means that the function $||x^\tau_\alpha-x_*||^{s-1}$ is continuous on $[\alpha_0;+\infty)$. Therefore, $\rho(\alpha)$ is also continuous on $[\alpha_0;+\infty)$.

Theorem 2.2. *Let X and X^* be strictly convex Banach spaces and $A: X\to X^*$ be a monotone-bounded hemicontinuous operator with $D(A)=X$. Assume that conditions* **(1)**-**(3)** *are satisfied, the operator U^s satisfies condition* (13). *Then*

(i) there exists at least a solution $\tilde{\alpha}$of the equation (30),
(ii) let μ, δ, $\varepsilon\to 0$. Then
(1) $\tilde{\alpha}\to 0$;
(2) if $0<p<q$ *then* $\dfrac{\mu+\delta+\varepsilon}{\tilde{\alpha}}\to 0$, $x^\tau_{\tilde{\alpha}}\to x_0\in S_0$*with x_*-minimal norm and there exist constants C_1, $C_2>0$ such that for sufficiently small μ, δ, $\varepsilon>0$ the relation*

$$C_1\le(\mu+\delta+\varepsilon)^p\tilde{\alpha}^{-1-q}\le C_2 \tag{32}$$

holds.

Proof.

(i) For $0 < \alpha < 1$, it follows from (7) that

$$\langle A_h x_\alpha^\tau + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x_* - x_\alpha^\tau \rangle + \varphi_\varepsilon(x_*) - \varphi_\varepsilon(x_\alpha^\tau) \geq -\mu g(\|x_\alpha^\tau\|)\|x_* - x_\alpha^\tau\|.$$

Hence,

$$\alpha\langle U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - x_*\rangle \leq \mu g(\|x_\alpha^\tau\|)\|x_* - x_\alpha^\tau\| + \varphi_\varepsilon(x_*) - \varphi_\varepsilon(x_\alpha^\tau) + \langle A_h x_\alpha^\tau - Ax_\alpha^\tau + Ax_\alpha^\tau - Ax_* + Ax_* - f + f - f_\delta, x_* - x_\alpha^\tau\rangle.$$

We invoke the condition **(1)**, the monotonicity of A, (8), (10), (12), and the last inequality to deduce that

$$\alpha\|x_\alpha^\tau - x_*\|^{s-1} \leq (h + \mu)g(\|x_\alpha^\tau\|) + C_0 + \|Ax_* - f\| + \delta. \tag{33}$$

It follows from (33) and the form of $\rho(\alpha)$ that

$$\begin{aligned}\alpha^q\rho(\alpha) &= \alpha^{1+q}(c + \|x_\alpha^\tau - x_*\|^{s-1}) \\ &= c\alpha^{1+q} + \alpha^q \times \alpha\|x_\alpha^\tau - x_*\|^{s-1} \\ &\leq c\alpha^{1+q} + \alpha^q[(h + \mu)g(\|x_\alpha^\tau\|) + C_0 + \|Ax_* - f\| + \delta].\end{aligned}$$

Therefore, $\lim_{\alpha\to+0} \alpha^q\rho(\alpha) = 0$.

On the other hand,

$$\lim_{\alpha\to+\infty} \alpha^q\rho(\alpha) \geq c \lim_{\alpha\to+\infty} \alpha^{1+q} = +\infty.$$

Since $\rho(\alpha)$ is continuous, there exists at leat one $\tilde{\alpha}$ which satisfies (30).

(ii) It follows from (30) and the form of $\rho(\tilde{\alpha})$ that

$$\tilde{\alpha} \leq c^{-1/(1+q)}(\mu + \delta + \varepsilon)^{p/(1+q)}.$$

Therefore, $\tilde{\alpha} \to 0$ as μ, δ, $\varepsilon \to 0$.

If $0 < p < q$, it follows from (30) and (32) that

$$\begin{aligned}\left[\frac{\mu + \delta + \varepsilon}{\tilde{\alpha}}\right]^p &= [(\mu + \delta + \varepsilon)^p\tilde{\alpha}^{-q}]\tilde{\alpha}^{q-p} \\ &= [c\tilde{\alpha} + \tilde{\alpha}\|x_{\tilde{\alpha}}^\tau - x_*\|^{s-1}]\tilde{\alpha}^{q-p} \\ &\leq c\tilde{\alpha}^{1+q-p} + \tilde{\alpha}^{q-p}[2\mu g(\|x_{\tilde{\alpha}}^\tau\|) + C_0 + \|Ax_* - f\| + \delta].\end{aligned}$$

So,

$$\lim_{\mu,\delta,\varepsilon\to 0}\left[\frac{\mu + \delta + \varepsilon}{\tilde{\alpha}}\right]^p = 0.$$

By Theorem 2.1 the sequence $x_{\tilde{\alpha}}^\tau$ converges to $x_0 \in S_0$ with x_*-minimal norm as μ, δ, $\varepsilon \to 0$.

Clearly,

$$(\mu + \delta + \varepsilon)^p\tilde{\alpha}^{-1-q} = \tilde{\alpha}^{-1}\rho(\tilde{\alpha}) = (c + \|x_{\tilde{\alpha}}^\tau - x_*\|^{s-1}),$$

therefore, there exists a positive constant C_2 such that (32). On the other hand, because $c > 0$ so there exists a positive constant C_1 satisfied (32). This finishes the proof.

□

Theorem 2.3. *Let X be a strictly convex Banach space and A be a monotone-bounded hemicontinuous operator with D(A) = X. Suppose that*

(i) for each h, δ, ε >0 conditions **(1)**-**(3)** *are satisfied;*
(ii) U^s *satisfies condition* (13);
(iii) A is an inverse-strongly monotone operator from X into X, Fréchet differentiable at some neighborhood of* $x_0 \in S_0$ *and satisfies*

$$||A(x) - A(x_0) - A'(x_0)(x - x_0)|| \leq \tilde{\tau}||A(x) - A(x_0)||; \tag{34}$$

(iv) there exists $z \in X$ *such that*

$$A'(x_0)^* z = U^s(x_0 - x_*);$$

then, if the parameter $\alpha = \alpha(\mu, \delta, \varepsilon)$ *is chosen by* (30) *with* $0 < p < q$, *we have*

$$||x^\tau_{\alpha(\mu,\delta,\varepsilon)} - x_0|| = O((\mu + \delta + \varepsilon)^{\mu_1}), \quad \mu_1 = \frac{1}{1+q}\min\left\{\frac{1+q-p}{s}, \frac{p}{2s}\right\}.$$

Proof. By an argument analogous to that used for the proof of the first part of Theorem 2.1, we have (21). The boundedness of the sequence $\{x^\tau_\alpha\}$ follows from (21) and the properties of $g(t)$, $d(t)$ and α. On the other hand, based on (20), the property of U^s and the inverse-strongly monotone property of A we get that

$$\|A(x^\tau_\alpha) - A(x_0)\|^2 \leq m_A^{-1}\Big\{\big[(h+\mu)g(\|x^\tau_\alpha\|) + \delta + \alpha\|x^\tau_\alpha - x_*\|^{s-1}\big]\|x_0 - x^\tau_\alpha\| + \varepsilon[d(\|x_0\|) + d(\|x^\tau_\alpha\|)]\Big\}.$$

Hence,

$$||A(x^\tau_\alpha) - A(x_0)|| = O(\sqrt{\delta + \mu + \varepsilon + \alpha}).$$

Further, by virtue of conditions (iii), (iv) and the last estimate, we obtain

$$\begin{aligned}\langle U^s(x_0 - x_*), x_0 - x^\tau_\alpha\rangle &= \langle z, A'(x_0)(x_0 - x^\tau_\alpha)\rangle \\ &\leq ||z||(\tilde{\tau}+1)||A(x^\tau_\alpha) - A(x_0)|| \\ &\leq ||z||(\tilde{\tau}+1)O(\sqrt{\delta + \mu + \varepsilon + \alpha}).\end{aligned}$$

Consequently, (21) has the form

$$\begin{aligned} m_s||x^\tau_\alpha - x_0||^s \leq &\frac{2\mu g(||x^\tau_\alpha||) + \delta}{\alpha}||x_0 - x^\tau_\alpha|| \\ &+ ||z||(\tilde{\tau}+1)O(\sqrt{\delta+\mu+\varepsilon+\alpha}) \\ &+ \frac{\varepsilon}{\alpha}[d(||x_0||) + d(||x^\tau_\alpha||)]. \end{aligned} \tag{35}$$

When α is chosen by (30), it follows from Theorem 2.1 that

$$\alpha(\mu, \delta, \varepsilon) \leq C_1^{-1/(1+q)}(\mu + \delta + \varepsilon)^{p/(1+q)}$$

and

$$\frac{\mu+\delta+\varepsilon}{\alpha(\mu,\delta,\varepsilon)} \le C_2(\mu+\delta+\varepsilon)^{1-p}\alpha^q(\mu,\delta,\varepsilon)$$
$$\le C_2C_1^{-q/(1+q)}(\mu+\delta+\varepsilon)^{1-p/(1+q)}.$$

Therefore, it follows from (35) that

$$m_s||x^\tau_{\alpha(\mu,\delta,\varepsilon)} - x_0||^s \le \tilde{C}_1(\mu+\delta+\varepsilon)^{1-p/(1+q)}||x^\tau_{\alpha(\mu,\delta,\varepsilon)} - x_0|| + \tilde{C}_2(\mu+\delta+\varepsilon)^{1-p/(1+q)} + \tilde{C}_3(\mu+\delta+\varepsilon)^{p/2(1+q)},$$

where $\tilde{C}_i$, $i = 1, 2, 3$, are the positive constants. Using the implication

$$a, b, c \ge 0, \quad s > t, \quad a^s \le ba^t + c \Rightarrow a^s = O(b^{s/(s-t)} + c),$$

we obtain

$$||x^\tau_{\alpha(\mu,\delta,\varepsilon)} - x_0|| = O((\mu+\delta+\varepsilon)^{\mu_1}).$$

Remark 2.1 If α is chosen a priori such that $\alpha \sim (\mu + \delta + \varepsilon)^\eta$, $0 < \eta < 1$, it follows from (35) that

$$m_s||x^\tau_{\alpha(\mu,\delta,\varepsilon)} - x_0||^s \le \tilde{C}_4(\mu+\delta+\varepsilon)^{1-\eta}||x_0 - x^\tau_{\alpha(\mu,\delta,\varepsilon)}|| + \tilde{C}_5(\mu+\delta+\varepsilon)^{\eta/2} + \tilde{C}_6(\mu+\delta+\varepsilon)^{1-\eta}.$$

Therefore,

$$||x^\tau_{\alpha(\mu,\delta,\varepsilon)} - x_0|| = O((\mu+\delta+\varepsilon)^{\mu_2}), \quad \mu_2 = \min\left\{\frac{1-\eta}{s}, \frac{\eta}{2s}\right\}.$$

Remark 2.2 Condition (34) was proposed in [13] for studying convergence analysis of the Landweber iteration method for a class of nonlinear operators. This condition is used to estimate convergence rates of regularized solutions of ill-posed variational inequalities in [14].

Remark 2.3 The generalized discrepancy principle for regularization parameter choice is presented in [15] for the ill-posed operator equation (4) when A is a linear and bounded operator in Hilbert space. It is considered and applied to estimating convergence rates of the regularized solution for equation (4) involving an accretive operator in [16].

Competing interests

The author declares that they have no competing interests.

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